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OPERATIONS RESEARCH

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OPERATIONS RESEARCH
AN INTRODUCTION

By

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Preface to the Second Edition

The authors feel highly encouraged and satisfied by the overwhelming response to the first edition of the book *Operations Research—An Introduction*. We are placing in the hands of the readers a thoroughly revised, improved and updated *second edition* of the book. Many new chapters have been added, existing chapters are thoroughly revised and a large number of problems from the various university and competitive examinations have been included both as solved examples and unsolved exercises. The book contains about 200 solved and 340 unsolved problems.

In preparing the second edition, the valuable comments and suggestions from the readers of various universities have been properly incorporated. We hope that the present volume will have wider scope and prove much more useful to all the readers—students, teachers and professionals of O.R.

The authors feel highly obliged to all who took the trouble of sending them valuable comments. Suggestions for further improvement of the book will be gratefully accepted.

PREM KUMAR GUPTA
D. S. HIRA

Preface to the First Edition

For more than thirty years, a new branch of science called operations research has been fast developing. The essence of operations research are models which help us arrive at optimum decisions. It is these models which constitute the subject matter of this book.

While writing on operations research, a particular problem may be approached in many different ways. Thus, the exposition may be intended for mathematicians, economists, engineers, socialogists, etc. This book is written primarily for Engineering graduates.

This book is simply an introduction to the vast subject of operations research. Efforts have been made to simplify the technical material without distorting it. The book does not require a high level of mathematical knowledge on the part of the reader. An elementary knowledge of differential and integral calculus is all that is required to understand the subject.

Each chapter begins with a number of important and interesting examples taken from a variety of fields. Almost every problem presents a new idea. The authors feel that knowing the various fields in which a model can help, the reader will gather more interest and incentive to know its theoretical and mathematical background and method of application. Additional examples towards the end of each chapter are provided to test the reader's understanding of the subject matter.

Every effort has been made to present the subject in easy, clear, lucid and systematic manner. References at the end of each chapter are given to cover more advanced extension of the topics presented.

The authors express their deep gratitude and thanks to Shri T. K. Kundra, Design Engineer, I. D. D. C., I. I. T., New Delhi for his inspiration, valuable suggestions, guidance and help every moment they sought.

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Suggestions for further improvement of the book will be gratefully accepted.

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Basics of Operations Research

The main purpose of this book is to provide the reader with basic concepts of operations research (abbreviated to OR). The subject is so vast that in any introductory text, such as this, the discussion of various topics has got to be limited. Complete volumes have been written on some of the topics discussed here. Attempt has been made to present a variety of material within a limited structure. References are indicated at the end of each chapter to provide the reader with exhaustive treatment of the different topics.

This chapter provides an overall view of the subject of operations research. It covers some general ideas on the subject, thus providing a perspective. The remaining chapters deal with specific ideas and specific methods of solving OR problems.

1.1 Development of Operations Research

(*i*) *Pre-World War II* : No science has ever been born on a specific day. Operations research is no exception. Its roots are as old as science and society. Though the roots of OR extend to even early 1800s, it was in 1885 when Ferderick W. Taylor emphasised the application of scientific analysis to methods of production, that the real start took place. Taylor conducted experiments in connection with a simple shovel. His aim was to find that weight load of material moved by shovel which would result in maximum of material moved with minimum of fatigue. After many experiments with varying weights, he obtained the optimum weight load, which though much lighter than that commonly used, provided maximum movement of material during a day. In 1917, A. K. Erlang, a Danish mathematician, published his work on the problem of congestion of telephone traffic. The difficulty was that during busy periods, telephone operators were unable to handle the calls the moment they were made, resulting in delayed calls. A few years after its appearance, his work was accepted by the British Post Office as the basis for calculating circuit facilities.

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During the 1930s, H. C. Levinson applied scientific analysis to the problems of merchandising. His work included scientific study of customers' buying habits, response to advertising and relation of environment to the type of article sold.

However, it was the First Industrial Revolution which contributed mainly towards the development of OR. Before this revolution, most of the industries were small scale, employing only a handful of men. The advent of machine tools—the replacement of man by machine as a *source of power* and improved means of transportation and communication resulted in fast flourishing industry. It became increasingly difficult for a single man to perform all the managerial functions (of planning, sale, purchase, production, etc.). Consequently, a division of management function took place. Managers of production, marketing, finance, personnel, research and development etc., began to appear. With further industrial growth, further subdivisions of management functions took place. For example, production department was sub-divided into sections like maintenance quality control, procurement, production planning, etc.

This industrial development, brought with it, a new type of problems called executive-type problems. These problems are a direct consequence of functional division of labour in an organization. In an organization, each functional unit (department or section) performs a part of the whole job and for its successful working, develops its own objectives. These objectives, though in the best interest of the individual department, may not be in the best interest of the organization as a whole. Infact, these objectives of individual departments may be inconsistent and clashing with each other. Consider, for example, the attitudes of the various departments of a business organization towards the inventory policy. The production department wants to have maximum production, associated with the lowest possible cost. This can be achieved by producing only one item continuously. Thus it is interested in long, uninterrupted production runs, because such runs minimise set-up and clean-up costs. Thus it prefers to have a large inventory in relatively few product lines.

The marketing department also wants a large but divers inventory so that a customer may be provided immediate delivery over a wide variety of products. It would also like to have a flexible production policy so as to meet special demands at a short notice.

The finance department wants to minimize inventory so as to minimize the unproductive capital investments 'tied up' in it. It also believes that inventories should rise and fall with rise and fall in company's sales,

The personnel department wants to hire good labour and to retain it. This is possible only when goods are produced continuously for inventory during slack periods also. In other words, it is interested in maintaining a constant production level resulting in large inventory.

To set an inventory policy which serves the interest of the organization as a whole and not that of any individual department is an executive type problem, which can be satisfactorily solved by the application of OR techniques. The decision which is in the best interest of the organization as a whole is called *Optimal* (*optimum*) *decision* and the one in the best interest of an individual department is called *sub-optimal decision*.

(*ii*) *World War II* : During World War II, the military management in England called on a team of scientists to study the strategic and tactical problems of air and land defence. This team of scientists included physicists, psychologists, engineers, mathematicians and others. Many of these problems were of executive type. The objective was to determine the most effective utilization of limited military resources. The application included the effective use of newly invented radar, allocation of British Air Force planes to missions and the determination of best patterns for searching submarines. This group of scientists formed the first OR team.

The name *operations research* (*or operational research*) was apparently coined because the team was carrying out research on (military) operations. The encouraging results of these efforts led to the formation of more such teams in British armed services and the use of such scientific teams soon spread to Western Allies—the United States, Canada and France. Thus though this science of operations research originated in England, the United States soon took the lead. In United States these OR teams helped in developing strategies for mining operations, inventing new flight patterns and planning of sea mines.

(*iii*) *Post-World War II* : Immediately after the war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their problems. Industrial operations research in U.K. and U.S.A. developed along different lines. In U.K., the critical economic situation required drastic increase in production efficiency and creation of new markets. Nationalisation of a few key industries further increased the potential field for OR. Consequently OR soon spread from military to Government, industrial, social and economic planning.

In U.S.A. the situation was different. Impressed by its dramatic success in U.K., defence operations research in U.S.A. was increased. Most of the war-experienced OR workers remained in military services. Industrial executives did not call for much help because they were returning to the peace-time situation and many of them believed that it was merely a new application of an old technique. Operations research has been shown by a variety of names in that country such as, operational analysis, operations evaluation, systems analysis, systems evaluation, systems research and management science.

The progress of industrial operations research in U.S.A. was due to advent of Second Industrial Revolution which resulted in *automation*—the replacement of man by machine as a *source of control*. This new revolution began around 1940s when electronic computers became commercially available. These electronic brains possessed tremendous computational speeed and information storage. But for these digital computers, operations research with its complex computational problems could not have achieved its promising place in all kinds of operational environments.

Today, the impact of operations research can be felt in many areas. This is shown by the ever increasing number of educational institutions offering this subject at degree level. The fast increasing number of management consulting firms speaks of the popularity of the subject. Of late, OR activities have spread to diverse fields such as hospitals, libraries, city planning, transportation systems, crime investigation, etc. Some of the Indian organisations using OR techniques are : Indian Airlines, Railways, Defence Organizations, Fertilizer Corporation of India, Delhi Cloth Mills, Tata Iron & Steel Co., etc.

1.2 Definition of Operations Research

Many definitions of OR have been suggested from time to time. On the other hand are put forward a number of arguments as to why it cannot be defined. Perhaps the subject is too young to be defined in an authoritative way. Some of the different definitions suggested are :

(1) Operations research is a *scientific method* of providing executive departments with a *quantitative basis* for decisions regarding the operations under their control. —Morse & Kimball

(2) Operations research, in the most general sense, can be characterised as the application of *scientific methods, techniques and tools* to problems involving the *operations of systems* so as to provide

these in control of the operations with *optimum solutions* to the problems.
— Churchman, Ackoff, Arnoff

(3) Operations research is applied *decision theory.* It uses any *scientific, mathematical or logical means* to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems.
—Miller and Starr

(4) Operations research is a *scientific approach* to problem solving for executive management. —H.M. Wagner

(5) Operations research is the art of giving *bad answers* to problems, to which, otherwise, *worse answers* are given.
—Thomas L. Saaty

(6) Operations research is an aid for the executive in making his decisions by providing him with the needed *quantitative information* based on the *scientific method of analysis*. —C. Kittel

(7) Operations research is the *systematic, method-oriented* study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a *sound, scientific and quantitative basis* for decision making.
—E.L. Arnoff & M. J. Netzorg

(8) Operations research is the application of *scientific methods* to problems arising from operations involving *integrated systems of men, machines and materials.* It normally utilizes the knowledge and skill of an *interdisciplinary research team* to provide the managers of such systems with *optimum operating solutions.*
—Fabrycky and Torgersen

(9) Operations research is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man-machine systems ; and operations research workers are actively engaged in applying this knowledge to practical problems in business, government and society.
—Operations Research Society of America

(10) Operations research is the application of *scientific method* by *interdisciplinary teams* to problems involving the control of organized (man-machine) systems so as to provide solutions which *best serve the purpose of the organization as a whole.*.
—Ackoff and Sasieni

(11) Operations research utilizes the planned approach (*updated scientific method*) and an *interdisciplinary team* in order to represent complex functional relationships as mathematical models for the

purpose of providing a *quantitative basis* for decision-making and *uncovering new problems* for quantitative analysis.

—Thierauf and Klekamp

1.3 Characteristics of Operations Research

The various definitions of operations research presented in section 1·2 bring out the essential characteristics of operations research. They are

(*i*) its systems (or executive) orientation,
(*ii*) the use of interdisciplinary teams,
(*iii*) application of scientific method, and
(*iv*) uncovering of new problems.

Let us consider each of these in some detail.

1.3.1 System (OR Executive) Orientation of OR

One of the most important characteristics of OR study is its concern with problems as a whole or its system orientation. This means that an activity by any part of an organization has some effect on the activity of every other part. The optimum operation of one part of a system may not be the optimum operation for some other part. Therefore, to evaluate any decison, one must identify all possible interactions and determine their impact on the organization as a whole.

Many problems that appear simple on the surface may not be really so. Take, for example, the inventory policy of an organization, already considered in section 1.1. Production department is interested in long, uninterrupted production runs since they reduce the set-up and clean-up costs. To solve the problem with this viewpoint is simple. However, these long runs will result in large raw material, in-process and finished product inventories in relatively few product lines. This will result in bitter conflict with finance, marketing and personnel departments. As already discussed finance department wants to have the minimum possible inventory ; marketing department, a large but diversified inventory while personnel department wants continuous production during slack periods also resulting in large inventories.

In view of the above difficulties, it is necessary that the problem be analysed with painstaking care and all parts of the organization affected be thoroughly examined. When all factors affecting the system (organization) are known, a *methematical model* can be prepared. A solution to this model will optimize the profits to the system as a whole. Such a solution is called an *optimal* (*optimum*) *solution.*

1.3.2 The use of Interdisciplinary Teams

A second characteristic of OR study is that it is performed y a team of scientists whose individual members have been drawn from different scientific and engineering disciplines. For example, one may find a mathematician, statistician, physicist, psychologist, economist and an engineer working together on an OR problem.

It has been recognised beyond doubt that people from different disciplines can produce more unique solutions with greater probability of success, than could be expected from the same number of persons from a single discipline. For example, when confronted with the problem of increasing production in a plant, the personnel psychologist will try to select better workers or improve their training; the mechanical engineer will try to improve the machines; the industrial engineer will try to simplify the operations or offer incentives; while the systems analyst will try to improve the flow of information through the plant. Thus the OR team can look at the problem from many different angles in order to determine which one (or which combination) of approaches is the best.

Another reason for the existence of OR teams is that knowledge is increasing at a very fast rate. No single person can collect all the useful scientific information from all disciplines. Different members of the OR team bring the latest scientific know-how to analyse the problem and help in providing better results.

1.3.3 Application of Scientific Method

A third distinguishing feature of OR is the use of scientific method to solve the problem under study. Most scientific research, such as chemistry and physics can be carried out well in the laboratories, under controlled conditions, without much interference from the outside world. However, the same is not true in the systems under study by OR teams. For example, no company can risk its failure in order to conduct a successful experiment. Though, experimentation on sub-systems is sometimes resorted to, by and large, a research approach that does not involve experimentation on the total system is preferred.

An operations rasearch worker is in the same position as the astronomer, since the latter can observe the system that he studies, but cannot manipulate it. Therefore, he constructs *representations of the systems and its operations* (*models*) on which he conducts his research. An OR worker also does the same. The construction of a model is described in section 1.8 and the reader is asked to refer it for further details.

1·3.4 Uncovering of new Problems

The fourth characteristic of oprations research, which is often overlooked, is that solution of an OR problem may uncover a number of new problems. Of course, all these uncovered problems need not be solved at the same time. However, in order to derive maximum benefit, each one of them must be solved. It must be remembered that OR is not effectively used if it is restricted to one-shot problems only. In order to derive full benefits, continuity of research must be maintained. Of course, the results of OR study pertaining to a particular problem need not wait until all the connected problems are solved.

1.4. Scientific Method in Operations Research

The scientific method in operations research consists of the following three phases.

(*i*) the judgement phase,
(*ii*) the research phase, and
(*iii*) the action phase.

Out of these, the research phase is the longest. However, the other two phases are also equally important as they provide the basis of the research phase. We shall now consider these phases in a little more detail.

1.4.1 The Judgement Phase

It consists of :

(*a*) *Determination of the operation* : An operation is a combination of different actions dealing with raw ma er als (e.g., men and machines) which form a structure from which action with regard to broader objectives is attained. For example, the act of assembling an engine is an opertion. It consists of many actions which contribute towards the completed assembly. any conceivable operation will always be associated with problems of its successful completion.

(*b*) *Determination of objectives and values associated with the operation* : In the judgement phase, due care must be given to define correctly the frame of references of the operation. Efforts should be made to find the type of situation, e.g., manufacturing, engineering, tactical, strategic, etc. The amount of risk involved, other areas affected by the solution, etc., must be determined. objectives and values, whether economic, social, aesthetic, etc., should be carefully determined so as to have a clear approach to the

solution of the problem. Further, one must determine the time limits for finding the solution, degree of accuracy desired in the results as well as necessity of feedback of the obtained information into the operation.

(c) *Determination of effectiveness measures* : Effectiveness measure (or measure of effectiveness) is a measure of success of a model in representing a problem and providing a solution. It is the connecting link between the objectives and the analysis required for corrective action. It tests the success of a solution and determines if there is need for improving the method of attack or even of altering the solution. Measure of effectiveness may be expressed as ratio or rate. For example, in traffic studies it may be expressed in terms of cars per hour, cars per accident or delay per car ; in bombing of ships it may be expressed as the number of hits per bomb, number of ships sunk per bomb. etc. Effectiveness measure must be chosen properly, since, an improper choice may result in completely wrong conclusions about the problem.

(d) *Formulation of the problem relative to the objectives* : Since every operation is study with problems, the operation analyst must determine the type of problems, its origin and causes. Problems are of many types :

1. *Remedial type* with its origin in actual or threatened accidents, e.g., airplane crashes, job performance hazards.

2. *Optimization type*, e.g. Performing a job more efficiently.

3. *Transference type* consisting of applying the new advances, improvements and inventions in one field to other fields. For example use of isotopes in medicines on one hand (to determine the rate of absorption of substances in certain parts of human body) and in machinery on the other (for testing wearing qualities in automobile tyres.)

4. *Prediction type*, e.g,, forecasting the problems associated with future developments and inventions.

Before selecting a problem for investigation, careful thought must be given to find whether the problem really exists. Hasty selection of problems often leads to wastage of time (devoted by the analyst) and wrong results.

1.4.2 The Research Phase

It includes

(a) *Observation and data collection for better understanding of the problem* : Many times, actual observations by trained observers

at the scene of operation may be difficult and dangerous too. If time permits, operational experiments simulating the actual problem should be set up. Where the information regarding the problem cannot be obtained, the analyst, with the incomplete data at hand, should try to find the missing parts of the problem.

(*b*) *Formulation of relevant hypotheses and models* : Tentative explanations, when formulated as propositions are called *hypotheses*. It is very important to state the hypothesis and its anticipated consequences before starting its verification. The formulation of a good hypothesis depends upon the sound knowledge of subject matter. A hypothesis must provide an answer to the problem in question. It must be capable of verification, otherwise, it may be refuted by empirical evidence.

An essential feature of OR is that is considers a problem as an entity rather than a collection of disconnected sub-operations. *Model* is the device which treats the problem as a whole ; it is essentially a hypothesis. Formulation of models may be based on pure theoretical considerations or on hypothesis derived from known facts and data. The time available for the development of a model is an important factor. For instance, when urgent answers are needed, little will be gained by developing elaborate models. Similarly, for rough estimates, a detailed development of model is only a wastage.

(*c*) *Analysis of available information and verification of hypothesis* : Most of the time that a scientist spends in training is devoted to learning how to analyse and interpret information. Qualitative as well as quantitative methods may be used for this purpose. For example, when it is required to find out work done by a force, a hypothesis is made. It is based on the knowledge of mechanics, giving $W = F \times S$, where S is the distance through which the point of application of the force F moves.

An hypothesis need not be proved for every possibility in order to be acceptable. Sampling methods are usually sufficient to verify it.

(*d*) *Prediction and generalization of results and consideration of alternative methods* : Once a model has been verified, a theory is developed from the model to obtain a complete description of the problem. This is done by studying the effects of changes in the parameters of the model. The theory so developed may be used to extrapolate into the future.

Lastly, the analyst determines the alternative methods of solving the problem and recommends a new research based on revised ohypthesis. The advantages of this approach are obvious and need

little justification if economic and time factors do not stand in the way.

1.4.3 The Action Phase

The action phase consists of making recommendations for remedial action to those who first posed the problem and who control the operations directly. These recommendations consist of a clear statement of the assumptions made, scope and limitations of the information presented about the situation, alternative courses of action, effects of each alternative as well as the preferred course of action.

A primary function of OR group is to provide an administrator with better understanding of the implications of the decisions he makes. The scientific method supplements his ideas and experiences and helps him to attain his goals more fully.

1·5. Necessity of Operations Research in Industry

After having studied as to what is operations research, we shall now try to answer as to why study OR or what is its importance or why its need has been felt by the industry.

As already pointed out, science of OR came into existence in connection with the war operations, to decide the strategy by which enemy could be harmed to the maximum possible extent with the help of the available warfare. War situation required reliable decision making. But its need has been equally felt by the industry due to the following reasons :

(*a*) *Complexity* : In a big industry, the number of factors influencing a decision have increased. Situation has become big and complex because these factors interact with each other in complicated fashion. There is, thus, great uncertainty about the outcome of interaction of factors like technological, environmental, competitive, etc. For instance, consider a factory production schedule which has to take into account

(*i*) customer demand,
(*ii*) requirements of raw materials,
(*iii*) equipment capacity and possibility of equipment failure, and
(*iv*) restrictions on manufacturing processes.

Evidently, it is not easy to prepare a schedule which is both economical and realistic. This needs mathematical models, which, in addition to optimisation, help to analyse the complex situation. With such models, complex problems can be split up into simpler parts,

each part can be analysed separately and then the results can be synthesized to give insights into the problem.

(*b*) *Scattered responsibility and authority* : In a big industry, responsibility and authority of decision making is scattered throughout the organization and thus the organization, if it is not conscious, may be following inconsistent goals. *Mathematical quantification* of OR overcomes this difficulty also to a great extent.

(*c*) *Uncertainty* : There is a great uncertainty about economic and general environment. With economic growth, uncertainty is also growing. This makes each decision costlier and time consuming. OR is, thus, quite essential from reliability point of view.

(*d*) *Knowledge explosion* : Knowledge is increasing at a very fast rate. Majority of the industries are not up-to-date with the latest knowledge and are, therefore, at a disadvantage. OR teams collect the latest information for analysis purposes which is quite useful for the industries.

1·6. Scope of Operations Research

Having known the definition of OR, it is easy to visualize the scope of operations research. Whenever there is a problem for optimization, there is scope for the application of OR. When we broaden the scope of OR, we find that really OR has been practised for hunreds of years before World War II.

In the field of industrial management, there is a chain of problems starting from the purchase of raw materials to the dispatch of finished goods. The management is interested in having an overall view of the methad of optimizing profits. In order to take decision on scientific basis, OR team will have to consider various alternative methods of producing the goods and the return in each case. OR study should also point out the possible changes in the overall structure like installation of a new machine, introduction of more automation etc.

In both developing and developed economies, OR approach is equally applicable. In developing economies, there is a great scope of developing an OR approach towards planning. The basic problem is to orient the planning so that there is maximum growth of per capita income in the shortest possible time, by taking into consideration the national goals and restrictions imposed by the country. The basic problem in most of the countries in Asia and Africa is to remove poverty and hunger as quickly as possible. There is, therefore, a great scope for economists, statisticians, administrators, techni-

cians, politicians and agriculture experts working together to solve this problem with an OR approach.

OR approach needs to be equally developed in agriculture sector on national or international basis. With population explosion and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with climatic conditions and available facilities. The problem of optimal distribution of water from the various water resources is faced by each developing country and a good amount of scientific work can be done in this direction.

OR approach is equally applicable to big and small organizations. For example, whenever a departmental store faces a problem like employing additional sales girls, purchasing an additional van, etc., techniques of OR can be applied to minimize cost and maximize benefit for each such decision.

OR is directly applicable to business and society. For instance, it is increasingly being applied in L.I.C. offices to decide the premium rates of various policies. It has also been extensively used in petroleum, paper, chemical, metal processing, aircraft, rubber, transport and distribution, mining and textile industries.

Thus we find that OR has a diversified and wide scope in the social, economic and industrial problems of today.

1·7. Operations Research and Decision Making

Operations research or management science, as the name suggests, is the science of managing. As is known, management is most of the time making decisions. It is thus a decision science which helps management to make better decisions. Decision is, in fact, a pivotal world in managing. It is not only the headache of management, rather all of us make decisions. We daily decide about minor to major issues. We choose to be engnieers, doctors, etc., a vital decision which is going to affect us throughout our lives We choose to purchase at a particular shop—a decision of relatively minor importance.

Decision making can be improved and, in fact, there is a scope of large scale improvement. The essential cnaracteristics of all decisions are

(*i*) objectives,

(*ii*) alternatives,

(*iii*) influencing factors.

Once these characteristics are known, one can think of improving the characteristics so as to improve upon the decision itself.

Let us consider a situation in which a decision has been taken to see a particular movie and the problem is to decide the conveyance. Three alternatives are available : rickshaw, autorickshaw and a local bus.

In the first level of decision making, autorickshaw is chosen as the mode of conveyance just by intuition, i.e., it is decided at random Evidently, it is a highly emotional and qualitative way of decision making.

In the second level of decision making, the three conveyances are compared and it is decided qualitatively that autorickshaw will be preferred since, though a little, costlier, it is time saving and more comfortable.

In the third level of decision making, the three alternatives are compared and it is suggested that autorickshaw will be chosen, as it will be taking only 1/3rd time than an ordinary rickshaw and shall be only 10% costlier while more comfortable. The local bus is rejected since it would not reach the theatre in time at all.

Though outcome of all these decisions is the same, still we can judge the quality of each decision. We may brand the first decision as 'bad' since it is highly emotional, while we may call the second decision as 'good' since it is scientific though qualitative. The third decision is doubtlessly the best as it is scientific and quantitative.

It is this *scientific quantification* used in OR, which helps management to make better decisions. Thus in OR, the essential features of decisions, namely, objectives, alternatives and influencing factors are expressed in terms of scientific quantifications or mathematical equations. This gives rise to certain mathematical relations, termed as a whole as *mathematical model.* Thus the essence of OR is such mathematical models. For different situations different models are used and this process is continuing since World War II when the term OR was coined.

1.8. Phases of Operations Research or Operations Research Approach or How Operations Research Works

Operations research, like all scientific research, is based on scientific methodology, which proceeds along the following lines :

1. Formulating the problem.
2. Constructing a model to represent the system under study
3. Deriving a solution from the model.

4. Testing the model and the solution derived from it.
5. Establishing controls over the solution.
6. Putting the solution to work, i.e. implementation.

1.8.1 Formulating the Problem

It is very essential that the problem at hand be clearly defined. It is almost impossible to get the 'right' answer from a 'wrong' problem.

In formulating a problem for OR study, analysis must be made of the four major components

(*a*) the environment,
(*b*) the decision maker,
(*c*) the objectives,
(*d*) alternative courses of action and constraints.

Out of the four components, *environment* is most comprehensive since it embraces and provides a setting for the other three. In general, environment is the framework within which a system of organised activity is directed to attain the prescribed objectives or goals. It involves physical, social and economic factors which may affect the problem under consideration.

Decision maker is the second component of the problem. Decision maker or research consumer or system operator is the person who is in actual control of the operations (system) under study. Before OR approach can be successful, the operation researcher (operations analyst) must study the decision maker and his relationship to the problem at hand.

Objectives are the third component of the problem to which analysis must be made. Objectives should be defined by taking into account the system (problem) as a whole. A common error is to identify the objectives, considering only a portion of the entire system. Under such conditions, what is considered best for this portion of the system, may actually prove harmful for the entire system. OR tries to take into account as broad a scope of objectives as possible.

Alternatives are the final components of the problem. The research problem is to determine which alternative course of action is most effective to achieve a certain set of objectives. Others affected by the decisions under study should also be identified. There must be complete agreement on these points between the persons initiating the OR study (operation researchers) and the persons

performing these operations. In addition, a measure of effectiveness must be agreed upon by the parties involved.

1.8.2 Constructing a Model for the Problem Under Study

After formulating the problem, the next step is to construct a model for the system under study. In OR study, it is usually a mathematical model. A mathematical model consists of a set of equations which describe the system or problem. These equations represent : (*i*) the *effectiveness function* and (*ii*) *constraints*. The effectiveness function, usually called the *objective function* is a mathematical expression of the objectives, i.e., mathematical expression of the cost or profit of the operation. *Constraints* or *restrictions* are mathematical expressions of the limitations on the fulfillment of the objectives.

The objective function and constraints are functions of two types of variables, *controllable* (also called *decision*) variables and *uncontrollable* variables. A variable that is directly under the control of the operations analyst is called controllable variable ; the values of these variables are to be determined. A variable that is not under the control of operations analyst is called uncontrollable variable. The general form of a mathematical model is,

$$E = f(x_i, y_i)$$

where E = effectiveness function

x_i = controllable variables

y_i = uncontrollable variables

f = relationships between E and x_i, y_i

It must not be forgotten that a model is only an *approximation* of the reality (real situation). Hence it may not include all the variables. This is often misunderstood by those who are not familiar with the OR approach.

A model helps to analyse a system without the interruption of the latter. It makes the problem more meaningful and clarifies important relationships among the variables. It also tells as to which of the variables are more important than the others. Once a model is formulated, it is possible to analyse the problem.

1.8.3 Deriving Solution from the Model

A solution may be extracted from a model either by conducting experiments on it, i.e., by *simulation* or by *mathematical analysis*. Some cases may require the use of a combination of simulation and

mathematical analysis. This depends upon the nature and complexity of the system under study.

Mathematical analysis for deriving an *optimum* solution from a model consists of two types of procedures : *analytic* and *numerical.* Analytic procedures makes use of the various branches of mathematics such as calculus or matrix algebra.

Numerical procedure consists of trying various values of controllable variables in the model comparing the results obtained and selecting that set of values of these variables which gives the best solution. These procedure vary from simple trial and error to complex *iteration.* During an iteration, successive trials of controllable variables tend to approach an optimum solution.

Since a model is an approximation of the real system or problem, the optimum solution for the model does not guarantee an optimum solution for the real problem. However, if the model is well formulated and tested, solution from the model will provide a good approximation to the solution of the real problem. This book is mainly devoted to the study of the various methods for finding these solutions.

1.8.4 Testing the Model and the Solution Derived from it

As already discussed, a model is never a perfect representation of reality. But, if properly formulated and correctly manipulated, it may be useful in predicting the effect of changes in control variables on the overall system effectiveness. The usefulness of a model is tested by determining how well it predicts the effect of these changes. Such an analysis is usually called *sensitivity analysis.* The utility or validity of the solution can be checked by comparing the results obtained without applying the solution with the results obtained when it is used.

1·8.5 Establishing Controls over the Solution

Life is not static, it is subjected to continuous, unceasing change. A solution which we felt was optimum today, may not be so tomorrow. A solution derived from a model remains a solution only so long as the uncontrolled (uncontrollable) variables retain their values and the relationship between the variables does not change. The solution itself goes 'out of control' if the values of one or more controlled variables varies or relationship between variables undergoes a

change. Therefore, controls must be established to indicate the limits within which the model and its solution can be considered as reliable. Also tools must be developed to indicate as to how and when the model or its solution will have to be modified to take the changes into account.

1.8.6 Putting the Solution to work (OR Implementation)

Finally, because the objective of OR is not merely to produce reports but to improve the system performance, the results of the research must be implemented. For this, solution obtained above should be translated into operating procedures which can be easily understood and applied by those who control the operations. Changes necessary in existing procedures and resources must be clearly indicated and should be implemented. After the solution has been applied to the system, OR group must study the response of the system to the changes made. Actual performance of the system may indicate some additional changes or modification to be made on the part of OR group.

The success of an OR study depends upon the co-operation received from the management at the implementation stage. One way of getting this co-operation is to make management an active participant in all phases of OR study. The importance of this phase cannot be overemphasized since it is from this phase that the benefits of an OR study will be realized.

These phases of OR study are not rigid rules; they are seldom conducted in the order presented. In many projects, for instance, the formulation of the problem is not complete until the project itself is virtually completed. Obviously, there is a considerable interplay between the different phases.

1.9. Types of Models

A model, as used in operations research, is defined as an idealized representation of a real-life system. Various types of models used in OR as well as in other sciences are : *iconic, analogue* and *symbolic.*

1·9.1 Iconic or Physical Models

In iconic or physical models, properties of the real system are represented by the properties themselves, frequently with a change of scale. Thus, iconic models resemble the system they represent but differ in size; they are *images.* For example, globes are used to represent the orientation and shape of various continents, oceans and other geographical feature of the earth. A model of the solar, system,

likewise, represents the sun and planets in space. Iconic models of atoms and molecules are commonly used in physics, chemistry and other sciences. However, these models are usually scaled up or down. For example, in a globe, the diameter of the earth is scaled down, but its shape, relative sizes of continents, oceans, etc., are approximately correct. On the other hand, a model of the atom is scaled up so as to make it visible to the naked eye. Iconic models may be two dimensional (photographs, maps, blue prints, etc.) or three dimensional (globes, automobiles, airplanes etc.). Ordinarily it is easier to work with the model of a building, earth, sun, atom, etc., than with the modelled entity itself. Iconic models are quite specific and concrete but difficult to manipulate for experimental purposes.

1·9·2 Analogue or Schematic Models

Analogue models can represent dynamic situations and are used more often than iconic models since they are analogous to the characteristics of the system under study. They use one set of properties to represent some other set of properties which the system under study possesses. After the model is solved, the solution is re-interpreted in terms of the original system.

For example, graphs are very simple analogues. They represent properties like force, speed, age, time, etc., in terms of distance. A graph is well suited for representing quantitative relationship between any two properties and predicts how a change in one property affects the other.

An *organizational chart* is a common schematic model. It represents the relationships existing between the various members of the organization. *A man-machine chart* is also a schematic model. It represents a time varying interaction of men and machines over a complete work cycle. *A flow process chart* is another schematic model which represents the order of occurrence of various events to make a product. Contour lines on a map are analogus of elevation. Flow of water through pipes may be taken as an analogue of the 'flow' of electricity through wires. Similarly, demand curves and frequency distribution curves used in statistics are examples of analogue models.

Transformation of properties into analogous properties increases our ability to make changes. Usually it is easier to change an analogue than to change an iconic model and also lesser number of changes are required to get the same results. For example, it is easier to change the contour lines on a two dimensional chart than to change the relief on a three dimensional one. In general, schematic models

are less specific and concrete but easier to manipulate than iconic models.

1.9.3 Symbolic or Mathematical Models

Symbolic models employ a set of mathematical symbols (letters, numbers, etc.) to represent the decision variables of the system under study. These variables are related together by mathematical equation (s)/inequation (s) which describe the properties of the system. A solution from the model is, then, obtained by applying well developed mathematical techniques

In many research projects, all the three types of models are used in sequence; iconic and analogue models are used as initial approximations, which are, then, refined into symbolic model.

Mathematical models differ from those traditionally used in physical sciences in two ways :

1. Since OR systems involve social and economic factors, these models use probabilistic elements.

2. They consist of two types of variables : controllable and uncontrollable. The objective is to select those values for controllable variables which optimize some measure of effectiveness. Therefore, these models are used in decision situations rather than in physical phenomena.

In OR, symbolic models are used wherever possible, not only because they are easier to manipulate but also because they yield more accurate results. Most of this text, therefore, is devoted to the formulation and solution of these mathematical models.

1.10 Types of Mathematical Models

Many OR models have been developed and applied to problems in business and industry. Some of these models are

1. Mathematical techniques
2. Statistical techniques
3. Inventory models
4. Allocation models
5. Sequencing models
6. Routing models
7. Competitive models
8. Queuing models
9. Dynamic programming models
10. Simulation techniques
11. Decision theory

12. Replacement models
13. Heuristic models
14. Combined methods

1.10.1 Mathematical Techniques

In principle, any mathematical technique can become a useful tool for operations analyst. Mathematical techniques most commonly employed are : differential equations, linear difference equations, integral equations, operator theory, vector and matrix theory. Detailed description of these techniques is beyond the scope of this book. For details, the reader is referred to any standard text on calculus.

1.10.2 Statistical Techniques

Some of the most commonly applied techniques come from probability theory and statistics. These include discrete and continuous probability renewal theory, Markov processes and stochastic processes. All these techniques are useful when dealing with uncertainty, errors, sampling, estimation and prediction. For some introductory text-books see references [5] and [10].

1.10.3 Inventory Models

Inventory models deal with idle resources like man, machines, materials and money. These models are concerned with two decisions: how much to order (*i.e.* produce or purchase) and when to order in order to minimize total cost. The total cost consists of carrying cost, replenishment cost and shortage cost. Economic order, quantity equations, linear, dynamic and quadratic programming are used to solve inventory models. Chapter 12 deals with these models.

1.10.4 Allocation Models

Allocation models are used to solve problems in which (*a*) there are a number of jobs to be performed and there are alternative ways of doing them and (*b*) resources or facilities are limited.

In such situations, the objective is to allot the resources to the jobs in such a way as to optimize the overall effectiveness (*i.e.* minimize the total cost or maximize the total profit). This is called *mathematical programming*. When the constraints are expressed as linear equations/inequations, this is called *linear programming*. Chapter 2 is devoted to this topic. If any of the constraints are non-linear, this is called *non-linear programming*. In addition to lineai and non-linear programming, there are other types of programming

—integer, quadratic, dynamic, convex, stochastic and parametric. Their difference lies in the type of data handled and the assumptions made.

The simplest type of allocation model involves the association of a number of jobs to the same number of resources (men). This is called *assignment model.* The assignment problem becomes more complex if some of the jobs require more than one resource or if the resource can be used for more than one job. Such a problem is called *transportation problem.* The transporation and assignment models are discussed in Chapter 3 and Chapter 4 respectively.

1.10.5 Sequencing Model

Sequencing refers to the determination of optimal sequence (order) for performing a set of jobs so as to minimize the total processing time (or cost). Only simple problems of this kind have been solved analytically. For others, simulation and heuristic methods have been used. Solution of sequencing models by analytical methods is taken up in Chapter 5.

For sequencing problems in which some precedence relationship is given, i.e. certain jobs must be completed before others can start, special techniques have been developed. The two most powerful techniques are PERT and CPM. Chapters 14 through 18 deal with these techniques.

1.10.6 Routing Models

Routing problems in networks are the problems which are related to sequencing. There are two important routing problems :

(*a*) the travelling salesman problem

(*b*) minimal path problem

In the travelling salesman problem, there are a number of cities a salesman has to visit. The distance (or time or cost) between every pair of cities is known. The salesman is to start from his home city, visit each city only once and return to his home city, and the problem is to find the shortest route in distance (or time or cost).

In the minimal path method, there are many routes available between one city (origin) and the other (destination) and the problem is to find the shortest route. Both of these models have been dealt within Chapter 5.

1.10.7 Competitive Models

These models are used when two or more individuals or organizations with conflicting objectives try to make decisions. In such situations a decision made by one decision maker affects the decision made by one or more of the remaining decision makers. Competitive models are applicable to a wide variety of situations such as two players struggling to win at chess, candidates fighting an election, two enemies planning war tactics, firms struggling to maintain their market shares, etc. Chapter 7 is devoted to such situations.

1.10.8 Queuing Models

Queuing problems involve the arrival of units to be serviced at one or more service facilities. There is either too much or too less demand on the facilities so that either the units or the facilities have to wait. Costs are associated with both types of waiting times. In either case, the problem is to either schedule arriving units or provide extra facilities or both so as to obtain an optimum balance between the costs associated with waiting time and idle time. Queuing models are discussed in Chapter 10.

1.10.9 Dynamic Programming Models

Dynamic programming models are used for situations that extend over a number of time periods. Regardless of what the previous decisions are, the dynamic model tries to determine the optimum decision for the periods that still lie ahead. Dynamic models involve manipulation of a large amount of information and require electronic computers. Chapter 8 deals with these models.

1.10.10 Simulation Techniques

Simulation is a representation of reality (problem, system, etc.) through the use of a model or other device, which will react in the same manner as reality under a given set of conditions. It is covered briefly in Section 10.8 as a part of queuing theory, while a more complete treatment is given in Chapter 13. Simulation is a very powerful tool and is used for problems which fail to be solved by direct analysis. Once the simulation model is designed, it takes little time to run a simulation on a computer.

1.10.11 Decision Theory

Decision theory plays an important role in helping managers make better decisions. Since, in the world we live, the course of future events cannot be predicted with absolute certainty, probabili-

ties are associated with these events. Decision theory covers three categories of decision making : under certainty, risk and uncertainty. For details, the reader is referred to reference [18].

1.10.12 Replacement Models

Replacement problems are generally of two types involving replacement of items that deteriorate with time and those that do not deteriorate but suddenly fail. The first category includes items like vehicles, machines, equipment, uniforms, etc. The problem consists of finding the optimum time for replacement so that the sum of the cost of new equipment, cost of maintaining efficiency on the old and the cost of loss of efficiency is minimum. These problems can be solved by calculus and dynamic programming. The second category includes items like electric bulbs, tubes, tyres, etc. The problem, here, is of finding which items to rep lace and whether or not to replace them in a group and, if so, when. The objective is to minimise the sum of the cost of the item, cost of replacing the item and the cost associated with failure of item. These problems can be solved by statistical sampling and probability theory. Chapter 11 is devoted to these problems.

1.10.13 Heuristic Models

According to Thierauf and Klekamp [reference (18)], heuristic models use rules of thumb or intuitive rules and guidelines (generally under computer control) to explore the most likely paths and to make educated guesses in arriving at a problem's solution. Thus, checking all the alternatives, so as to obtain the optimum one, is not required. Heuristic models seem to be quite promising for future OR work. They bridge the gap between strictly analytical formulations and the operating principles which managers are habitual of using. The reader wishing more detail on the subject should consult reference [18].

1.10.14 Combined Methods

Real systems may not involve only one of the models discussed above. A production control problem, for example, normally consists of a combination of inventory, allocation and queuing models.

The usual method of solving such combined models consists of 'solving' them one at a time in some logical sequence. However, OR combines these models and constructs some type of *master model*, which takes into account the interaction of individual models.

Lastly, it must be emphasized that the above classification does not cover all OR problems. However, it does cover most of them,

It is expected that in the days to come more and more of new processes will be revealed and subjected to mathematical analysis.

1.11 Constructing the Model

It was pointed out in previous sections that formulation of the problem required analysis of the system under study. This analysis shows the various phases of the system and the way it can be controlled. With the formulation of the problem, the first stage in model construction is over. The next step is to define a measure of effectiveness, i.e., the next step is to construct a model in which effectiveness of the system is expressed as a function of the variables defining the system. The general form of OR model is

$$E = f(x_i, y_j)$$

where E = effectiveness of the system

x_i = variables of the system that can be controlled

y_j = variables of the system that cannot be controlled but do effect E

Deriving of solution from such a model consists of determining those values of control variables x_i, for which the measure of effectiveness is optimized. Optimization includes both maximization (in case of profits, production capacity, etc.) and minimization (in case of losses, cost of production, etc.).

Varisous steps in the construction of a model are

1. Selecting components of the system
2. Pertinence of components
3. Combining the components
4. Substituting symbols

1.11.1 Selecting Components of the System

All the components of the system which contribute towards the effectiveness measure of the system should be listed.

1.11.2 Pertinence of Components

Once a complete list of components is prepared, the next step is to find whether or not to take each of these components into account. This is determined by finding the effect of various alternative courses of action on each of these components. Generally, one or more components (e.g., fixed costs) are independent of the changes made among the various alternative courses of action. Such components may be temporarily dropped from consideration.

1.11.3 Combining the Components

It may be convenient to group certain components of the system. For example, the purchase price, freight charges and receiving cost of a raw material can be combined together and called 'raw material acquisition cost'. The next step is to determine, for each component remaining on the modified list, whether its value is fixed or variable. If a component is variable, various aspects of the system affecting its value must be determined. For instance manufacturing cost usually consists of

(*i*) the number of units manufactured, and

(*ii*) the cost of manufacturing a unit.

Once each variable component in the modified list has been broken down like this, symbols may be assigned to each of these sub-components.

The foregoing steps will be clear from the example considered below :

A newsboy wants to decide the number of newspapers he should order to maximize his expected profit. He purchases a certain number of newspapers everyday and sells some or all of them. He earns a profit on each paper sold. He can return the unsold papers, but at a loss. The number of persons who buy newspapers varies from day to-day.

To construct the model for this problem, we identify the various relevant components (variables) and then assign symbols to them.

Let N = number of newspapers ordered per day

A = profit earned on each newspaper sold

B = loss on each newspaper returned

D = demand i.e. number of newspapers sold per day (if $N \geqslant D$)

p (D) = probability that the demand will be equal to D on any randomly selected day

P = net profit per day

If $D > N$ i.e., demand is more than the number of newspapers ordered, the profit to the newsboy is

$$P\ (D > N) = NA$$

If on the other hand, demand is less than the number ordered, the profit is

$$P\ (D \leqslant N) = DA - (N - D)\ B$$

$\therefore$ Net expected profit per day, $\bar{p}$ can be expressed as

$$\bar{p} = \sum_{D=0}^{D=N} p(D)\,[DA-(N-D)\,B] + \sum_{D=N+1}^{\infty} p(D).\,NA$$

This is a decision model of the risk type. Here, $\bar{p}$ is the measure of performance, N is the controlled variable, D is an uncontrollable variable, while A and B are uncontrollable constants. Solution of this model consists of finding that value of N which maximizes $\bar{p}$.

1.12. Approximations (Simplifications) in OR Models

While constructing a model one comes across two conflicting objectives :

(*i*) the model should be as easy to solve as possible,

(*ii*) it should be as accurate as possible.

Moreover, the management must be able to understand the solution of the model and must be capable of using it. Obviously one must pay due care to the mathematical complexity of the solution. Therefore, while constructing the model, the reality (problem under study) should be simplified but only to the point where there is no significant loss of accuracy. Some of the common simplifications include

1. Omitting certain variables
2. Aggregating variables
3. Changing the nature of variables
4. Changing the relationship between variables, and
5. Modifying constraints

1.12.1 Omitting Certain Variables

Clearly, variables having a large effect on system's performance cannot be omitted. However, it requires a lot of study to decide which variables have and which do not have large effects. For instance, in production and inventory control models, the effect of production-un sizes on in-process inventory costs is usually negligible as compared to effects of other variables and is, therefore, neglected.

1.12.2 Aggregating Variables

Most problems involve a large number of decision variables. For instance, some inventory problems involve the purchase of more than a million items. For solving such problems, the controlled variables are grouped into 'families'. A family is, then, supposed to consist of all identical members. One principle of 'family' formation is

1. Low usage, low cost
2. Low usage, high cost
3. High usage, low cost
4. High usage, high cost

1.12.3 Changing the Nature of Variables

The nature of variables may be changed in three ways :

(*i*) by treating a variable as constant,

(*ii*) by treating a discrete variable as continuous

(*iii*) by treating a continuous variable as discrete.

A variable may be treated as constant with its value equal to the mean of the variable's distribution. For example, in most production quantity models set up cost is treated as constant.

From both analytical and computational viewpoints it is easier to treat a discrete variable as continuous. Most of OR techniques deal with continuous variables. Even if the discrete variables are few in number, the computational difficulties become quite large. For instance, withdrawals of items from stock that are actually discrete are assumed as continuous at a constant rate, over a planning period.

However, for processes in which time between events is a relevant variable, considerable simplification may be obtained by assuming that events occurring within a certain period occur instantaneously at the beginning or end of the period.

1.12.4 Changing the Relationship between Variables

Models can be simplified by modifying the functional form of the model. Non-linear functions require a complex solution method. The most powerful computational techniques are applicable only to models having linear functions. Therefore, nonlinear functions are usually approximated to linear functions (*e.g.*, in linear programming). Many times, a curve is approximated by a series of straight lines (*e.g.*, in non-linear programming). Quadratic functions are used as approximations since their derivatives are linear (*e.g.*, in quadratic programming). Discrete functions (*e.g.*, binomial and Poisson) are sometimes approximated by continuous normal functions.

1.12.5 Modifying Constraints

Constraints can be subtracted, added or modified to simplify the model. If it is not possible to solve a model with all the constraints, they can be temporarily ignored and a 'solution' obtained.

If this 'solution' happens to satisfy these constraints, it is accepted. If it does not, constraints are added, one at a time, with increasing complexity, until a solution satisfying the constraints is obtained. A general rule regarding constraints is that when they are dropped the solution derived from the model becomes optimistic (it gives better performance than the 'true' solution). On the other hand, adding of constraints makes the solution pessimistic.

1.13. Role of Computers in Operations Research

It was said in section 1.1 that the computer played a vital role in the development of OR. But for the computer, OR would not have achieved its present position. It is because in most OR techniques, computations are so complex and involved that these techniques would be of no real use in the absence of the computer. Most large scale applications of OR techniques which require only a few minutes on the computer, may take weeks, months and even years to yield the same results manually.

No doubt, the computer is an essential and integral part of OR. Today, OR methodology and computer methodology are growing in parallel. It appears that in the coming years the line dividing the two methodologies will disappear and the two sciences will combine to form a more general and comprehensive science.

1·14. Difficulties in Operations Research

The previous sections have brought out the positive side of OR only. However, there is also the need to point out the negative side. Certain common traps and pitfalls can and have, ruined the otherwise good work. Some of these pitfalls are quite obvious while others are so subtle and hidden that extreme care is required to locate their presence.

In the very first phase of OR—the problem formulation phase—a number of pitfalls can and do arise. It is necessary that the right problem be selected and it must be completely and accurately defined Is the right problem being solved ? Is the scope considered wide and proper ? Will it result in optimization or only sub-optimization Will the solution properly reflect the objectives as well as the imposed constraints ? Are proper effectiveness measures being used ? This phase of problem formulation is perhaps the most important and toughest part of OR study.

Secondly, data collection may also consume a very large portion of time and money spent on OR study.

Thirdly, the whole study by operations analyst is based on his observations in the past. Strictly speaking, these observations can only be related to the laws that operated in the past, as there is no evidence that the laws will continue to operate in future also. If the laws are applied to the future, it clearly amounts to extrapolation in time.

Fourthly, the operation researcher, while making observations, may affect the behaviour of the system he is studying. Moreover, however comprehensive his experiments may be, his observations can never be more than a sample of the whole. These difficulties present special hazards to operation researcher. His aim is to find out what happens in a working organization. He can get the information in two ways : by direct observations or from the previous records. The behaviour of an organization depends upon the activities of the persons in it and the very fact that they are being observed is bound to affect their behaviour. On the other hand, accuracy of previous records is always doubtful and they seldom provide the complete information in all the points sought.

Perhaps the greatest difficulty in OR, however, is created by the time factor. The managers have to make decisions one way or the other, and a fairly good solution to the problem at the right time may be much more useful than the perfect solution too late. Further, the cost involved is also an important factor. Sometimes, some simple application of OR may yield a good solution quickly and it may be unwise to spend a lot of money and effort to produce a slightly better solution much later.

Other pitfalls in problem solving include

(*i*) warping the problem to fit a standard model, tool or technique

(*ii*) failure to test the model and solution before implementation

(*iii*) failure to establish proper controls

It is the responsibility of OR scientist to translate his highly specialised and technical thoughts, ideas and concepts into simple operational procedures capable of being easily understood by the management and workers alike. He must also ascertain that the new proposals are properly implemented. But for the proper implementation, the whole OR study becomes useless.

Lastly, what may appear to be a pitfall is the fact that OR study may raise more questions than it answers. However, this may

ultimately result in more deep insight into the system, yielding further benefits and improvements.

1·15. Limitations of Operations Research

1. Mathematical models which are essence of OR do not take into account qualitative factors or emotional factors which are quite real. All influening factors, which cannot be quantified, find no place in mathematical models.

2. Mathematical models are applicable to only specific categories of problems.

3. Being a new field, generally there is a resistance from the employees to the new proposals.

4. Management, who has to implement the advised proposals, may itself offer a lot of resistance due to conventional thinking.

5. Young enthusiasts, overtaken by its advantages and exactness, generally forget that OR is meant for men and not that men are meant for it.

Thus at the implementation stage, the decision cannot be governed by quantitative considerations alone. It must take into account the delicacies of human relationships. That is, in addition to being a pure scientist, one has to be tactful and learn the art of getting the decisions implemented. This art can be achieved by experience as well as by getting training in social sciences, particularly psychology.

In fact, many managers may make a joke of OR as they think that the decisions made otherwise may be better. But being aware of its limitations, they need to be convinced of its utility, which doubtlessly forms the essential guideline for making better decisions.

1·16. Bibliographic Notes

The literature on operations research is quite extensive. One of the earliest and comprehensive text book is by C. Churchman, R. Ackoff and E. Arnoff [2]. This book deals with the subject matter quite exhaustively. The book edited by J. McCloskey and F. Trefethen [9] is also quite extensive and lays good emphasis on the history of OR.

A very good exposition of OR methods is given in the book by M. Sasieni, A. Yaspan and L. Friedman [13].

Books by A. Kaufmann [8] and T. Saaty [11] deal with the mathematical methods used in OR. Both present the material on a high mathematical level.

A collective work presented by Abe Shuchman [16] gives a very exhaustive treatment of the subject. It needs very little mathematical background. Text books by R. Thierauf and R. Klekamp [18] and H. Taha [17] are other good books on the subject of operations research.

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PART I

Allocation Models

Allocation models are used to solve a class of problems which arise when

(a) there are a number of activities (jobs) to be performed and there are alternative ways of doing them.

(b) resources or facilities are not available for performing each activity in the most effective manner.

In such situations, the objective is to allot the resources to the jobs in such a way as to optimise the overall effectiveness (i.e., to either minimize the total cost or maximize the total returns).

The number of possible alternative ways (choices) of allocating the resources to the activities can be finite or infinite. If a problem has finite number of choices, all the different choices can be found, at least, theoretically. Unfortunately, in most of the practical situations, numbers of such alternatives is very large and enumerating all of them is a very lengthy process. For example, there are 10 ways of aliocating 10 riders to 10 horses.

Clearly, for many practical situations, solutions in principle (e.g., enumeration) are insufficient. And naturally, the first situations to be discussed were comparatively simple ones where the effectiveness function and constraints are expressed as linear functions of the allocations. The analysis of these situations is called *linear programming*. This technique of linear programming can be used for three different groups of problems. They are

1. *Assignment Problems* : This group comprises problems concerning what happens when we associate each of a number of 'origins' with each of the *same* number of destinations'. Each source (origin) is to be associated with one and only one job (destination) and associations are to be made in such a way as to maxmize (or minimize) the total effectiveness. Resources are not divisible among jobs, nor are jobs divisible among resources.

2. *Transportation Problems* : This group of problems concerns as to what happens to the effectiveness function when we associate each of a number of origins with each of a ***possibly different*** number

of destinations (jobs). The total movement from each origin and the total movement to each destination is given and it is desired to find how the associations be made subject to the limitations on totals. In such problems, resources can be divided among jobs or some jobs may be done with a combination of resources but resources and jobs must be expressed in terms of only one kind of unit.

3. *Simplex Problems* : This group of problems includes the above two groups. The computations are quite lengthy and methods of groups 1 and 2 should be used whenever possible. Generally, a simplex problem involves maximization or minimization of a linear function of a set of non-negative variables, subject to a set of linear inequalities relating the variables.

We shall first discuss (in Chapter 2) how linear programming is used to solve simplex problems which are most general in nature. The role played by linear programming in solving transportation problems and assignment problems will be dealt with in Chapter 3 and Chapter 4 respectively.

2

Linear Programming

2·1. Introduction

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as *Objective function*, subject to a set of linear equations and/or inequations known as *restrictions* or *constraints*. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The restrictions may be imposed by different sources such as market demand, production processes and equipment, storage capacity, raw material availability, etc. By linearity is meant a mathematical expression in which the variables do not have powers.

It was in 1947 that George Dantzing and his associates found out a technique for solving military planning problems while they were working on a project for U.S. Air Force. This technique consisted of representing the various activities of an organization as a linear programming (L.P.) model and arriving at the optimal programme by minimizing a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as "Simplex Method" to solve linear programming problems.

2.2. Requirements for a Linear Programming Problem

All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. If the supply of these resources were unlimited, the need for management tools like linear programming would not arise at all. Supply of resources being limited, the management must find the best allocation of its resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent. However, this involves a number of problems which can be overcome by quantitative methods, particularly the linear programming.

Generally speaking, linear programming can be used for optimization problems if the following conditions are satisfied :

1. There must be a well defined objective function (profit, cost or quantities produced) which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.

2. There must be restrictions on the amount or extent of attainment of the objective and these restrictions must be capable of being expressed as linear equalities or inequalities in terms of variables.

3. There must be alternative courses of action. For example, a given product may be processed by two different machines and problem may be as to how much of the product to allocate to which machine.

4. Another necessary requirement is that decision variables should be interrelated and non-negative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are generally illogical.

5. As stated earlier, the resources must be in limited supply. For example, if a firm starts producing greater number of a particular product, it must make smaller number of other products as the total production capacity is limited.

2.3. Examples on the Applications of Linear Programming

In this section we illustrate the application of linear programming to various situations with the help of examples. The objective is to familiarize the reader with some of the areas where this technique may be applied. In these examples the stress is laid on the analysis of the situation and formulation of the L.P. model rather than its solution. The solution of such problems is discussed separately under articles 2.4 and 2.10.

Situation 1 :

The first, most typical situation to which linear progamming may be applied is the following :

There are many targets which can be achieved with the help of different types of resources. The main feature of this situation is that to achieve each individual target, different resources can be simultaneously used, while they can also be used to fulfil other targets. Moreover, the allocations of individual resources are determined in advance. The objective, therefore, is to decide the *correct allocation of resources* to achieve the individual targets. We shall describe this situation with the help of three examples.

EXAMPLE 2.3.1 (Production Allocation Problem)

A firm produccs three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines is given in the table below.

Table 2.1

Machine	*Time per unit (minutes)*			*Machine capacity (minutes/day)*
	Product 1	*Product 2*	*Product 3*	
M_1	2	3	2	440
M_2	4	—	3	470
M_3	2	5	—	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs. 3 and Rs. 6 respectively. It is assumed that all the amounts produced are consumed in the market.

Formulation of Linear Programming Model

Step 1 :

From the study of the situation find the *key-decision* to be made. It this connection, looking for variables helps considerably. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2 :

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 respectively.

Step 3 :

Express the *feasible alternatives* mathematically in terms of variables. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1, x_2 and x_3,

$$\text{where} \quad x_1, x_2, x_3 \geqslant 0 \qquad \text{...(2.1)}$$

Since negative production has no meaning and is not feasible.

Step 4 :

Mention the *objective* quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., to maximize $Z = 4x_1 + 3x_2 + 6x_3$ ...(2.2)

Step 5 :

Put into words the *influencing factors* or *restrictions* (or *constraints*). These occur generally because of constraints on availability (resources) or requirements (demands). Express these restrictions also as linear equalities/inequalities in terms of variables.

Here, restrictions are on the machine capacities and can be mathematically expressed as

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leqslant 440 \\ 4x_1 + 0x_2 + 3x_3 &\leqslant 470 \\ 2x_1 + 5x_2 + 0x_3 &\leqslant 430 \end{aligned} \quad \text{...(2.3)}$$

The complete linear programming problem is then given by

$$\begin{aligned} \text{maximize } Z &= 4x_1 + 3x_2 + 6x_3 \\ \text{subject to } & 2x_1 + 3x_2 + 2x_3 \leqslant 440 \\ & 4x_1 + 3x_3 \leqslant 470 \\ & 2x_1 + 5x_2 \leqslant 430 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

EXAMPLE 2.3-2 (Production Planning Problem)

A factory manufactures a product each unit of which consists of 5 units of part A and 4 units of part B. The two parts A and B require different raw materials of which 120 units and 240 units respectively are available. These parts can be manufactured by three different methods. Raw material requirements per production run and the number of units for each part produced are given below.

Table 2.2.

Method	*Input per run (units)*		*Output per run (units)*	
	Raw material 1	*Raw material 2*	*Part A*	*Part B*
1	7	5	6	4
2	4	7	5	8
3	2	9	7	3

Determine the number of production runs for each method so as to maximize the total number of complete units of the final product.

Formulation of Linear Programming Model

Step 1 :

The *key-decision* to be made is to determine the number of production runs for each method.

Step 2 :

Let x_1, x_2, x_3 represent the number of production runs for method 1, 2 and 3 respectively.

Step 3 :

Feasible alternatives are the sets of values of

$$x_1, x_2 \;\&\; x_3, \quad \text{where} \quad x_1, x_2, x_3 \geqslant 0 \qquad \text{...(2.4)}$$

Since negative number of production runs has no meaning and is not feasible.

The *objective* is to maximize the total number of units of the final product. Now, the total number of unit of part A produced by different methods is $6x_1+5x_2+7x_3$ and for part B is $4x_1+8x_2+3x_3$. Since each unit of the final product requires 5 units of part A and 4 units of part B, it is evident that the maximum number of units of the final product cannot exceed the smaller value of

$$\frac{6x_1+5x_2+7x_3}{5} \text{ and } \frac{4x_1+2x_2+3x_3}{4}.$$

Thus the objective is to maximize

$$Z=\text{Min}\left(\frac{6x^1+5x_2+7x_3}{5}, \frac{4x_1+8x_2+3x_3}{4}\right) \qquad \text{...(2.5)}$$

Step 5 :

Restrictions are on the availability of raw materials. They are, for

$$\text{raw material 1,} \quad 7x_1+4x_2+2x_3 \leqslant 120$$

$$\text{and raw material 2,} \quad 5x_1+7x_2+9x_3 \leqslant 240 \qquad \text{...(2.6)}$$

The above formulation violates the linear programming properties since the objective function is non-linear. (Linear relationship between two or more variables is the one in which the variables are directly and precisely proportional). However, the above model can be easily reduced to the generally acceptable linear programming format.

Let $y = \text{Min}\left(\frac{6x_1+5x_2+7x_3}{5}, \frac{4x_1+8x_2+3x_3}{4}\right)$

It follows that $\frac{6x_1+5x_2+7x_3}{5} \geqslant y$ and $\frac{4x_1+8x_2+3x_3}{4} \geqslant y$

i.e., $6x_1+5x_2+7x_3-5y \geqslant 0$

and $4x_1+8x_2+3x_3-4y \geqslant 0$.

Thus the above problem reduces to the following L.P. problem :

maximize $Z = y$

subject to
$$6x_1+5x_2+7x_3-5y \geqslant 0$$
$$4x_1+8x_2+3x_3-4y \geqslant 0$$
$$7x_1+4x_2+2x_3 \leqslant 120$$
$$5x_1+7x_2+9x_3 \leqslant 240$$

where $x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0$.

EXAMPLE 2·3-3. (Advertising Media Selection Problem)

An advertising company wishes to plan its advertising strategy in three different media—television, radio and magazines. The purpose of advertising is to reach as large a number of potential customers as possible. Following data has been obtained from market survey :

Table 2·3

	Television	*Radio*	*Magazine I*	*Magazine II*
Cost of an advertising unit	Rs. 30,000	Rs. 20,000	Rs. 15,000	Rs. 10,000
No. of potential customers reached per unit	2,00,000	6,00,000	1,50,000	1,00,000
No. of female customers reached per unit	1,50,000	4,00,000	70,000	50,000

The company wants to spend not more than Rs. 450,000 on advertising. Following are the further requirements that must be met :

(*i*) at least 1 million exposures take place among female customers,

(*ii*) advertising on magazines be limited to Rs. 1,50,000.

(*iii*) at least 3 advertising units be bought on magazine I and 2 units on magazine II, and

(*iv*) the number of advertising units on television and radio should each be between 5 and 10.

Formulate a L.P. model for the problem.

Formulation of Linear Programming Model

Step 1 :

The *key decision* to be made is to determine the number of advertising units to be bought in television, radio, magazine I and magazine II.

Step 2 :

Let x_1, x_2, x_3, x_4 represent the number of these advertising units in television, radio, magazine I and magazine II respectively.

Step 3 :

Feasible alternatives are sets of values of x_1, x_2, x_3, x_4, where x_1, x_2, x_3, x_4 all $\geqslant 0$.

Step 4 :

The *objective* is to maximize the total number of potential customers.

i.e., maximize $Z = 10^5(2x_1 + 6x_2 + 1{\cdot}5x_3 + x_4)$.

Step 5 :

Constraints are

On the advertising budget : $30{,}000\,x_1 + 20{,}000\,x_2 + 15{,}000\,x_3 + 10{,}000\,x_4 \leqslant 4{,}50{,}000$

or $30\,x_1 + 20\,x_2 + 15\,x_3 + 10\,x_4 \leqslant 450$

On number of female customers reached by the advertising campaign : $150{,}000x_1 + 400{,}000x_2 + 70{,}000x_3 + 50{,}000\,x_4 \geqslant 1{,}000{,}000$

or $15x_1 + 40x_2 + 7x_3 + 5x_4 \geqslant 100$

On expenses on magazine advertising : $15{,}000x_1 + 10{,}000x_2 \leqslant 1{,}50{,}000$

or $15x_1 + 10x_2 \leqslant 150$

On no. of units on magazines : $x_3 \geqslant 3$

$x_4 \geqslant 2$

On no. of units on television : $5 \leqslant x_1 \leqslant 10$

On no. of units on radio : $5 \leqslant x_2 \leqslant 10$

Thus the complete L.P. problem is

$$\text{maximize } Z = 10^5(2x_1 + 6x_2 + 1{\cdot}5x_3 + x_4)$$

subject to

$$30x_1+20x_2+15x_3+10x_4 \leqslant 450$$
$$15x_1+40x_2+7x_3+5x_4 \geqslant 100$$
$$15x_1+10x_2 \leqslant 150$$
$$x_3 \geqslant 3$$
$$x_4 \geqslant 2$$
$$x_1 \geqslant 5$$
$$x_1 \leqslant 10$$
$$x_2 \geqslant 5$$
$$x_2 \leqslant 10$$

where $x_1, x_2, x_3, x_4 \geqslant 0$.

EXAMPLE 2·3·4. **(Inspection Problem)**

A company has two grades of inspectors, 1 and 2 to undertake quality control inspection. At least 1,500 pieces must be inspected in an 8-hour day. Grade 1 inspector can check 20 pieces in an hour with an accuracy of 96%. Grade 2 inspector checks 14 pieces an hour with an accuracy of 92%.

The daily wages of grade 1 inspector are Rs 5 per hour while those of grade 2 inspectors are Rs. 4 per hour. Any error made by an inspector costs Rs. 3 to the company. If there are, in all, 10 grade 1 inspectors and 15 grade 2 inspectors in the company, find the optimal assignment of inspectors that minimizes the daily inspection cost.

Formulation of L.P. Model

Step 1 :

The *key decision* to be made is to determine the number of grade 1 and grade 2 inspectors for assignment.

Step 2 :

Let x_1 and x_2 represent the number of these inspectors.

Step 3 :

Feasible alternatives are sets of values of x_1 and x_2 where $x_1 \geqslant 0$, $x_2 \geqslant 0$.

Step 4 :

The *objective* to minimize the daily cost of inspection. Now the company has to incur two types of costs ; wages paid to the inspectors and the cost of their inspection errors. The cost of grade 1 inspector/hour is

$$\text{Rs. } (5+3 \times 0{\cdot}03 \times 20) = \text{Rs. } 6{\cdot}80$$

Similarly, cost of grade 2 inspector/hour is

$$\text{Rs. } (4+3 \times 0{\cdot}03 \times 14) = \text{Rs. } 5{\cdot}26$$

∴ The objective function is

minimize $Z = 8(6{\cdot}8x_1 + 5{\cdot}26x_2) = 54{\cdot}4x_1 + 42{\cdot}08x_2$.

Step 5 :

Constraints are

On the number of grade 1 inspectors : $x_1 \leqslant 10$

On the number of grade 2 inspectors : $x_2 \leqslant 15$

On the number of pieces to be inspected daily : $20 \times 8x_1 + 14 \times 8x_2 \geqslant 1{,}500$

or $16x_1 + 112x_2 \geqslant 1{,}500$.

Thus the complete L.P. problem is

$$\text{minimize } Z = 54 \cdot 4x_1 + 42 \cdot 08x_2$$

$$\text{subject to} \quad 160x_1 + 112x_2 \geqslant 1{,}500$$

$$x_1 \leqslant 10$$

$$\text{where } x_1, x_2 \geqslant 0.$$

EXAMPLE 2·3·5.

A chemical company produces two products, X and Y. Each unit of product X requires 3 hours on operation I and 4 hours on operation II, while each unit of product Y requires 4 hours on operation I and 5 hours on operation II. Total available time for operations I and II is 20 hours and 26 hours respectively. The production of each unit of product Y also results in two units of a by-product Z at no extra cost.

Product X sells at profit of Rs. 10/unit, while Y sells at profit of Rs. 20/unit. By-product Z brings a unit profit of Rs. 6 if sold ; in case it cannot be sold, the destruction cost is Rs. 4/unit. Forecasts indicate that not mere than 5 units of Z can be sold. Determine the quantities of X and Y to be produced, keeping Z in mind, so that the profit earned is maximum.

Formulation of L.P. Model

Step 1 :

The *key decision* to be made is to determine the number of units of products X, Y and Z to be produced.

Step 2 :

Let the number of units of products X, Y and Z produced be x_1, x_2, x_z, where

x_z = number of units of Z produced

= number of units of Z sold + number of units of Z destroyed

$= x_3 + x_4$ (say).

Step 3 :

Feasible alternatives are sets of values of x_1, x_2, x_3 and x_4, where $x_1, x_2, x_3, x_4 \geqslant 0$.

Step 4 :

Objective is to maximize the profit. Objective function (profit function) for products X and Y is linear because their profits (Rs. 10/unit and Rs. 20/unit) are constants irrespective of the number of units produced. A graph between the total profit and quantity produced will be a straight line. However, a similar graph for product Z is non-linear since it has slope +6 for first part, while a slope of −4 for the second. However, it is piece-wise linear, since it is linear in the regions (0, 5) and (5, ∞). Thus splitting x_2 into two parts, viz. the number of units of Z sold (x_3) and number of units of Z destroyed (x_4) makes the objective function for product Z also linear.

Thus the objective function is

$$\text{maximize } G = 10x_1 + 20x_2 + 6x_3 - 4x_4$$

Step 5 :

Constraints are

On the time available on operation I : $3x_1 + 4x_2 \leqslant 20$

On the time available on operation II : $4x_1 + 5x_2 \leqslant 26$

On the number of units of product Z sold ; $x_3 \leqslant 5$

On the number of units of product Z produced :

$$2x_2 = x_3 + x_4 \quad \text{or} \quad -2x_2 + x_3 + x_4 = 0$$

Thus the linear programming problem becomes

$$\text{maximize } G = 10x_1 + 20x_2 + 6x_3 - 4x_4$$

$$\text{subject to} \quad 3x_1 + 4x_2 \leqslant 20$$
$$4x_1 + 5x_2 \leqslant 26$$
$$-2x_2 + x_3 + x_4 = 0$$
$$x_3 \leqslant 5$$

where x_1, x_2, x_3, x_4, all $\geqslant 0$.

Situation 2 :

The second situation is that certain targets have been set and for each target a minimum degree of implementation has been fixed in advance. On the other hand, the types of resources for the implementation of these targets are mentioned but the allocation of these resources is not given. The problem is to ensure that minimum targets are at least fulfilled with minimum cost on resources. This, therefore, is a situation in which the resources must be so allocated as

to fulfil the targets to the fixed level. This type of situation is called *choice of resources* situation. We shall describe this situation with the help of a few examples.

EXAMPLE 2-3-6.

A firm produces an alloy having the following specifications :

(*i*) specific gravity $\leqslant 0\cdot 9$

(*ii*) chromium $\geqslant 8\%$

(*iii*) melting point $\geqslant$ 450°C

Raw materials A, B and C having the properties shown in the table can be used to make the alloy.

Table 2·4

Property	*Properties of raw material*		
	A	*B*	*C*
Specific gravity	0·92	0·97	1 04
Chromium	9%	13%	16%
Melting point	540°C	490°C	480°C

Cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C. Find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Formulation of Linear Programming Model

Step 1 :

Key decision to be made is how much (percentage) of raw materials A, B and C be used for making the alloy.

Step 2 :

Let the percentage contents of A, B and C be x_1, x_2 and x_3 respectively.

Step 3 :

Feasible alternatives are sets of values of x_j, where $j=1$, 2 and 3. ...(2.7)

Step 4 :

Objective is to minimize the cost

i.e., minimize $Z=90x_1+280x_2+40x_3$ ...(2·8)

Step 5 :

Constraints are imposed by the specifications required for the alloy. They are

$$\left.\begin{array}{l} 0{\cdot}92x_1+0.97x_2+1.04x_3\leqslant 1{\cdot}9 \\ 9x_1+13x_2+16x_3\geqslant 8 \\ 540x_1+490x_2+480x_3\geqslant 450 \\ \text{Also} \quad x_1+x_2+x_3=103. \end{array}\right\} \quad ...(2{\cdot}9)$$

The above equations/inequations represent the L.P. model for the given problem.

EXAMPLE 2.3-7. (Fluid Blending Problem)

An oil company produces two grades of gasoline P and Q which it sells at Rs. 3 and Rs. 4 per litre. The refinery can buy four different crudes with the following constituents and costs :

Table 2·5

Crude	*Constituents*			*Price/litre*
	A	*B*	*C*	
1	0·75	0.15	0.10	Rs. 2.00
2	0·20	0·30	0.50	Rs. 2.25
3	0.70	0·10	0.20	Rs. 2.50
4	0.40	0·60	0.50	Rs. 2.75

The Rs. 3 grade must have at least 55 per cent of A and not more than 40 per cent of C. The Rs. 4 grade must not have more than 25 per cent of C. Determine how the crudes should be used so as to maximize the profit.

[Pb. Univ. B.Sc. Mech. Engg. Dec., 1978]

Formulation of Linear Programming Model

Step 1 :

Key decision to be made is how much of each crude be used for making each of the two grades of gasoline.

Step 2 :

Let these quantities be represented by x_{ij}, where i=crude 1, 2, 3, 4 and j=grade P, Q.

Step 3 :

The various *alternatives* are sets of values of x_{ij} ($\geqslant 0$), where $i=$ 1, 2, 3, 4 and $j=$P, Q.

Thus x_{1p}=amount of crude 1 used in gasoline of grade P

$$\begin{array}{lllllllll}
x_{2p}= & ,, & ,, & 2 & ,, & ,, & ,, & ,, & ,, \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

x_{1q}=amount of crude 1 used in gasoline of grade Q

$$\begin{array}{lllllllll}
x_{2q}= & ,, & ,, & 2 & ,, & ,, & ,, & ,, & ,, \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

where $x_{1p}, x_{2p}, x_{3p}, x_{4p}, x_{1q}, x_{2q}, x_{3q}, x_{4q}$ all$\geqslant$O. ...(2·10)

Step 4 :

Objective is to maximize the profit, which obviously is

$$3(x_{1p}+x_{2p}+x_{3p}+x_{4p})+4(x_{1q}+x_{2q}+x_{2q}+x_{4q})-2(x_{1p}+x_{1q})$$
$$-2.25(x_{2p}+x_{2q})-2.50(x_{3p}+x_{3q})--2.75(x_{4p}+x_{4q})$$

i.e., maximize $Z=x_{1p}=0.75x_{2p}+0{\cdot}50x_{3p}+0.25x_{4p}+2x_{1q}+1.75x_{2q}$
$+1.50x_{3q}+1.25x_{4q}$. ...(2·11)

Step 5 :

Constraints are on the quantities of constituents A and C to be allowed in the two grades of gasoline.

$$\left.\begin{array}{ll}
\textit{i.e.,} & 0.75x_{1p}+0.20x_{2p}+0.70x_{3p}+0.40x_{4p} \geqslant 0.55 \\
& \qquad\qquad\qquad (x_{1p}+x_{2p}+x_{3p}+x_{4p}) \\
& 0.10x_{1p}+0.50x_{2p}+0.20x_{3p}+0.50x_{4p} \leqslant 0.40 \\
& \qquad\qquad\qquad (x_{1p}+x_{2p}+x_{3p}+x_{4p}) \\
\text{and} & 0.10x_{1q}+0.50x_{2q}+0.20x_{3q}+0.50x_{4q} \leqslant 0.25 \\
& \qquad\qquad\qquad (x_{1q}+x_{2q}+x_{3q}+x_{4q}).
\end{array}\right\} \quad ...(2.12)$$

Thus the problem is to maximize equation (2.11) subject to constraints (2.12) and non-negativity restrictions (2.10).

EXAMPLE 2.3-8. (Diet Problem)

A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in table 2.6.

Table 2.6

Food type	Yield per unit			Cost per unit
	Proteins	*Fats*	*Carbohydrates*	
1	p_1	f_1	c_1	d_1
2	p_2	f_2	c_2	d_2
3	p_3	f_3	c_3	d_3
4	p_4	f_4	c_4	d_4
Minimum daily requirement	P	F	C	X

Formulation of L.P. Model

Step 1 :

Key decision is to decide the number of units used of 1st, 2nd, 3rd and 4th type of food.

Step 2 :

Let these units be x_1, x_2, x_3 and x_4 respectively.

Step 3 :

Feasible alternatives are sets of values of $x_j \geqslant 0$, where $j=1, 2, 3, 4.$ ...(2.13)

Step 4 :

Objective is to minimize the cost

i.e., minimize $Z = x_1d_1 + x_2d_2 + x_3d_3 + x_4d_4$ ...(2.14)

Step 5 :

Constraints are that the minimum daily requirements of the various constituents must be fulfilled.

i.e., for proteins, $x_1p_1 + x_2p_2 - x_3p_3 + x_4p_4 \geqslant P$

for fats, $x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 \geqslant F$

and for carbohydrates $x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 \geqslant C.$ ...(2.15)

Thus the L.P. problem is to minimize equation (2.14) subject to constraints (2.15) and non-negativity restrictions (2.13).

EXAMPLE 2.3-9 (War Strategy Problem)

The strategic bomber command receives instructions to interrupt the enemy tank production. The enemy has four key plants located in separate cities, and destruction of any one plant will effectively halt the production of tanks. There is an acute shortage of fuel, which limits the supply to 45,000 litres for this particular mission. Any bomber sent to any particular city must have at least enough fuel for the round trip plus 100 litres.

The number of bombers available to the commander and their descriptions are as follows :

Bomber type	*Description*	*Km./litre*	*Number available*
A	Heavy	2	40
B	Medium	2·5	30

Information about the location of the plants and their probability of being attacked by a medium bomber and a heavy bomber is given below.

Plant	*Distance from base (km.)*	*Probability of destruction by a heavy bomber*	*Probability of destruction by a medium bomber*
1	400	0·10	0·08
2	450	0·20	0·16
3	500	0·15	0·12
4	600	0·25	0·20

How many of each type of bombers should be dispatched, and how should they be allocated among the four targets in order to maximize the probability of success ?

Formulation of L.P. Model

Step 1 :

Key decision to be made is how many of each type of bombers be sent to which plant.

Step 2 :

Let the bombers sent be x_{ij}, where i=type A, B and j=plant 1, 2, 3, 4.

Step 3 :

Feasible alternatives are sets of values $x_{ij} \geqslant 0$, where i=A, B & j=1, 2, 3, 4.

i.e., $x_{A1}, x_{A2}, x_{A3}, x_{A4}, x_{B1}, x_{B2}, x_{B3}, x_{B4}$, all $\geqslant 0$ ...(2·16)

Step 4 :

Objective is to maximize the probability of success in destroying at least one plant and this is equivalent to minimizing the probability of not destroying any plant. Let Q denote this probability.

Then, $Q = (1-0{\cdot}1)^{x}A_1 \,.(1-0{\cdot}2)^{x}A_2 \,.\, (1-0{\cdot}15)^{x}A_3 \,.\, (1-0{\cdot}25)^{x}A_4.$

$(1-0{\cdot}08)^{x}B_1 \,(1-0{\cdot}16)^{x}B_2 \,.\, (1-0{\cdot}12)^{x}B_3.\, (1-0{\cdot}20)^{x}B_4.$

Here the objective function is non-linear but it can be reduced to the linear form.

Now, minimizing Q is equivalent to minimizing log Q and log Q is linear. Moreover, minimizing log Q is equivalent to maximizing $-\log Q$ or maximizing $\log 1/Q$. Taking base of log as 10,

$$\log 1/Q = -(x_{A1} \log 0{\cdot}9 + x_{A2} \log 0{\cdot}8 + x_{A3} \log 0{\cdot}85 + x_{A4} \log 0{\cdot}75 + x_{B1} \log 0{\cdot}92 + x_{B2} \log 0{\cdot}84 + x_{B3} \log 0{\cdot}88 + x_{B4} \log 0{\cdot}80)$$

Therefore, the objective is to maximize

$$\left.\begin{aligned} \log 1/Q = 0.0457x_{A1} + 0.09691x_{A2} + 0.07041x_{A3} \\ +0.12483x_{A4} + 0.03623x_{B1} + 0.06558x_{B2} \\ +0.05538x_{B3} + 0.09691x_{B4}. \end{aligned}\right\} \quad ...(2.17)$$

Step 5 :

Constraints are

(*a*) due to limited supply of fuel

$$\left(2\times\frac{400}{2}+100\right)x_{A1}+\left(2\times\frac{450}{2}+100\right)x_{A2}+\left(2\times\frac{500}{2}+100\right)x_{A3}+\left(2\times\frac{600}{2}+100\right)x_{A4}+$$
$$\left(2\times\frac{400}{2.5}+100\right)x_{B1}+\left(2\times\frac{450}{2.5}+100\right)x_{B2}+\left(2\times\frac{500}{2.5}+100\right)x_{B3}+\left(2\times\frac{600}{2.5}+100\right)x_{B4}\leqslant 45,000$$

i.e., $500x_{A1}+550x_{A2}+600x_{A3}+700x_{A4}+420x_{B1}+460x_{B2}+500x_{B3}+580x_{B4}\leqslant 45,000.$ ...(2·18*a*)

(*b*) due to limited number of air crafts

$$\left.\begin{aligned} x_{A1}+x_{A2}+x_{A3}+x_{A4}\leqslant 40 \\ x_{B1}+x_{B2}+x_{B3}+x_{B4}\leqslant 30. \end{aligned}\right\} \quad ...(2.18\ b)$$

Thus the L.P. problem is to maximize equation (2·17) subject to in-equations (2.18*a*), (2·18*b*) and (2·16).

EXAMPLE 2.3-10 (Production Scheduling Problem)

A company wants to plan the next week's production of its three products A, B and C. These products are made on three machines—lathes, drills and grinders. Time available on lathes, drills and grinders for the next week is 200 hrs., 250 hrs. and 300 hrs. respectively. The products can be made through different alternative routes shown in the table below. The products sell in the market at Rs. 20, Rs. 15 and Rs. 25 per unit respectively.

(*a*) Formulate the L.P. model assuming unlimited market demand for the products.

(*b*) There is a fixed order (that has to be satisfied) of 250 units of A, 200 units of B and 150 units of C.

The customer pays Rs. 20, Rs. 15 and Rs. 25 per unit of products A, B and C in the fixed order and is willing to pay Rs. 15, Rs. 10 and Rs. 20 per unit for the extra units of A, B and C respectively. Construct the model that maximizes the sales revenue.

(*c*) If not more than 200 units of C can be sold in the market, what modifications would be required in the model ?

(*d*) If there is possibility of using overtime, how can it be taken into consideration ?

Machines	*Product A*			*Product B*		*Product C*			*Machine hours available*
	Route			*Route*		*Route*			
		2	3	1	2	1	2	3	
Lathes	0·5	0·7	0·3		0·5	0·6	0·5	0·3	200
Drills	0·5	0·3	0·2	0·4	0·3	0·7	0·4	0·1	250
Grinders	0·6	0·4	0·6	0·7	0·5	0·4	0·3		300

(*a*) **Formulation of L.P. Model**

Step 1 :

Key decision to be made is to determine the number of units of products A, B and C to be manufactured through first, second and third routes.

Step 2 :

Let the number of units of products A, B and C manufactured through first, second and third routes be x_{A1}, x_{A2}, x_{A3} ; x_{B1}, x_{B2} and x_{C1}, x_{C2}, x_{C3} respectively.

Step 3 :

Feasible alternatives are sets of values of these variables, where each of them is non-negative.

i.e., $x_{A1}, x_{A2}, x_{A3}, x_{B1}, x_{B2}, x_{C1}, x_{C2}, x_{C3}$, all $\geqslant 0$.

Step 4 :

Objective is to maximize the sales revenue.

i.e., maximize $Z = 20\,(x_{A1}+x_{A2}+x_{A3})+15\,(x_{B1}+x_{B2})$
$+25\,(x_{C1}+x_{C2}+x_{C3})$.

Step 5 :

Constraints are on the machine hours available for each machine. They are

for lathes : $0.5x_{A1}+0.7x_{A2}+0.3x_{A3}+0.5x_{B2}+0.6x_{C1}+0.5x_{C2}$
$+0.3x_{C3}\leqslant 200$

for drills : $0.5x_{A1}+0.3x_{A2}+0.2x_{A3}+0.4x_{B1}+0.3x_{B2}+0.7x_{C1}$
$+0.4x_{C2}+0.1x_{C3}\leqslant 250$ and

for grinders : $0.6x_{A1}+0.4x_{A2}+0.6x_{A3}+0.7x_{B1}+0.5x_{B2}+0.4x_{C1}$
$+0.3x_{C2}\leqslant 300.$

Thus the L.P. model is to maximize Z subject to the constraints and non-negativity restriction mentioned above.

(*b*) The fixed order is for 250 units of A, 200 units of B and 150 units of C. The total number of units of product A produced are $x_{A1}+x_{A2}+x_{A3}$ and in order to satisfy the fixed order it must be $\geqslant 250$ *i.e.*, for lathes : $x_{A1}+x_{A2}+x_{A3}\geqslant 250$

for drills : $x_{B1}+x_{B2}\geqslant 200$ and

for grinders : $x_{C1}+x_{C2}+x_{C3}\geqslant 150$

These are, then, the additional constraints to be satisfied (along with the constraints of step 5).

The new objective function is slightly more complicated and may be written as

maximize $Z_1=250\times 20+15\,(x_{A1}+x_{A2}+x_{A3}-250)+200\times 15+10\,(x_{B1}+x_{B2}-200)+150\times 25+20\,(x_{C1}+x_{C2}+x_{C3}-150)$.

The problem is, thus, to maximize Z_1 subject to the above six constraints while satisfying the non-negativity condition.

(*c*) This market limitation results in a new constraint

$x_{C1}+x_{C2}+x_{C3}\leqslant 200$

and the problem is to maximize Z_1 while satisfying this 7th (additional) constraint also.

(*d*) Let x_{A10}, x_{A20} and x_{A30} represent the number of units of product A manufactured during overtime through routes 1, 2 and 3 respectively. The overtime machine hours available need to be given in the problem, which will result in three more constraints. The objective function representing sales revenue will have to be replaced by profit function as production during overtime is less profitable than regular production. Though the objective function becomes more complex and number of constraints becomes large, the problem remains a linear programming problem.

Situation 3 :

The first situation which we considered was related to allocation of resources. The inputs of different resources were determined in advance and the problem was to allocate them to each individual goal so as to achieve the highest fulfilment of the goals.

The second situation related to the choice of resources. Lower (and in some cases upper) limits were set in advance for individual goals and the problem was to determine the optimum set of resources which will achieve the lower limits (or will not exceed the upper limits).

The third situation which we shall consider now is, in a certain sense, synthesis of the previous two ones. Thus, while on one hand, there are allocations of resources which enable to achieve the goals set, lower limits are provided for the goals to be achieved. We are to make the most effective allocation of the resources for the individual goals. We shall call this type of situation as *resource allocation-choice situation.* On one hand, we are to allocate the available resources to individual goals so as not to exceed the limit set on these resources; on the other hand, we have to choose the resources for individual goals such that these goals are achieved, at least to some extent, as specified, in advance. We shall study an example to make this situation clear.

EXAMPLE 2.3-11 (Transportation Problem)

A dairy firm has three plants located throughout a state. Daily milk production at each plant is as follows :

Plant 1 — 6 million litres

Plant 2 — 1 million litres

Plant 3 — 10 million litres

Each day the firm must fulfil the needs of its four distribution centres. Minimum requirement of each centre is as follows :

Distribution centre 1 — 7 million litres

Distribution centre 2 — 5 million litres

Distribution centre 3 — 3 million litres

Distribution centre 4 — 2 million litres

Cost of shipping one million litre of milk from each plant to each distribution centre is given in hundreds of rupees in the table below.

Table 2.7

Distribution Centres

		1	2	3	4
	1	2	3	11	7
Plants	2	1	9	6	1
	3	5	8	15	9

The dairy firm wishes to decide as to how much should be the shipment from which plant to which distribution centre so that the cost of shipment may be minimum. (See chapter on 'Transportation Models' for formulation of L. P. model).

Situation 4 :

The fourth and last situation is called the *assignment problem*. The assignment problem is strictly connected with allocation problem. In both cases the objective is to achieve the targets with the help of available resources in specified amounts. However, the operating conditions are different. While in the allocation problem each target can be achieved in one way only (e.g., to manufacture a certain product, a definite amount of various resources has got to be used to produce each unit of the product), in assignment problem, the individual targets can be achieved in different ways.

Generally, there are a certain number of targets (say n) to be achieved and there are an equal number of available resources. Each of these n resources can be used to attain any of the n targets. Resources are to be assigned to different targets so that each target is achieved to a lesser or greater extent. Moreover, an additional condition has to be satisfied that once particular resource is allocated to fulfil a given target, it is not to be used, even in part, to fulfil any other target. We shall now explain this situation with the help of an example.

EXAMPLE 2.3-12 (Assignment Problem)

A machine tool company wants to make 4 sub-assemblies through 4 contractors. Each contractor is to receive only one sub-assembly. The cost of each sub-assembly is determined by the bids submitted by each contractor and is shown below in hundreds of rupees.

Table 2.8

Contractors

		1	2	3	4
	1	16	14	15	18
Sub-assemblies	2	12	13	16	14
	3	14	13	11	12
	4	16	18	15	17

Assign the different sub-assemblies to contractors in such a way so as to minimize the total cost. (See chapter on 'Assignment Models' for formulation of L. P. model).

EXAMPLE 2. 3-13 (Travelling Salesman Problem)

A salesman wishes to visit cities A, B, C, D & E. He does not want to visit any city twice before completing his tour of all the cities and wishes to return to the point of starting journey. Cost of going from one city to another in rupees is given below.

Table 2.9

	A	B	C	D	E
A	0	2	5	7	1
B	6	0	3	8	2
C	8	7	0	4	7
D	12	4	6	0	5
E	1	3	2	8	0

Find the least cost route.

EXAMPLE 2.3-14 (**Make or buy decisions**)

There are five engine parts E_1, E_2, E_3, E_4 and E_5 which can either be manufactured or bought from outside. Six machines M_1, M_2, M_3, M_4, M_5, M_6 are available to produce the five parts. t_{ij} is the time taken if part i is machined on machine j. Not more than 40 hours are available on each machine, while 800 of each component are required. The production cost of part i on machine j is c_{ij} and the purchase cost of part i is c_i. Formulate an L. P. model to produce or purchase the parts so as to minimize the cost.

EXAMPLE 2.3-15

A machine shop processes 5 items in sequence. The set-up cost involved for producing an item depends on the item presently in process and the set-up required for producing the new item. C_{ij}, the cost of changing the set-up from item i to j are as follows :

Table 2.10

		To item				
		A	B	C	D	E
	A	—	4	7	3	4
	B	4	—	6	3	4
From item	C	7	6	—	7	5
	D	3	3	7	—	7
	E	4	4	5	7	—

If the objective is to minimize the total set up cost, formulate an L.P. model to determine the optimum sequence in which the products should be produced.

2.4. Graphical Solution of two-variable L.P. Problems

A linear programming problem with only two variables presents a simple case, for which the solution can be derived using a graphical method. We shall explain this method by taking a few specific situations.

EXAMPLE 2.4-1

A firm manufactures two products A & B on which the profits earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hrs. 30 mins. while machine M_2 is available for 10 hrs. during any working day. Find the number of units of products A and B to be manufactured to get maximum profit.

Formulation of Linear Programming Model

Step 1 :

Key decision is to determine the extent (number of units) of manufacturing the products A and B.

Step 2 :

Let these extents be x_1 and x_2 respectively.

Step 3 :

Feasible alternatives are sets of values of

x_1, x_2, where $x_1 \geqslant 0$ and $x_2 \geqslant 0$. ...(2.19)

Step 4 :

Objective is to maximize the profit

i.e., maximize $Z = 3x_1 + 4x_2$. ...(2.20)

Step 5 :

Constraints are on the time available for machines M_1 and M_2

$$\left.\begin{array}{l}\text{i.e., for machine } M_1,\ 1.x_1 + 1.x_2 \leqslant 450 \\ \text{and for machine } M_2,\ 2.x_1 + 1.x_2 \leqslant 600.\end{array}\right\} \quad ...(2.21)$$

Thus the problem is to maximize equation (2.20) subject to relations (2.21) and (2.19). This will be done graphically.

Solution of L.P. Model

The non-negativity restrictions $x_1 \geqslant 0$ and $x_2 \geqslant 0$ imply that values of the variables x_1 and x_2 can lie only in the first quadrant (x_1, x_2 plane). This is shown by shaded area of figure 2.1. Other quadrants do not satisfy the non-negativity restrictions and hence the pt. (x_1, x_2) cannot lie in them. Therefore, a number of alternatives are eliminated.

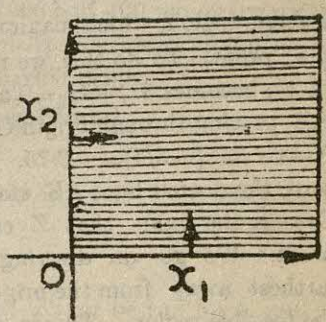

Figure 2.1,

The effect of the remaining constraints can now be added to figure 2.1. This is done by plotting all the restrictions with their inequality sign changed into equality sign. The direction in which each

constraint holds good is then determined from the direction of the inequality and is indicated by an arrow on its associated straight line. The constraint conditions define the boundary of the region containing feasible solution.

For example, the next constraints are $x_1+x_2 \leqslant 450$ and $2x_1+x_2 \leqslant 600$. We plot lines $x_1+x_2=450$ and $2x_1+x_2=600$ as shown in figure 2·2. Any point lying on or below the line $x_1+x_2=450$ satisfies the constraint $x_1+x_2 \leqslant 450$. Similarly any point lying on or below the line $2x_1+x_2=600$ satisfies the constraint $2x_1+x_2 \leqslant 600$. This is clearly indicated by the direction of arrowheads. The shaded area in

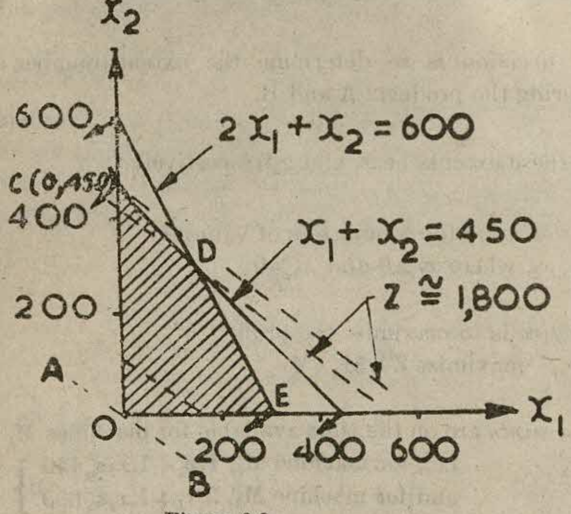

Figure 2·2.

the figure satisfies both the constraints $x_1+x_2 \leqslant 450$ and $2x_1+x_2 \leqslant 600$ and also the non-negativity restrictions $x_1 \geqslant 0$, $x_2 \geqslant 0$. This area is called the *solution space* or the *region of feasible solutions*. Any point in this shaded region is a *feasible solution* to the given problem.

To Find Optimal Solution

Method 1 : Our problem is to find the point (or points) in the feasible region, which maximizes (or maximize) the objective function (i.e., profit). To do this, we notice that when Z is made zero, equation (2·20) becomes $3x_1+4x_2=0$ and this is represented by the dotted line AB passing through origin O. Its slope i.e., tangent of the angle with X-axis is $(-3/4)=-0{\cdot}75$. As the value of Z is increased from zero, the dotted line AB starts moving to the right, parallel to itself. Greater the value that Z can assume, more will be the company's profit. We go on drawing lines parallel to this line till the line is farthest away from the origin and passes through only one point of the feasible region. This is the point where maxima is attained. It is possible that such a line may be one of the edges of the feasible

region. In that case every point on that edge gives the same maximum value of the objective function.

In the present example, maximum is obtained at the corner point C (0, 450), which means that only product B should be manufactured and 450 unit of this product B should be produced. The daily profit will be Z = Rs. $(0+4\times450)$ = Rs. 1,800. This solution corresponding to pt. C (0, 450), which maximizes the objective function, is called *optimal solution*.

Method 2 : The four vertices of the convex region OCDE are O (0, 0), C (0, 450), D (180, 300) and E (300, 0). Values of the objective function $Z=3x_1+4x_2$ at these vertices are

$Z(O)=0$, $Z(C)=1{,}800$, $Z(D)=540+1{,}200=1{,}740$, $Z(E)=900$.

Thus the maximum value of Z is Rs. 1,800 and it occurs at the vertex C (0, 450).

Hence the solution to the problem is

$$x_1=0,\ x_2=450 \text{ and } Z_{max}=\text{Rs. } 1{,}800.$$

EXAMPLE 2.4-2.

Mohan-Meakins Breveries Ltd. has two bottling plants, one located at Solan and the other at Mohan Nagar. Each plant produces three drinks, whisky, beer and fruit juices named A, B and C respectively. The number of bottles produced per day are as follows :

	Plant at	
	Solan (S)	Mohan Nagar (M)
Whisky A	1,500	1,500
Beer B	3,000	1,000
Fruit juices C	2,000	5,000

A market survey indicates that during the month of April, there will be a demand of 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of fruit juices. The operating costs per day for plants at Solan and Mohan Nagar are 600 and 400 monetary units. For how many days each plant be run in April so as to minimize the production cost, while still meeting the market demand ?

Formulation of Linear Programming Model

Step 1 :

Key decision is to determine the number of days for which each plant must be run in April.

Step 2 :

Let the plants at Solan and Mohan Nagar be run for x_1 and x_2 days.

Step 3 :

Feasible alternatives are sets of values of $x_1 \geqslant 0$ and $x_2 \geqslant 0$ which meet the objective. ...(2.22)

Step 4 :

Objective is to minimize the production cost.

i.e., minimize $Z = 600x_1 + 400x_2$. ...(2.23)

Step 5 :

Constraints are on the demand.

$$\left.\begin{array}{ll} \text{i.e., for whisky,} & 1{,}500x_1 + 1{,}500x_2 \geqslant 20{,}000 \\ \text{for beer,} & 3{,}000x_1 + 1{,}000x_2 \geqslant 40{,}000 \\ \text{and for fruit juices,} & 2{,}000x_1 + 5{,}000x_2 \geqslant 44{,}000 \end{array}\right\} \quad ...(2.24)$$

Thus we are to optimize equation (2.23), subject to constraints represented by relations (2·24) and nonnegativity restrictions (2·22).

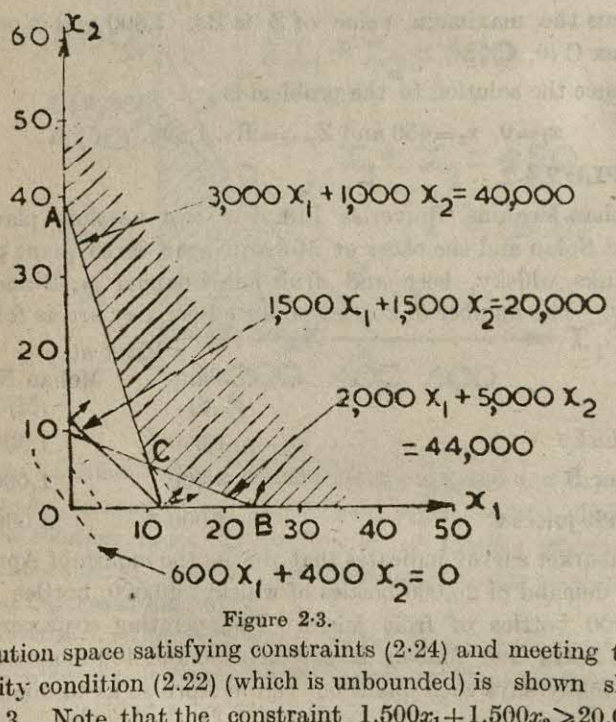

Figure 2·3.

The solution space satisfying constraints (2·24) and meeting the nonnegativity condition (2.22) (which is unbounded) is shown shaded in figure 2.3. Note that the constraint $1{,}500x_1 + 1{,}500x_2 \geqslant 20{,}000$ does not affect the solution space since it is dominated by the constraint $3{,}000x_1 + 1{,}000x_2 \geqslant 40{,}000$. The constraint $1{,}500x_1 + 1{,}500x_2 \geqslant 20{,}000$ is called a *redundant constraint*.

To Find Optimal Solution :

Method 1. The objective function, when $Z=0$, gives the equation $600x_1 + 400x_2 = 0$ which is shown by the dotted line AB passing through the origin O. As Z is increased from zero, the dotted line AB moves to the right, parallel to itself. Since we are interested in

minimizing Z, we increase the value of Z till the dotted line passes through the *nearest corner* of the shaded region from the origin. This gives the **minimum** value of Z, while keeping x_1 and x_2 within the region of feasible solutions. The coordinates of this point C are (12, 4). Thus production cost will be minimum if plants at Solan and Mohan Nagar are run for 12 days and 4 days respectively, giving the production cost as $600 \times 12 + 400 \times 4 = 7,200 + 1,600 = 8,800$ monetary units. Substituting the values of x_1 and x_2 in constraints (2·24) we find that market demand is also met.

Method 2 : The three vertices of the convex set OABC are A (0, 40), B (22, 0) and C(12, 4).

Values of the objective function $Z = 600x_1 + 400x_2$ at these vertices are

$Z(A) = 16,000$, $Z(B) = 13,200$ and $Z(C) = 7,200 + 1,600 = 8,800$.

Thus the minimum value of Z is 8,800 monetary units and it occurs at the vertex C (12, 4).

Hence the solution to the problem is

$$x_1 = 12 \text{ days}$$
$$x_2 = 4 \text{ days}$$
and $$Z_{min} = 8,800 \text{ monetary units}$$

The above examples indicate that the search for the optimum is reduced to finding *only the vertices* (corner points) of the solution space. Mathematically, a corner point is known as an *extreme point*. Once all the extreme points are known, the one that gives the best value of the objective function is the optimum. Sections 2·9 and 2·10 show that the *simplex method* consists of determining some of these vertices (extreme points) in a selective manner.

2.5. Some Exceptional Cases

In section 2.4 we discussed two linear programming problems and optimal solution for either of them was unique. However, it may not be so for every problem. In general a linear programming problem may have

(*i*) a definite and unique optimal solution

(*ii*) an infinite number of optimal solutions

(*iii*) an unbounded solution

(*iv*) no solution

The first case was covered in the previous section. A few examples are presented here to cover the remaining three cases.

EXAMPLE 2.5-1

A firm uses lathes, milling machines and grinding machines to

produce two machine parts. Table 2·11 represents the machining times required for each part, the machining times available on different machines and the profit on each machine part.

Table 2.11

Types of machine	Machining time required for the machine part (minutes) I	II	Maximum time available per week (minutes)
Lathes	12	6	3,000
Milling machines	4	10	2,000
Grinding machines	2	3	900
Profit per unit	Rs. 40	Rs. 100	

Find the number of parts I and II to be manufactured per week to maximize the profit.

Formulation of Linear Programming Model

Step 1 :

Key decision is to determine the number of machine parts I and II to be manufactured per week.

Step 2 :

Let the number of parts I and II manufactured per week be x_1 and x_2 respectively.

Step 3 :

Feasible alternatives are sets of values of of x_1 and x_2, where

$$x_1 \geqslant 0,\ x_2 \geqslant 0. \qquad \ldots(2\cdot25)$$

Step 4 :

Objective is to maximize the profit.

i.e., maximize $Z = 40\,x_1 + 100\,x_2$ ...(2·26)

Step 5 :

Constraints are on the time available on each machine.

$$\left.\begin{array}{ll}\text{Thus for lathes,} & 12x_1 + 6x_2 \leqslant 3{,}000 \\ \text{for milling machines,} & 4x_1 + 10x_2 \leqslant 2{,}000 \\ \text{and for grinding machines,} & 2x_1 + 3x_2 \leqslant 900\end{array}\right\} \qquad \ldots(2\cdot27)$$

...interested

Thus the problem is to determine the values of x_1 and x_2 which meet the non-negativity condition (2·25), satisfy the constraints (2·27) and maximize equation (2·26).

Solution of L.P. Model

The solution space satisfying the constraints (2·27) and meeting the non-negativity condition (2·25) is shown shaded in Fig. 2·4. Note that the constraint $2x_1+3x_2 \leqslant 900$ does not affect the solution space and is thus a redundant constraint.

The four vertices of the convex set OABC are O(0, 0). A(0, 200), B(187·5, 125), C(250, 0).

Values of the objective function $Z=40x_1+100x_2$ at these vertices are

$Z(O)=0$, $Z(A)=20{,}000$, $Z(B)=20{,}000$, $Z(C)=10{,}000$.

Thus maximum value Z occurs at two vertices A and B of the convex shaded region OABC.

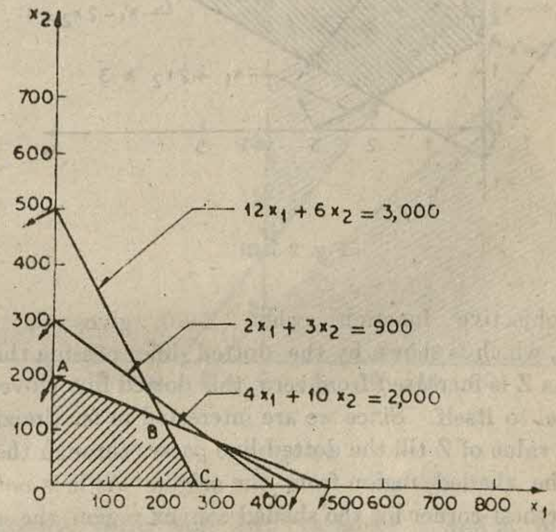

Fig. 2·4.

∴ The two points A and B give the maximum value of Z. It follows that every linear convex combination of these points will also give the same maximum value of Z. Therefore, there is no unique optimal solution to the problem and any point between A and B on the line AB can be taken as an optimal solution with a profit value of Rs. 20,000.

EXAMPLE 2·5·2

Maximize $Z=5x_1+4x_2$

subject to $x_1-2x_2\leqslant 1$

$x_1+2x_2\geqslant 3$

$x_1,\ x_2\geqslant 0.$

Solution. The solution space satisfying the constraints $x_1-2x_2\leqslant 1$, $x_1+2x_2\geqslant 3$ and the non-negativity condition $x_1\geqslant 0$, $x_2\geqslant 0$ is shown shaded in Fig. 2·5. This shaded convex region is unbounded.

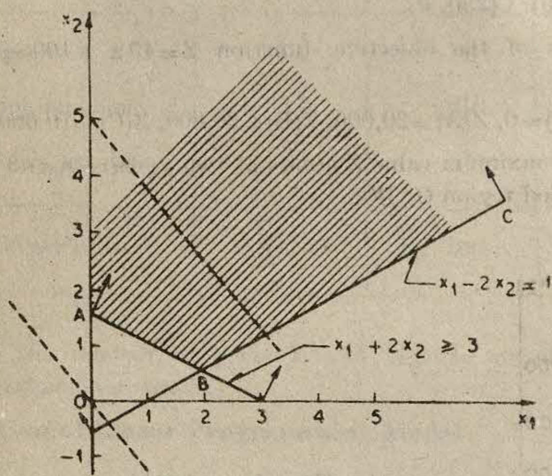

Fig. 2·5.

The objective function, when $Z=0$, gives the equation $5x_1+4x_2=0$, which is shown by the dotted line passing through the origin O. As Z is increased from zero, this dotted line moves to the right, parallel to itself. Since we are interested in maximizing Z, we increase the value of Z till the dotted line passes through the farthest corner of the shaded region from the origin. As it is not possible to get the farthest corner for the shaded convex region, the maximum value of Z cannot be found as it occurs at infinity only. The problem, therefore, has an unbounded solution.

Remark : An easy method to plot the dotted line for objective function $Z=5x_1+4x_2$ is to assign any value to Z, say $5\times 4=20$, and to plot the line $5x_1+4x_2=20$.

A dotted line is then drawn through the origin parallel to this line.

EXAMPLE 2·5·3

Maximize $\quad Z=3x+2y$

subject to $\quad -2x+3y\leqslant 9$

$3x-2y\geqslant -20$

$x, y\geqslant 0.$

Solution. Fig. 2·6 indicates two shaded regions, one satisfying the constraint $-2x+3y\leqslant 9$ and the other satisfying the constraint $3x-2y\geqslant -20$. These two shaded regions in the first quadrant do not overlap with the result that there is no point (x, y) common to both the shaded regions. The problem cannot be solved graphically (or by any other method of solving L.P. problems) *i.e.*, the solution of the problem does not exist.

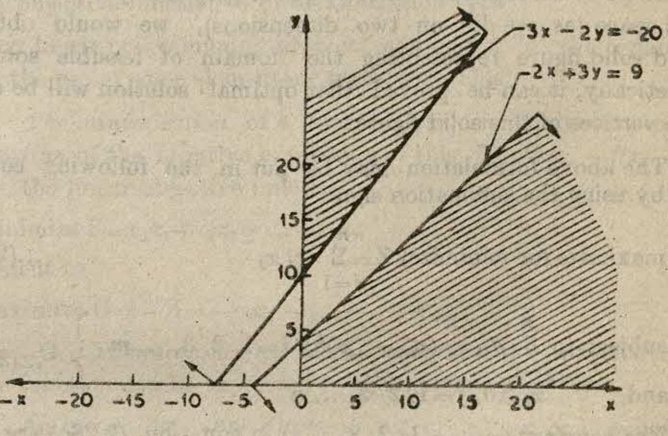

Fig. 2·6.

Evidently, there were two variables x_1 and x_2 in the above examples and the problems were, therefore, two dimensional and were simple to be represented (by the two axes lying in a plane) and solved graphically. Now, as the number of variables increases to 3, 4, ... we come across 3-dimensional, 4-dimensional, ... problems which become quite laborious to be solved by graphical methods. In such cases *simplex technique* helps us in

(*i*) starting with a feasible solution

(*ii*) searching optimal solution in a systematic way.

2.6. General Mathematical Formulation for Linear Programming

The general linear programming problem can be expressed as follows :

Find the values of decision variables $x_1, x_2, x_3, \ldots, x_n$ which satisfy the constraints

$$\left.\begin{array}{l} a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 + \cdots + a_{1n} x_n\ (\leqslant, =, \geqslant) b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 + \cdots + a_{2n}\, x_n\ (\leqslant, =, \geqslant) b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}\, x_1 + a_{m2} x_2 + a_{m3} x_3 + \cdots + a_{mn}\, x_n\ (\leqslant, =, \geqslant) b_m \end{array}\right\} \qquad \ldots(2.28)$$

$$\text{and } x_j \geqslant 0, \text{ where } j = 1, 2, 3, \ldots, n \qquad \ldots(2.29)$$

and maximize or minimize the objective function (profit, loss, cost, etc.) which is a linear function of x_j, such as

$$Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots + c_n x_n \qquad \ldots(2.30)$$

If we could represent these relations graphically on an n-dimensional space (as we did on two dimensions), we would obtain a shaded solid figure representing the domain of feasible solutions. Theoretically, it can be proved that optimal solution will be on one of the vertices of this solid figure.

The above formulation may be put in the following compact form by using the summation sign :

$$\text{maximize (or minimize) } Z = \sum_{j=1}^{n} c_j\, x_j \qquad \ldots(2.28\,a)$$

$$\left.\begin{array}{ll} \text{subject to} & \sum_{j=1}^{n} a_{ij}\, x_j\ (\leqslant, =, \geqslant) b_i,\ i = 1, 2, 3, \ldots, m \\ \text{and} & x_j \geqslant 0,\ j = 1, 2, 3, \ldots, n \end{array}\right\} \qquad \ldots(2.29\,a)$$

The constants c_j ($j = 1, 2, 3 \ldots, n$) in equation (2.28a) are called *cost coefficients* ; the constants b_i ($i = 1, 2, 3, \ldots, m$) in the constraint conditions are called *stipulations* and the constants a_{ij} ($i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$) are called *structural coefficients.*

2.7. Canonical and Standard forms of Linear Programming Problem

After formulating the linear programming problem, the next step is to obtain its solution. But before any analytic method is used to obtain the solution, the problem must be available in a particular form. Two forms are dealt with here, the *canonical form* and the *standard form*. While the canonical form is helpful in dealing with *duality theory*, discussed in section 2.15, the standard

form is used to develop the general procedure for solving any linear programming problem.

2.7-1 The Canonical Form

The general linear programming problem discussed in section 2.6 can always be put in the following form, called the canonical form.

$$\text{maximize } Z=\sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}\, x_j \leqslant b_i \quad , \quad i=1, 2, \ldots, m$$

$$x_j \geqslant 0, \qquad j=1, 2, \ldots, n.$$

The characteristics of this form are

(*a*) all decision variables are non-negative

(*b*) all constraints are of the ($\leqslant$) type

(*c*) objective function is of maximization type

Any linear programming problem can be put in the canonical form by the use of some elementary transformations.

1. *The minimization* of a function, $f(x)$, is equivalent to the *maximization* of the negative expression of this function, $-f(x)$. For example, the linear objective function

$$\text{minimize } Z=c_1x_1+c_2x_2+\ldots+c_nx_n$$

is equivalent to

$$\text{maximize } G=-Z=-c_1x_1-c_2x_2-\ldots-c_nx_n,$$

with $Z=-G$. Therefore, for all linear programming problems the objective function can be expressed in the maximization form.

2. An inequality in one direction ($\leqslant$ or $\geqslant$) can be changed to an inequality in the opposite direction ($\geqslant$ or $\leqslant$) by multiplying both sides of the inequality by -1. For example, the linear constraint

$$a_1x_1+a_2x_2 \geqslant b$$

is equivalent to

$$-a_1x_1-a_2x_2 \leqslant -b.$$

Also

$$p_1x_1+p_2x_2 \leqslant q$$

is equivalent to

$$-p_1x_1-p_2x_2 \geqslant -q.$$

3. An equation may be replaced by two weak inequalities

in opposite directions. For example, $a_1x_1+a_2x_2=b$ is equivalent to the two simultaneous constraints

$$a_1x_1+a_2x_2\leqslant b \quad \text{and} \quad a_1x_1+a_2x_2\geqslant b$$

$$\text{or } a_1x_1+a_2x_2\leqslant b \quad \text{and} \quad -a_1x_1-a_2x_2\leqslant -b.$$

4. So far, we have assumed the decision variables $x_1, x_2, \ldots, x_n$ to be all non-negative. It is possible, in actual practice, that a variable may be unconstrained in sign, i.e., it may be positive or negative (it may vary from $-\infty$ to $+\infty$). If a variable is unconstrained, it is expressed as the difference between two non-negative variables. For example, if x is a negative variable, then it can be expressed as

$$x=x'-x''$$

where $x'\geqslant 0$ and $x''\geqslant 0$.

2.7-2 The Standard Form

The characteristics of the standard form are

1. All the constraints are expressed in the form of equations, except the non-negativity constraints which remain inequalities ($\geqslant 0$).

2. The right hand side of each constraint equation is non-negative.

3. All the decision variables are non-negative.

4. The objective function is of the maximization or minimization type.

The inequality constraints are changed to equality constraints by adding or subtracting a non-negative variable from the left-hand sides of such constraints. These new variables are called *slack variables* or simply *slacks*. They are added if the constraints are ($\leqslant$) and subtracted if the constraints are ($\geqslant$). Since in the case of ($\geqslant$) constraints the subtracted variable represents the surplus of left-hand side over right-hand side, it is commonly known as *surplus variable* and is, in fact, a negative slack. In our discussion, however, we shall always use the name "slack" variable and its sign will depend on the inequality sign in the constraint. Both decision variables and slack variables are called *admissible variables* and are treated in the same manner while finding a solution to a problem.

For example, the constraint

$$a_1x_1+a_2x_2\leqslant b \quad , \quad b\geqslant 0$$

is changed in the standard form to $a_1x_1+a_2x_2+s_1=b$, where $s_1\geqslant 0$. Also constraint

$$p_1x_1+p_2x_2\geqslant q \quad , \quad q\geqslant 0$$

is changed to $p_1x_1+p_2x_2-s_2=q$, where $s_2 \geqslant 0$.
The quantities s_1 and s_2 are variables and their values depend upon the values assumed by other $x's$ in a particular equation.

Before trying for the solution of the linear programming problem, it must be expressed in the standard form. The information given by the standard form is then expressed in the *"table form"* or *"matrix form"*.

Let us consider the general linear programming problem

$$\text{maximize } Z=\sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j \leqslant b_i,\ (b_i \geqslant 0),\ i=1, 2, 3,\ldots, m$$

$$x_j \geqslant 0, \qquad j=1, 2, 3,\ldots, n.$$

This is expressed in the standard form as

$$\text{maximize } Z=\sum_{j=1}^{n} c_jx_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j+s_i=b_i, \qquad i=1, 2, 3,\ldots, m$$

$$x_j \geqslant 0, \qquad j=1, 2, 3,\ldots, n$$

$$s_i \geqslant 0, \qquad i=1, 2, 3,\ldots, m.$$

Such an L.P. problem formed after the introduction of slack or surplus variables is called *reformulated L.P. problem.*

The above information is then expressed in the form of table shown below.

Table 2·12

Objective Value	*Decision Variables*	*Slack Variables*	*R.H.S. constraints*	
Z	$x_1\ x_2 \ldots x_n$	$s_1\ s_2 \ldots s_m$	0	
1	$-c_1\ -c_2 \ldots -c_n$	0 0...0	0	Z-equation
0	$a_{11}\ a_{12} \ldots a_{1n}$	1 0...0	b_1	Constraint equations
0	$a_{21}\ a_{22} \ldots a_{2n}$	0 1...0	b_2	
$\vdots$	$\vdots\ \vdots\ \vdots$	$\vdots\ \vdots\ \vdots$	$\vdots$	
0	$a_{m1}\ a_{m2} \ldots a_{mn}$	0 0...1	b_m	

where Z-equation is obtained by expressing the objective function as

$$Z-\sum_{j=1}^{n} c_j x_j=0.$$

Now, solving the L.P. problem means determining the set of non-negative values of variables x_j and s_i which will maximize Z while satisfying the constraint equations. The concept is simple but we have a set of m equations with n unknowns and an infinite number of solutions is possible. Clearly, a hit and trial method for finding the optimal solution is not feasible. There is a definite need for an efficient and systematic procedure which will yield the desired solution in a finite number of trials. An iterative procedure called *simplex technique* helps us to reach the optimal solution (if it exists) in a finite number of iterations.

EXAMPLE 2.7-1

Express the following linear programming problem in the standard form :

$$\begin{aligned} \text{maximize} \quad & Z=3x_1+2x_2+5x_3 \\ \text{subject to} \quad & 2x_1-3x_2 \leqslant 3 \\ & x_1+2x_2+3x_3 \geqslant 5 \\ & 3x_1+2x_3 \leqslant 2 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0. \end{aligned}$$

Solution. Here x_1 and x_2 are restricted to be non-negative, while x_3 is unrestricted.

Let us express x_3 as $=x'_3-x''_3$, where $x'_3 \geqslant 0$ and $x''_3 \geqslant 0$. Thus the above constraints can be written as

$$\begin{aligned} & 2x_1-3x_2 \leqslant 3 \\ & x_1+2x_2+3x'_3-3x''_3 \geqslant 5 \\ & 3x_1+2x'_3-2x''_3 \leqslant 2 \end{aligned}$$

where $x_1 \geqslant 0,\ x_2 \geqslant 0,\ x'_3 \geqslant 0,\ x''_3 \geqslant 0.$

Introducing the slack variables, the standard form is

$$\begin{aligned} \text{maximize} \quad & Z=3x_1+2x_2+5x'_3-5x''_3 \\ \text{subject to} \quad & 2x_1-3x_2+s_1=3 \\ & x_1+2x_2-3x'_3-3x''_3-s_2=5 \\ & 3x_1+2x'_3-2x''_3+s_3=2. \end{aligned}$$

where $x_1 \geqslant 0,\ x_2 \geqslant 0,\ x'_3 \geqslant 0,\ x''_3 \geqslant 0,\ s_1 \geqslant 0,\ s_2 \geqslant 0$ and $s_3 \geqslant 0.$

EXAMPLE 2.7-2

Express the following linear programming problem in the

standard form :

Determine x_1, x_2, x_3 so as to

$$\text{maximize } Z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} \text{subject to} \quad & 2x_1 + 3x_2 - 2x_3 \leqslant 40 \\ & 4x_1 - 2x_2 + x_3 \leqslant 24 \\ & x_1 - 5x_2 - 6x_3 \geqslant 2 \\ & x_1 \geqslant 0. \end{aligned}$$

Solution. Here only x_1 is restricted to be non-negative, while x_2 and x_3 are unrestricted. Let us express

$$x_1 \text{ as} = y_1, \text{ where } y_1 \geqslant 0$$

$$x_2 = y_2 - y_3, \ y_2, y_3 \geqslant 0$$

and $\quad x_3 = y_4 - y_5, \ y_4, y_5 \geqslant 0.$

Thus the given constraints can be written as

$$\begin{aligned} 2y_1 + 3y_2 - 3y_3 - 2y_4 + 2y_5 &\leqslant 40 \\ 4y_1 - 2y_2 + 2y_3 + y_4 - y_5 &\leqslant 24 \\ y_1 - 5y_2 + 5y_3 - 6y_4 + 6y_5 &\geqslant 2 \end{aligned}$$

where y_1, y_2, y_3, y_4, y_5, all $\geqslant 0$.

Introducing the slack variables, the standard form is

$$\text{maximize} \quad Z = 3y_1 + 2y_2 - 2y_3 + 5y_4 - 5y_5$$

$$\begin{aligned} \text{subject to} \quad & 2y_1 + 3y_2 - 3y_3 - 2y_4 + 2y_5 + s_1 = 40 \\ & 4y_1 - 2y_2 + 2y_3 + y_4 - y_5 + s_2 = 24 \\ & y_1 - 5y_2 + 5y_3 - 6y_4 + 6y_5 - s_3 = 2 \end{aligned}$$

where $\quad y_1, y_2, y_3, y_4, y_5, s_1, s_2, s_3$, all $\geqslant 0$.

EXAMPLE 2·7·3

Reformulate the problem into standard form :

$$\text{minimize} \quad Z = 2x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \quad & 2x_1 - 3x_2 - x_3 = -4 \\ & 3x_1 + 4x_2 - x_4 = -6 \\ & 2x_1 + 5x_2 + x_5 = 10 \\ & 4x_1 - 3x_2 + x_6 = 18 \end{aligned}$$

where $\quad x_3, x_4, x_5, x_6$, all $\geqslant 0$.

Solution. Here x_3, x_4, x_5, x_6 (which are all non-negative) are the slack variables. The decision variables x_1, x_2 are unrestricted in sign.

Putting $\quad x_1 = y_1 - y_2$, $x_2 = y_3 - y_4$, $x_3 = y_5$, $x_4 = y_6$, $x_5 = y_7$ and $x_6 = y_8$, the problem in standard form is

$$\text{minimize} \quad Z = 2y_1 - 2y_2 + 3y_3 - 3y_4$$

$$\begin{aligned} \text{subject to} \quad & 2y_1 - 2y_2 - 3y_3 + 3y_4 - y_5 = -4 \\ & 3y_1 - 3y_2 + 4y_3 - 4y_4 - y_6 = -6 \end{aligned}$$

$$2y_1-2y_2+5y_3-5y_4+y_7=10$$
$$4y_1-4y_2-3y_3+3y_4+y_8=18$$

where $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8$, all $\geqslant 0$.

Remark. When $x_1=y_1-y_2$, it can be seen that for any value of x_1, there will be an infinite number of combinations of (y_1, y_2) which satisfy this equation. However, if values of y_1 and y_2 are given, there will be only one value of x_1. Therefore, if an optimal solution to the new problem is obtained which contains specific values of $y_1, y_2, \ldots$ the corresponding unique values of $x_1, x_2, \ldots$ will also give an optimal solution for the given problem. Thus an optimal solution to the new problem is also an optimal solution to the original problem.

2.8. Simplex Method

Simplex method, also called simplex technique or simplex algorithm was developed by G.B. Danzig, an American mathematician. It has the advantage of being universal, i.e., any linear model for which the solution exists, can be solved by it. In principle, it consists of starting with a certain solution of which all that we know is that it is feasible, i.e., it satisfies the non-negativity conditions $(x_j \geqslant 0, j=1, 2, 3, \ldots n)$. We then, improve upon this solution at consecutive stages, until, after a certain finite number of stages, we arrive at the optimal solution.

The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less (or more as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, this method leads to an optimal vertex in a finite number of steps. The basis of the simplex method consists of two fundamental conditions :

1. *The feasibility condition* : It ensures that if the starting solution is basic feasible, only basic feasible solutions will be obtained during computation.

2. *The optimality condition.* It guarantees that only better solutions (as compared to the current solution) will be encountered.

2.9. Development of Simplex Method

Simplex method makes use of the following three points in achieving a systematic reduction from an infinite number of solutions to a finite number of promising solutions :

1. If there are m equality constraints and n is the number of variables $(m \leqslant n)$, a start for the optimal solution is made by putting

n-m unknowns (out of n unknowns) equal to zero and then solving for the m equations in remaining m unknowns, provided that the solution exists and is unique. The n-m zero variables are called *nonbasic* variables and the remaining m variables are called *basic* variables which form a *basic solution*. If the solution yields all non-negative basic variables, it is called *basic feasible solution* ; otherwise it is *infeasible*. This step reduces the number of alternatives for the optimal solution from infinite to a finite number, whose maximum limit can be .

$$^{n}C_{m}=\frac{n\,!}{(n-m)\,!\,m\,!}$$

The resulting number of alternative solutions is still too large to be computationally feasible and is reduced by the 2nd condition.

2. We know that in a linear programming problem, all the variables must be non-negative. Since the basic solutions selected by condition 1 above are not necessarily non-negative, the number of alt ernatives can be further reduced by eliminating all *infeasible basic solutions* (solutions having variables less than zero). In the simplex method this is achieved by starting with a basic solution which is non-negative ($\geqslant 0$). A condition, called *feasibility condition is* then provided which ensures that the next basic solution to be selected from all the possible basic solutions is always feasible ($\geqslant 0$). This solution is called *basic feasible solution*. If all the basic variables are greater than zero (>0), the solution is called *non-degenerate*, if some of them are zero, the solu tion is called *degenerate*. It will be shown in section 2.10 that a new basic feasible solution can be obtained from a previous one by setting *one* of the m basic variables equal to zero and replacing it by a new non-basic variable. The basic variable set equal to zero is called a "*leaving variable*", while the new one is called an "*entering variable*".

3. The entering variable can be so selected that it improves the value of objective function so that the new solution is better than the previous one. This is achieved by the use of another condition called *optimality condition* which selects that entering variable which produces the largest *per unit* gain in the objective function. This procedure is repeated successively until no further improvement in the value of the objective function is possible. The final solution is, then called an *optimal basic feasible solution* or simply *optimal solution*. This is the solution which satisfies the objective function equation, the constraints as well as the non-negativity conditions. This is, of course, true only if the objective function has a finite value.

The foregoing discussion shows that simplex method procedure

is principally a screening process since it eliminates the solutions that are not promising solutions for the optimal solution.

It will not be out of place to further elaborate a few important terms with reference to equations (2.28), (2.29) and (2.30) of the general linear programming problem.

Solution : x_j($j=1, 2, ..., n$) is a solution of the general linear programming problem if it satisfies the constraints (2.28).

Feasible solution : x_j ($j=1, 2, ..., n$) is a feasible solution of the linear programming problem if it satisfies conditions (2.28) and (2.29).

Basic solution : The solution of m basic variables when each of the $n-m$ non-basic variables is set equal to zero is called basic solution.

Basic feasible solution : A feasible solution is called a basic feasible solution if it has no more than m positive x_j. In other words, it is a basic solution which also satisfies the non-negativity condition (2.29).

Non-degenerate basic feasible solution : A basic feasible solution is said to be non-degenerate if it has exactly m positive (non-zero) x_j. The solution, on the other hand, is degenerate if one or more of the m basic variables vanish.

Optimal solution : A basic feasible solution is said to be optimal or optimum if it also optimizes the objective function [equation (2.30)] while satisfying conditions (2·28) and (2·29).

EXAMPLE 2.9-1

Find all the basic solutions to the following problem :

$$\text{maximize} \quad Z=x_1+2x_2+3x_3$$
$$\text{subject to} \quad x_1+2x_2+3x_3=4$$
$$2x_1+3x_2+5x_3=7.$$

Also find which of the basic solutions are

(*i*) basic feasible

(*ii*) non-degenerate basic feasible

(*iii*) optimal basic feasible.

Solution : Since $n=3$ and $m=2$ in this problem, a basic solution can be obtained by setting any of the $(n-m)$ variables equal to zero and then solving the constraint equations. The total number of basic solutions is

$$\frac{3\,!}{2\,!\,1\,!}=3.$$

Out of these three solutions, solutions in which all basic variables (x_j) are $\geqslant 0$ will be basic feasible; solutions in which all basic variables are >0 will be non-degenerate basic feasible and the basic feasible solution that optimizes the objective function will be the optimal basic feasible solution. Table 2·13 gives a summary of the characteristics of the various basic solutions.

It may be seen that the first two solutions are basic feasible; they are also non-degenerate basic feasible solutions. The first solution, of course, is the optimal one.

Thus the optimal solution is $x_1=2$, $x_2=1$, $x_3=0$ with $Z_{max}=5$.

EXAMPLE 2.9-2

A firm manufactures four different machine parts M_1, M_2, M_3 and M_4 made of copper and zinc. The amounts of copper and zinc required for each machine part, their *exact* availability and the profit earned from one unit of each machine part are as follows :

	M_1 (*kg*)	M_2 (*kg*)	M_3 (*kg*)	M_4 (*kg*)	*Availability* (*kg*)
Copper	5	4	2	1	100
Zinc	2	3	8	1	75
Profit (*Rs.*)	12	8	14	10	

How many of each part be manufactured to maximize profit ? For this problem find

(*i*) basic solutions

(*ii*) basic feasible solutions

(*iii*) non-degenerate basic feasible solutions

(*iv*) optimal basic feasible solution.

Solution : Let x_1, x_2, x_3 and x_4 represent the quantities to be manufactured of machine parts M_1, M_2, M_3 and M_4 respectively. Then the linear programming problem is

$$\text{maximize } Z=12x_1+8x_2+14x_3+10x_4$$

$$\text{subject to} \quad 5x_1+4x_2+2x_3+x_4=100$$

$$2x_1+3x_2+8x_3+x_4=75$$

where x_1, x_2, x_3, x_4, all $\geqslant 0$.

Here $n=4$ and $m=2$. A basic solution can be obtained by setting any of the $(n-m=2)$ non-basic variables equal to zero and then solving the constraint (containing the basic variables) equations. The total number of basic solutions is

$$\frac{4\,!}{2\,!\;2\,!}=6.$$

Table 2.14 gives a summary of the characteristics of the various solutions.

Table 2·13

S. No. of the basic solution	*Basic variables*	*Non-basic variables*	*Values of the basic variables given by the constraint equations*	*Value of the objective function*	*Is the solution feasible ? (are all $x_j \geqslant 0$?)*	*Is the solution non-degenerate ? (are all basic variables >0 ?)*	*Is the solution feasible and optimal ?*
1	x_1, x_2	x_3	$x_1+2x_2=4$ $2x_1+3x_2=7$ $\therefore\ x_1=2,\ x_2=1$	5	Yes	Yes	Yes
2	x_1, x_3	x_2	$x_1+3x_3=4$ $2x_1+5x_3=7$ $\therefore\ x_1=1,\ x_3=1$	4	Yes	Yes	No
3	x_2, x_3	x_1	$2x_2+3x_3=4$ $3x_2+5x_3=7$ $\therefore\ x_2=-1,\ x_3=2$	3	No	No	No

Table 2.14

S. No. of basic solution	Basic variables	Non-basic variables	Values of the basic variables given by the constraint equations	Value of the objective function	Is the solution feasible ? (are all $x_j \geqslant 0$?)	Is the solution non-degenerate ? (are all basic variables >0 ?)	Is the solution feasible and optimal ?
1	x_1, x_2	x_3, x_4	$5x_1+4x_2=100$ $2x_1+3x_2=75$ $\therefore x_1=0, x_2=25$	200	Yes	No	Yes
2	x_1, x_3	x_2, x_4	$5x_1+2x_3=100$ $2x_1+8x_3=75$ $\therefore x_1=325/18,$ $x_3=175/36$	$\frac{5,125}{18}$	Yes	Yes	No
3	x_1, x_4	x_2, x_3	$5x_1+x_4=100$ $2x_1+x_4=75$ $\therefore x_1=25/3,$ $x_4=175/3$	$\frac{2,050}{3}$	Yes	Yes	Yes

S. No. of basic solution	Basic variable	Non-basic variables	Values of the basic variables given by the constraint equations	Value of the objective function	Is the solution feasible ? (are all $x_j \geqslant 0$?)	Is the solution non-degenerate ? (are all basic variables >0 ?)	Is the solution feasible and optimal ?
4	x_2, x_3	x_1, x_4	$4x_2+2x_3=100$ $3x_2+8x_3=75$ $\therefore x_2=25, x_3=0$	200	Yes	No	No
5	x_2, x_4	x_1, x_3	$4x_2+x_4=100$ $3x_2+x_4=75$ $\therefore x_2=25, x_4=0$	200	Yes	No	No
6	x_3, x_4	x_1, x_2	$2x_3+x_4=100$ $8x_3+x_4=75$ $\therefore x_3=-25/6,$ $x_4=325/3$	1,025	No	No	No

From the table the following inferences can be drawn :

1. Basic solutions are no. 1, 2, 3, 4, 5 and 6.
2. Basic feasible solutions are no. 1, 2, 3, 4 and 5.
3. Non-degenerate basic feasible solutions are no. 2 and 3.
4. Optimal basic feasible solution is no. 3, which gives

$$x_1=25/3,\ x_2=0,\ x_3=0,\ x_4=\frac{175}{3} \text{ and } Z_{max}=\frac{2{,}050}{3}.$$

2·10 Examples on the Applications of simple Technique

The simplex method or technique is an iterative procedure for solving the linear programming problems. As discussed in section 2·9 it consists in

(*i*) having a basic feasible solution

(*ii*) testing whether it is an optimal solution or not

(*iii*) improving the first trial solution by a set of rules, and repeating the sequences till an optimal solution is obtained.

The computation procedure requires at the most m non-zero variables (equal to number of constraints) in the solutions at any time, if at any stage the number of non-zero variables becomes less than m, the problem is said to degenerate. This technique will be made clear by considering a few examples.

EXAMPLE 2·10·1

Three grades of coal A, B and C contain phosphorus and ash as impurities. In a particular industrial process, fuel up to 100 ton (maximum) is required which should contain ash not more than 3% and phosphorus not more than 0·03%. It is desired to maximize the profit while satisfying these conditions. There is an unlimited supply of each grade. The percentage of impurities and the profits of grades are given below.

Coal	*Phosphorus* (%)	*Ash* (%)	*Profits in rupees per ton*
A	0·02	3·0	12·00
B	0·04	2·0	15·00
C	0·03	5·0	14·00

Find the proportions in which the three grades be used.

Formulation of the Linear Programming Model

Step 1 :

Key decision to be made is as to how much of each grade of

coal be used. Let x_1, x_2 and x_3 respectively be the amounts in tons of grades A, B, and C used.

Step 2 :

Feasible alternatives are sets of values of x_1, x_2, x_3, where $x \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0.$ ...(2·31)

Step 3 :

Objective is to maximize the profit

i.e., maximize $Z = 12x_1 + 15x_2 + 14x_3$. ...(2·32)

Step 4 :

Constraints are

(*i*) phosphorus content must not exceed 0·03%.

Therefore $0{\cdot}02x_1 + 0{\cdot}04x_2 + 0{\cdot}03x_3 \leqslant 0{\cdot}03\ (x_1 + x_2 + x_3)$

or $2x_1 + 4x_2 + 3x_3 \leqslant 3(x_1 + x_2 + x_3)$

or $-x_1 + x_2 \leqslant 0$ ...(2·32*a*)

(*ii*) ash content must not exceed 3%.

Therefore $3x_1 + 2x_2 + 5x_3 \leqslant 3(x_1 + x_2 + x_3)$

or $-x_2 + 2x_3 \leqslant 0$ ...(2·32*b*)

(*iii*) total quantity of fuel required is not more than 100 ton

Therefore $x_1 + x_2 + x_3 \leqslant 100.$ ...(2·32*c*)

Thus the problem is to maximize eqn. (2·32) subject to constraints (2·32 *a*), (2·32 *b*), (2·32 *c*) and non-negativity restriction (2·31).

Solution of the L. P. Model

Step 1. Make the Problem as N+S Co-ordinate Problem

First of all, we observe whether all b (R.H.S. constants) are positive or not. If not, they can be made positive by multiplying both sides of the constraints by -1. By doing so, direction of inequality will also change. In this example, however, since all b_i are positive, this step is not necessary.

In order to solve the problem by simplex method, the inequalities (2·32 *a*), (2·32 *b*) and (2·32 *c*) are converted into equalities by introducing new non-negative variables (called slacks or slack variables), s_1, s_2 and s_3 in them. The slacks contribute zero to the objective function. Therefore, the following equations result :

$$-x_2 + x_2 + s_1 = 0 \qquad ...(2{\cdot}33\ a)$$

$$-x_2 + 2x_3 + s_2 = 0 \qquad ...(2{\cdot}33\ b)$$

and $$x_1 + x_2 + x_3 + s_3 = 100 \qquad ...(2{\cdot}33\ c)$$

and the objective function becomes

maximize $Z = 12x_1 + 15x_2 + 14x_3 + 0s_1 + 0s_2 + 0s_3$. ...(2·34)

where x_1, x_2, x_3, s_1, s_2 and s_3, all $\geqslant 0$. ...(2·34a)

Step 2 : Make N Co-ordinates Assume Zero Value

We shall start with a feasible solution, which we shall get by assuming that the profit earned is zero. This will be so when non-basic or decision variables x_1, x_2 and x_3 are each equal to zero, *i.e.*, if $x_1=x_2=x_3=0$, we get [from equations (2·33 *a*), (2·33 *b*), (2·33 *c*)], $s_1=0$, $s_2=0$ and $s_3=100$, as the first feasible solution.

The above information can be expressed in the form of a simple matrix or table 2·15.

The non-basic variables x_1, x_2 and x_3 (in terms of which Z is expressed), are all zero. If any of them is made positive, Z will increase. It shows that Z at this stage is not maximum. It can be increased by changing the basis, *i.e.*, by including x_1, x_2 or x_3 in place of some basic variable (s_1, s_2 or s_3) which forms the present basis.

Table 2·15

		Body Matrix			Identity Matrix				
Objective function c_j		12	15	14	0	0	0		
e_i variable in current solution		x_1	x_2	x_3	s_1	s_2	s_3	b	
0	s_1	−1	1	0	1	0	0	0	Coefficients
0	s_2	−1	0	2	0	1	0	0	
0	s_3	1	1	1	0	0	1	100	

In table 2·15 c_j row gives the coefficients of profit function [equation (2·34)]. This row may be omitted in the succeeding tables. The second, third and fourth rows give the coefficients of equations (2·33 *a*), (2·33 *b*), (2·33 *c*) respectively. The variables in the first basic feasible solution are written in a column under 'variables in current solution' and the values of these variables are written in '*b*'-column.

Although two basic variables have zero values ($s_1=0$, $s_2=0$), *i.e.*, the solution is degenerate, lhe problem can still be solved by imagining $s_1=\epsilon$ and $s_2=\epsilon$ where ϵ (epsilon) is a very small positive number.

Step 3 : Perform Optimality Test

By performing optimality test, we can find whether the current feasible solution can be improved or not. This can be done by computing c_j-E_j, where $E_j=\Sigma e_i a_{ij}$. Here a_{ij} represents matrix elements in the *i*th row and *j*th column. If c_j-E_j is positive under any column, at least one better solution is possible. For example, for

$j=1$ *i.e.*, the first column, $c_j=12$, $E_j=\Sigma e_i a_{ij}=0\times(-1)+0\times(-1)+0\times(1)=0$.

$\therefore$ $c_j-E_j=12-0=12$. Similarly c_j-E_j for other columns can be computed and is shown below in table 2·16.

Table 2·16

	c_j	12	15	14	0	0	0	
e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-1	1	0	1	0	0	ϵ
0	s_2	-1	0	2	0	1	0	ϵ
0	s_3	1	1	1	0	0	1	100
E_j	$=\Sigma e_i a_{ij}$	0	0	0	0	0	0	Profit lost/ton
c_j-E_j		12	15	14	0	0	0	Net profit/ton

$\because$ c_j-E_j is positive, it follows that the current feasible solution can be improved.

Step 4 : Iterate Towards an Optimal Solution

Sub-step (*i*) : This is done by interchanging one of the non-basic variables with one of the slacks. Improvement can be made by

(*a*) finding which variable should be made zero in the current feasible solution (basis matrix) and

(*b*) finding which variable should enter the solution (basis matrix) by making it positive.

By observing c_j-E_j for different columns and marking the column for maximum possible positive c_j-E_j suggests the variable at the top of the column to be the one which should enter the solution or which should replace the slack. This variable is also termed as the *incoming variable* and the column in which it occurs is called *key column* and is marked as 'K'. If more than one variable appears with the same maximum value, any one of these variables may be selected arbitrarily as the incoming variable.

A positive value indicates that the profit earned by the firm can be increased. A negative value indicates the amount by which profit would decrease if one unit of the variable for that column were brought into the solution. The largest positive value in the c_j-E_j row is the basis for selecting the column 'K' since we want to maximize the profit. When no more positive values remain in the c_j-E_j row (values are zero or negative), the profit is maximum.

Now the elements under column 'b' are divided by the corresponding elements of the column 'K' and the row containing the minimum positive quotient is marked. The current variable (slack) is to be made zero. This variable is called *outgoing variable* and the corresponding row is called *key row* (or *pivot row*). If there is more than one row having the same minimum positive quotient, the row with the maximum element in the key column is selected as the key row. The element lying at the intersection of the key column and key row is called *key element* (or *pivot element*). This element must be made unity by multiplying/dividing the key row by a common multiplying factor.

If all the quotients are negative or zero, the value of incoming variable, whatever it may be, can be made as large as we like without violating the feasibility condition. In such a case the problem has an unbounded solution and the iteration stops.

The above procedure has been represented in Table 2·17. The row of objective function c_j has been omitted.

Table 2·17

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	-1	(1)	0	1	0	0	ϵ	ϵ ←(key row)
0	s_2	-1	0	2	0	1	0	ϵ	∞
0	s_3	1	1	1	0	0	1	100	100
$E_j=\Sigma e_i a_{ij}$		0	0	0	0	0	0		
c_j-E_j		12	15	14	0	0	0		
			↑ K						

In Table 2·17, maximum value of c_j-E_j is 15 under 'x_2'—column. This column is the key column and has been marked 'K'; x_2 is the incoming variable. To decide the outgoing variable, we divide the elements under 'b'-column by the corresponding elements of 'x'_2'-column and the quotients are represented under 'θ'-column. Under this column, ϵ is the least positive quotient and the row containing it is marked as key row. s_1 is, therefore, the outgoing variable and (1) is the key element.

Sub-step (*ii*) : Now all the elements in 'K' column are made zero except the key element which will be unity. This is done by subtractin or adding the proper multiples of key row to, the other rows. This new tabie will have s_1 replaced by x_2. This completes the first stage. The resulting table is shown below.

Table 2·18

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b
15	x_2	-1	1	0	1	0	0	ϵ
0	s_2	-1	0	2	0	1	0	ϵ
0	s_3	2	0	1	-1	0	1	$100-\epsilon$
$E_j=\Sigma e_i\,a_{ij}$		-15	15	0	15	0	0	
c_j-E_j		27	0	14	-15	0	0	

2nd feasible solution

The second feasible solution gives $x_1=0$, $x_2=\epsilon$, $x_3=0$ and $z=0$.

Step 5 :

Repeat step 3 for table 2·18 which gives the 2nd feasible solution. On finding the value of c_j-E_j for various columns, we find that it is positive for some of them. Hence at least one better solution exists. Values of c_j-E_j have been entered in Table 2·19.

Step 6 :

Repeat step 4.

Table 2·19

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b	θ	
15	x_2	-1	1	0	1	0	0	ϵ	$-\epsilon$	
0	s_2	-1	0	2	0	1	0	ϵ	$-\epsilon$	
0	s_3	(2)	0	1	-1	0	1	$100-\epsilon$	$50-\frac{\epsilon}{2}$	←key row
$E_j=\Sigma e_i\,a_{ij}$		-15	15	0	15	0	0			
c_j-E_j		27	0	14	-15	0	0			
		↑ K								

(*key element* 2)

Table 2·20 can be written by making key element unity.

Table 2·20

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b	
15	x_2	-1	1	0	1	0	0	ϵ	(*Key element*
0	s_2	-1	0	2	0	1	0	ϵ	'1', *entering variable* x_1;
0	s_3	(1)	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$50-\frac{\epsilon}{2}$	*leaving, variable* s_3)

Replace s_3 by x_1

Table 2·21

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
15	x_2	0	1	1/2	1/2	0	1/2	$50+\frac{\epsilon}{2}$	$100+\epsilon$
0	s_2	0	0	(5/2)	$-1/2$	1	1/2	$50-\frac{\epsilon}{2}$	$20+\frac{\epsilon}{5}$ ← (key row)
12	x_1	1	0	1/2	$-1/2$	0	1/2	$50+\frac{\epsilon}{2}$	$100-\epsilon$
$E_j=\Sigma e_i a_{ij}$		12	15	27/2	3/2	0	27/2		
c_j-E_j		0	0	1/2	$-3/2$	0	$-27/2$		
				↑ K				*(key element* 5/2)	

3rd feasible solution

The third feasible solution gives $x_1=50$, $x_2=50$, $x_3=0$ and Z=Rs. 1,350.

Step 7 :

Repeat step 3. Key element is 5/2. Make it unity.

Table 2·22

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b	
15	x_2	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$50+\frac{\epsilon}{2}$	
0	s_2	0	0	(1)	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$20+\frac{\epsilon}{5}$	*(Entering variable* x_3,
12	x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$50-\frac{\epsilon}{2}$	*leaving variable* s_2)

Replace s_2 by x_3.

Table 2·23

e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	s_3	b
15	x_2	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$40+\frac{2\epsilon}{5}$
14	x_3	0	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$20+\frac{\epsilon}{5}$
12	x_1	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$40-\frac{3\epsilon}{5}$
$E_j=\Sigma e_i a_{ij}$		12	15	14	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{68}{5}$	*Optimal solution*
c_j-E_j		0	0	0	$-\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{68}{5}$	

As $c_j - E_j$ is not positive under any column in Table 2·23, the solution given by it is an optimal solution.

From this table

$$x_1 = 40 \text{ tons}$$
$$x_2 = 40 \text{ tons}$$
$$x_3 = 20 \text{ tons}$$
$$\text{and } Z_{max} = \text{Rs. } 1{,}360.$$

Thus quantities of grades A, B and C of coal to be used are 40, 40 and 20 tons respectively.

To conclude this problem, the following is the summary of the feasibility and optimality conditions :

1. *Feasibility condition* : The leaving (outgoing) variable is the basic variable corresponding to the minimum positive quotient obtained by dividing the elements under column 'b' by the corresponding elements of the column 'k'.

2. *Optimality condition* : The entering (incoming) variable is the non-basic variable corresponding to the maximum positive (maximum negative) value of $c_j - E_j$ in a maximization (minimization) problem.

EXAMPLE 2·10·2

Show that there is an unbounded solution to the following L.P. problem :

maximize $Z = 4x_1 + x_2 + 3x_3 + 5x_4$

subject to the constraints
$$4x_1 - 6x_2 - 5x_3 - 4x_4 \geqslant -20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 \leqslant 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leqslant 20$$
$$x_1, x_2, x_3, x_4 \geqslant 0.$$

Solution

Step 1 : Make the problem as N+S co-ordinate problem

Since $b_1 = -20$, first it will be made positive by multiplying the first constraint throughout by -1, giving

$$-4x_1 + 6x_2 + 5x_3 + 4x_4 \leqslant 20$$

An error is usually made in neglecting the slack variables while checking the optimality condition. It is assumed that these variables usually do not appear in the objective function, and hence they do not affect its value. This is, however, incorrect. In the constraints, any change in the values of the slacks directly affects the values of the decision variables which, in turn, affect the value of the objective function. Therefore, the slacks should also be treated in exactly the same way as the decision variables.

In order to solve the problem by simplex method, the above inequations are converted into equations by introducing slack or basic variables in them. The slack variables contribute zero to the objective function. Therefore, the following equations result :

$$-4x_1+6x_2+5x_3+4x_4+s_1=20 \quad \text{...(2.35)}$$

$$-3x_1-2x_2+4x_3+x_4+s_2=10 \quad \text{...(2.36)}$$

$$\text{and } -8x_1-3x_2+3x_3+2x_4+s_3=20 \quad \text{...(2.37)}$$

and the objective function becomes

$$\text{maximize } Z=4x_1+x_2+3x_3+5x_4+0s_1+0s_2+0s_3 \quad \text{...(2.38)}$$

$$\text{where} \quad x_1, x_2, x_3, x_4, s_1, s_2, s_3, \text{ all} \geqslant 0. \quad \text{...(2·38}a)$$

Step 2 : Make N co-ordinates assume zero value

We shall start with a feasible solution, which we shall get by assuming that the objective function Z is zero. This will be so when non-basic variables x_1, x_2, x_3, x_4 are each equal to zero, giving $x_1=0$ $x_2=0$, $x_3=0$, $x_4=0$ and $s_1=20$, $s_2=10$, $s_3=20$ [from equations (2.35), (2.36) and (2.37)] as the first basic feasible solution.

The above information can be expressed in the form of a table, called simplex table (table 2·24). The non-basic variables x_1, x_2, x_3, x_4 are all zero. If any of them is made positive, Z will increase. It shows that Z at this stage is not maximum. It can be increased by changing the basis *i.e.*, by including x_1, x_2, x_3, x_4 in place of some basic variable (s_1, s_2 or s_3) which forms the present basis.

Table 2.24

		Body matrix				Identity matrix				
Objective function c_j		4	1	3	5	0	0	0		
c_i	variables in current solution	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b	
0	s_1	−4	6	5	4	1	0	0	20	*Coefficients*
0	s_2	−3	−2	4	1	0	1	0	10	
0	s_3	−8	−3	3	2	0	0	1	20	

First basic feasible solution

Step 3 : Perform optimality test

By performing the optimality test we can find whether the current feasible solution can be improved or not. This can be done

by computing $c_j - E_j$, where $E_j = \Sigma\ e_i\ a_{ij}$. Here, a_{ij} represents matrix element in the ith row and jth column. If $c_j - E_j$ is positive under any column, at least one better solution is possible. This is shown in table 2.25.

Table 2.25

Objective function c_j		4	1	3	5	0	0	0			
e_i	variables in current solution	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b	θ	
0	s_1	-4	6	5	(4)	1	0	0	20	5	←key row
0	s_2	-3	-2	4	1	0	1	0	10	10	
0	s_3	-8	-3	3	2	0	0	1	20	10	
	$E_j = \Sigma e_i\ a_{ij}$	0	0	0	0	0	0	0			loss in evaluation/unit
	$c_j - E_j$	4	1	3	5	0	0	0			net evaluation/unit
					↑ K						

Since $c_j - E_j$ is positive under 'x_1', 'x_2', 'x_3' and 'x_4'-columns, the initial basic feasible solution can be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (*i*). Mark the key column, key row and key element as shown in table 2.25 and explained in example 2.10-1. x_4 is the incoming non-basic variable, while s_1 is outgoing basic variable. Make the key element unity as shown in table 2.26.

Table 2.26

e_i	current solution variables	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b
0	s_1	-1	$\frac{3}{2}$	$\frac{5}{4}$	(1)	$\frac{1}{4}$	0	0	5
0	s_2	-3	-2	4	1	0	1	0	10
0	s_3	-8	-3	3	2	0	0	1	20

Sub-step (ii) : Replace s_1 by x_4. This is shown in tabe 2·27.

Table 2.27

Objective function c_j		4	1	3	5	0	0	0		
e_i	current solution variables	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b	θ
5	x_4	-1	$\frac{3}{2}$	$\frac{5}{4}$	1	$\frac{1}{4}$	0	0	5	-5
0	s_2	-2	$-\frac{7}{2}$	$\frac{11}{4}$	0	$-\frac{1}{4}$	1	0	5	$-\frac{5}{2}$
0	s_3	-7	-6	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	10	$-\frac{10}{7}$
$E_j=\Sigma e_i\, a_{ij}$		-5	$\frac{15}{2}$	$\frac{25}{4}$	5	$\frac{5}{4}$	0	0		
c_j-E_j		9	$\frac{-13}{2}$	$\frac{-13}{4}$	0	$\frac{-5}{4}$	0	0		
		$\uparrow K$								

2nd basic feasible solution

Step 5 :

Repeat step 3. x_1-column is the key column. To find key row we divide the elements in column 'b' by the corresponding elements of column 'k' and get column 'θ'. Since all the quotients in column 'θ' are negative, the value of incoming non-basic variable x_1 can be made as large as we like without violating the feasibility condition. The problem, therefore, has an unbounded solution and further iteration stops.

EXAMPLE 2.10-3

Show by simplex method, that the following L.P.P. has infinite number of non-basic feasible optimal solutions :

maximize $Z=4x_1+10x_2$

subject to $2x_1+x_2\leqslant 10$

$2x_1+5x_2\leqslant 20$

$2x_1+3x_2\leqslant 18$

$x_1, x_2 \geqslant 0.$

Solution

Step 1 : **Make the problem as N+S co-ordinate problem**

In order to solve the problem by simplex method, the above inequations are converted into equations by introducing slack or basic

variables in them. The slack variables contribute zero to the objective function.

Therefore, the following equations result :

$$2x_1+x_2+s_1=10 \quad ...(2.39)$$

$$2x_1+5x_2+s_2=20 \quad ...(2.40)$$

$$2x_1+3x_2+s_3=18 \quad ...(2\cdot41)$$

and the objective function becomes

$$\text{maximize } Z=4x_1+10x_2+0s_1+0s_2+0s_3 \quad ...(2\cdot42)$$

$$\text{where } x_1, x_2, s_1, s_2, s_3, \text{ all} \geqslant 0. \quad ...(2\cdot42a)$$

Step 2 : Make N co-ordinates assume zero value

We shall start by a basic feasible solution which we shall get by assuming that the objective function Z is zero. This will be so when non-basic variables x_1 and x_2 are each equal to zero, giving $x_1=0$, $x_2=0$ and $s_1=10$, $s_2=20$, $s_3=18$ [from equations (2·39), (2·40) and (2·41)] as the first basic feasible solution. The above information is put in the form of a table (table 2·28).

Table 2·28

		Body matrix		Identity matrix				
Objective function c_j		4	10	0	0	0		
e_i	variables in current solution	x_1	x_2	s_1	s_2	s_3	b	
0	s_1	2	1	1	0	0	10	Coefficients
0	s_2	2	5	0	1	0	20	
0	s_3	2	3	0	0	1	18	

First basic feasible solution

Step 3 : Perform Optimality test

By performing optimality test we can find whether the current basic feasible solution can be improved or not. This can be done by computing $c_j - E_j$, where $E_j = \Sigma e_i a_{ij}$. If $c_j - E_j$ is positive under any column, at least one better solution is possible. This is shown in table 2.29.

Table 2.29

Objective function c_j		4	10	0	0	0			
e_i	current solution variables	x_1	x_2	s_1	s_2	s_3	b	θ	
0	s_1	2	1	1	0	0	10	10	
0	s_2	2	(5)	0	1	0	20	4	←key row
0	s_3	2	3	0	0	1	18	6	
$E_j=\Sigma e_i\, a_{ij}$		0	0	0	0	0			
c_j-E_j		4	10 ↑K	0	0	0			

Since c_j-E_j is positive under x_1 and x_2-columns, the initial basic feasible solution can be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (i) : Mark the key column, key row and key element as shown in table 2·29 and explained in example 2.10-1. x_2 is incoming non-basic variable and s_2 is outgoing basic variable. The key element is 5. Make it unity. This is shown in table 2.30.

Table 2.30

e_i	c.s.v.	x_1	x_2	s_1	s_2	s_3	b
0	s_1	2	1	1	0	0	10
0	s_2	$\frac{2}{5}$	(1)	0	$\frac{1}{5}$	0	4
0	s_3	2	3	0	0	1	18

Sub-step (ii) : Replace s_2 by x_2. This is shown in table 2·31.

Table 2.31

Objective function c		4	10	0	0	0	
e_i	c.s.v.	x_1	x_2	s_1	s_2	s_3	b
0	s_1	$\frac{8}{5}$	0	1	$-1/5$	0	6
10	x_2	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	4
0	s_3	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1	6
$E_j=\Sigma e_i\, a_{ij}$		4	10	0	2	0	
c_j-E_j		0	0	0	-2	0	

Basic feasible optimal solution

Since $c_j - E_j$ is either zero or negative under columns of all variables, table 2.31 gives the basic feasible optimal solution, which is

$$x_1 = 0$$

$$x_2 = 4$$

$$\text{and } Z_{max} = 40.$$

It is clear from table 2.31 that element of $c_j - E_j$ row (net evaluation/unit) under column 'x_1' (corresponding to non-basic variables x_1) is zero. It indicates the existence of an alternative optimal basic feasible solution. Choosing x_1 as the incoming variable and slack s_1 as the outgoing variable, we get simplex tables 2.32 and 2.33.

Table 2.32

Objective function c_j		4	10	0	0		
e_i	current solution variables	x_1	x_2	s_1	s_2	s_3	b
0	s_1	(1)	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$
10	x_2	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	4
0	s_3	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1	6

Table 2.33

Objective function c_j		4	10	0	0	0	
e_i	c.s.v.	x_1	x_2	s_1	s_2	s_3	b
4	x_1	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$\frac{15}{4}$
10	x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{5}{2}$
0	s_3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3
$E_j = \Sigma e_i a_{ij}$		4	10	0	2	0	
$c_j - E_j$		0	0	0	-2	0	

The solution given by table 2.33 is also optimal with values

$$x_1 = \frac{15}{4}$$

$$x_2 = \frac{5}{2}$$

and $Z_{max}=4\times\frac{15}{4}+10\times\frac{5}{2}=40$, which is the same as the previous one. The above two are the basic feasible optimal solutions. Now if two basic feasible optimal solutions are known, an infinite number of non-basic feasible optimal solutions can be derived by taking any weighted average of these two solutions.

For example, if

$$x_1=\begin{bmatrix}0\\0\\4\\6\\6\end{bmatrix} \text{ and } x_2=\begin{bmatrix}0\\ \frac{15}{4}\\ \frac{5}{8}\\0\\3\end{bmatrix}$$

then $$x^*=\lambda\, x_1+(1-\lambda)\, x_2$$

$$=\begin{bmatrix}0\\ \frac{15}{4}(1-\lambda)\\ 4\lambda+\frac{5}{8}(1-\lambda)\\ 6\lambda\\ 6\lambda+3(1-\lambda)\end{bmatrix} \text{ where } 0\leqslant\lambda\leqslant1$$

$$=\begin{bmatrix}0\\ \frac{15}{4}-\frac{15}{4}\lambda\\ \frac{5}{8}+\frac{27}{8}\lambda\\ 6\lambda\\ 3+3\lambda\end{bmatrix} \text{ where } 0\leqslant\lambda\leqslant1.$$

It can be verified that x^* gives the same optimum value of 40 for Z, for all values $0\leqslant\lambda\leqslant1$.

2.11 Artificial Variable Technique for finding the First Basic Feasible Solution

In the previous problems, the slack variables readily provided the initial basic feasible solution. There are, however, many L.P. problems where slack variables cannot provide such a solution. In these problems at least one of the constraints is of (=) or ($\geqslant$0) type. There are two (closely related) methods available to solve such problems :

1. The "big M-method" or "M-technique" or the "method of penalties" due to A. Charnes.
2. The "two phase" method due to Danzig, Orden and Wolfe.

2.11-1 The Big M-method

This method consists of the following basic steps :

Step 1

Express the linear programming problem in the standard form as discussed in section 2.7-2.

Step 2

Add non-negative variables to the left hand side of all the constraints of (= or ⩾) type. These variables are called *artificial variables.* The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. To achieve this, these artificial variables are assigned a large penalty (—M for maximization problems and +M for minimization problems) in the objective function.

Step 3 :

Continue with the regular steps of simplex method.

The artificial variables are a computational device. They keep the starting equations in balance and provide a mathematical trick for getting a starting solution. By having a high penalty cost it is ensured that they will not appear in the final solution *i.e.,* they will be driven to zero when the objective function is optimized by using simplex method.

While making iterations, using simplex method, one of the following three cases may arise :

1. Column 'variables in current solution' contains no artificial variables. In this case, continue iterations till an optimum basic feasible solution is obtained or it is found that the problem has an unbounded solution.

2. Column 'variables in current solution' contains at least one artificial variable at zero level (zero value under column '*b*'). Also coefficient of each M in $c_j - E_j$ row (net evaluation row) is —ve for maximization (or +ve for minimization) problem. In this case the current basic feasible solution is optimum though degenerate.

3. Column 'variables in current solution' contains at least one artificial variable *not* at zero level (non-zero value in column '*b*'). Also coefficient of each M in $c_j - E_j$ row is —ve for maximization (or +ve for minimization) problem. In such a case the current basic feasible solution is not optimal since the objective function will contain unknown quantity M. Such a solution is called ***pseudo-optimum*** basic feasible solution.

Lastly, whenever at any stage of simplex method, an artificial variable leaves the basis, it is dropped and all entries in the column of this variable are omitted from the succeeding tables. The above three cases will now be explained with the help of examples.

EXAMPLE 2.11-1

Maximize $Z=3x_1-x_2$

subject to $2x_1+x_2 \geqslant 2$

$x_1+3x_2 \leqslant 3$

$x_2 \leqslant 4$

and $x_j \geqslant 0 \, ; j=1, 2.$

Solution

Step 1 : Make the problem as N+S co-ordinate problem

First of all we observe that all b_i are positive.

At first thought it appears that we should introduce slack variables s_1, s_2 and s_3 (for converting inequalities into equalities) as shown below.

$$2x_1+x_2-s_1=2 \qquad \text{...(2.43}a\text{)}$$

$$x_1+3x_2+s_2=3 \qquad \text{...(2.43}b\text{)}$$

and $$x_2+s_3=4 \qquad \text{...(2.43}c\text{)}$$

where $$x_1, x_2, s_1, s_2, s_3, \text{ all} \geqslant 0. \qquad \text{...(2.43}d\text{)}$$

Step 2 : Make N co-ordinates assume zero value

Putting $x_1=0$ and $x_2=0$, we get $s_1=-2$, $s_2=3$ and $s_3=4$, as the first basic feasible solution. But negative values for slack variables are unacceptable. Therefore, we introduce *artificial variables* A_i (A_1, A_2, A_3,...) and the above equations can be written as

$$2x_1+x_2-s_1+A_1=2 \qquad \text{...(2.44}a\text{)}$$

$$x_1+3x_2+s_2=3 \qquad \text{...(2.44}b\text{)}$$

and $$x_2+s_3=4 \qquad \text{...(2.44}c\text{)}$$

where $$x_1, x_2, s_1, s_2, s_3, A_1, \text{ all} \geqslant 0, \qquad \text{...(2.44}d\text{)}$$

which gives the first basic feasible solution.

Now artificial variables with values greater than zero destroy the equality required by the general linear programming model. Therefore, A_i (A_1, A_2, A_3,...) must not appear in the final solution. To achieve this, these artificial variables are assigned a large penalty (a large negative value, $-M$) in the objective function, which can be written as

$$\text{maximize } Z=3x_1-x_2-MA_1+os_1+os_2+os_3. \qquad \text{...(2.45)}$$

Thus the objective is to maximize equation (2.45) subject to constraints (2.44*a*), (2.44*b*), (2.44*c*) and non-negativity restriction (2.44*d*). The above information can be represented in the form of a simple table or matrix.

Table 2.34

Objective function c_j		3	−1	0	0	0	−M	
e_i	variables in current solution	x_1	x_2	s_1	s_2	s_3	A_1	b
−M	A_1	2	1	−1	0	0	1	2
0	s_2	1	3	0	1	0	0	3
0	s_3	0	1	0	0	1	0	4

Step 3 : Perform optimality test

By performing optimality test we can find whether the current basic feasible solution can be improved or not. This is done by computing $c_j - E_j$, where $E_j = \Sigma e_i a_{ij}$. This is shown in table 2·35.

Table 2·35

e_i	current solution variables	x_1	x_2	s_1	s_2	s_3	A_1	b	θ
−M	A_1	(2)	1	−1	0	0	1	2	1←key row
0	s_2	1	3	0	1	0	0	3	3
0	s_3	0	1	0	0	1	0	4	∞
$E_j = \Sigma e_i a_{ij}$		−2M	−M	M	0	0	M		
$c_j - E_j$		3+2M	−1+M	−M	0	0	−2M		
		↑ K							

First basic feasible solution

As $c_j - E_j$ is positive under some columns, the current basic feasible solution can be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (*i*) : Mark the key column, key row and key element as shown in table 2·35 and explained in example 2·10-1. x_1 is the incoming variable while A_1 is the outgoing variable. Make the key element unity as shown in table 2·36.

Table 2·36

e_i	current solution variables	x_1	x_2	s_1	s_2	s_3	A_1	b
−M	A_1	(1)	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1
0	s_2	1	3	0	1	0	0	3
0	s_3	0	1	0	0	1	0	4

Sub-step (ii) : Replace A_1 by x_1. Omit column 'A_1'. This is shown in table 2·37.

Table 2·37

e_i	current solution variables	x_1	x_2	s_1	s_2	s_3	b	θ
3	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	-2
0	s_2	0	$\frac{5}{2}$	$\left(\frac{1}{2}\right)$	1	0	2	4←(key row)
0	s_3	0	1	0	0	1	4	
$E_j=\Sigma e_i a_{ij}$		3	$\frac{3}{2}$	$-\frac{3}{2}$	0	0		
c_j-E_j		0	$-\frac{5}{2}$	$\frac{3}{2}$	0	0		
				↑ K				*2nd basic feasible solution*

Step 5 : Repeat step 3. The key row, key column and key element are shown in table 2·37. Make the key element unity as shown in table 2·38.

Table 2·38

e_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	b	
3	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	
0	s_2	0	5	(1)	2	0	4	
0	s_3	0	1	0	0	1	4	*Key element unity*

Step 6 : Repeat step 4. Replacing s_2 by s_1 we get table 2·39.

Table 2·39

e_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	b
3	x_1	1	3	0	1	0	3
0	s_1	0	5	1	2	0	4
0	s_3	0	1	0	0	1	4
$E_j=\Sigma e_i a_{ij}$		3	9	0	3	0	
c_j-E_j		0	-10	0	-3	0	

Optimal basic feasible solution

$\because$ c_j-E_j is either negative or zero under all columns, the optimal solution has been obtained. Optimum values are $x_1=3$ and $x_2=0$.

Also $s_1=4$, $s_2=0$ and $s_3=4$, $A_1=0$.

$$\therefore \quad Z_{max}=3x_1-2x_2+0s_1+0s_2+0s_3-MA_1$$
$$=3\times3-2\times0+0+0+0-0$$
$$=9.$$

EXAMPLE 2·11-2

Minimize $Z=2y_1+3y_2$

subject to restrictions $y_1+y_2\geqslant5$

$y_1+2y_2\geqslant6$

$y_1, y_2\geqslant0.$

Solution : Step 1 : Make the problem as N+S co-ordinate problem

First of all we observe that all b_i are non-negative.

In order to solve the problem by simplex method, the first step is to convert inequations into equations and at first sight it appears the slack variables should be introduced as follows :

$$y_1+y_2-s_1=5 \qquad \ldots(2\cdot46\ a)$$
$$y_1+2y_2-s_2=6 \qquad \ldots(2\cdot46\ b)$$
$$y_1, y_2, s_1, s_2, \text{ all}\geqslant0 \qquad \ldots(2\cdot46\ c)$$

and the objective is to maximize $Z=-2y_1-3y_2+0s_1+0s_2$. ...(2·47)

Step 2 : Make N co-ordinates assume zero value

Putting $y_1=0$, $y_2=0$, we get $s_1=-5$, $s_2=-6$ as the first basic feasible solution. But negative values for the slack variables are unacceptable. Therefore, we introduce artificial variables A_i (A_1, A_2) and the above equations can be written as

$$y_1+y_2-s_1+A_1=5 \qquad \ldots(2\cdot48\ a)$$
$$y_1+2y_2-s_2+A_2=6 \qquad \ldots(2\cdot48\ b)$$

where $y_1, y_2, s_1, s_2, A_1, A_2, \text{ all}\geqslant0.$ ...(2·48 c)

Now artificial variables with values greater than zero destroy the equality required by the linear programming model. Therefore, A_i (A_1, A_2) should not appear in the final solution. To achieve this, we assign a large penalty to these artificial variables (a large negative value, —M) in the objective function, which is written as

maximize $Z=-2y_1-3y_2+0s_1+0s_2-MA_1-MA_2$. ...(2·49)

Thus the problem is to maximize equation (2·49) subject to constraints (2·48 *a*), (2·48 *b*) and non-negativity restriction (2·48 *c*).

The above information can be represented in the form of a matrix or table shown below.

Table 2·40

Objective function c_j		-2	-3	0	0	$-M$	$-M$	
e_i	variables in current solution	y_1	y_2	s_1	s_2	A_1	A_2	b
$-M$	A_1	1	1	-1	0	1	0	5
$-M$	A_2	1	2	0	-1	0	1	6

First basic feasible solution

Step 3 : Perform Optimality test

By performing optimality test we can find whether the current feasible solution can be improved or not. This is done by computing $c_j - E_j$, where $E_j = \Sigma e_i a_{ij}$. This is shown in table 2.41.

Table 2·41

Objective function c_j		-2	-3	0	0	$-M$	$-M$		
e_i	current solution variables	y_1	y_2	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	1	1	-1	0	1	0	5	5
$-M$	A_2	1	(2)	0	-1	0	1	6	3← key row
$E_j = \Sigma e_i a_{ij}$		$-2M$	$-3M$	M	M	$-M$	$-M$		
$c_j - E_j$		$-2+2M$	$-3+3M$ ↑ K	$-M$	$-M$	0	0		

Step 4 : Iterate towards an optimal solution

Sub-step (*i*) : Mark key column, key row and key-element as shown in table 2.41 and explained in example 2.10-1. The key element is (2). Incoming variable is y_2 and outgoing variable is A_2. Make the key element unity.

Table 2·42

e_i	C.S.V.	y_1	y_2	s_1	s_2	A_1	A_2	b
$-M$	A_1	1	1	-1	0	1	0	5
$-M$	A_2	1/2	(1)	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3

Sub-step (*ii*) : Replace A_2 by y_2. Omit column 'A_2'.

Table 2·43

e_i	C.S.V.	y_1	y_2	s_1	s_2	A_1	b	θ
$-M$	A_1	$(\frac{1}{2})$	0	-1	$\frac{1}{2}$	1	2	4←(key
-3	y_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	3	6 row)
$E_j = \Sigma e_i a_{ij}$		$-\frac{M+3}{2}$	-3	M	$\frac{3-M}{2}$	$-M$		
$c_j - E_j$		$\frac{-1+M}{2}$ ↑ K	0	$-M$	$\frac{-3+M}{2}$	0		

2nd basic feasible solution

Step 5 :

Repeat step 3. The key column, key row and key element are shown in table 2.43. Make the key element unity as shown in table 2·44.

Table 2·44

e_i	C.S.V.	y_1	y_2	s_1	s_2	A_1	b
$-M$	A_1	(1)	0	-2	1	2	4
-3	y_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	3

Step 6 :

Repeat step 4. Replace A_1 by y_1. Omit column 'A_1'.

Table 2·45

	c_1	-2	-3	0	0		
e_i	C.S.V	y_1	y_2	s_1	s_2	b	
-2	y_1	1	0	-2	1	4	
-3	y_2	0	1	1	-1	1	*Optimal basic*
$E_j=\Sigma e_i a_{ij}$		-2	-3	1	1		*feasible solution*
c_j-E_j		0	0	-1	-1		

As c_j-E_j is not positive under any column, the optimal solution has been achieved. Here, $y_1=4$ and $y_2=1$,
so that minimum value of Z is$=2\times4+3\times1=11$*

EXAMPLE 2.11-3

Maximize $Z=x_1+2x_2+3x_3-x_4$
subject to the constraints

$$x_1+2x_2+3x_3=15$$
$$2x_1+x_2+5x_3=20$$
$$x_1+2x_2+x_3+x_4=10$$
$$x_1, x_2, x_3, x_4\geqslant 0.$$

[*Meerut B.Sc. (Math.) 1971, M.A. (Math.) 1977*]

Solution :

Step 1 : Make the problem as N+S coordinate problem

First of all we observe that all b_i are non-negative. Introducing artificial variables $A_i (i=1, 2, 3)$, the above constraint equations can be written as

$$x_1+2x_2+3x_3+A_1=15 \quad \text{...(2.50a)}$$
$$2x_1+x_2+5x_3+A_2=20 \quad \text{...(2.50b)}$$
$$x_1+2x_2+x_3+x_4+A_3=10 \quad \text{...(2.50c)}$$

where $x_1, x_2, x_3, x_4, A_1, A_2, A_3$, all $\geqslant 0$. ...(2·50d)

Now artificial variables with their values greater than zero destroy the equality required by the general linear programming model. Therefore A_i $(i=1, 2, 3)$ must not appear in the final solution. To achieve this, these artificial variables are assigned a large penalty

*Although this problem has been solved by converting the minimization problem into the maximization form, the problem could be solved in the minimization form also. This is done by selecting the non-basic variable having the largest negative value (in the c_j-E_j row) as the incoming variable. When all the elements in the (c_j-E_j) row are positive, the optimal solution is reached.

(a large negative value, $-M$) in the objective function, which can be written as

$$\text{maximize } Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3. \quad \ldots(2.51)$$

Thus the problem is to find the values of x_1, x_2, x_3 and x_4 which maximize equation (2.51) subject to constraints (2.50*a*), (2.50*b*) and non-negativity restriction (2.50*c*).

Step 2 : Make N Co-ordinates assume zero value

Putting $x_1 = x_2 = x_3 = x_4 = 0$ in the equations for constraints, $A_1 = 15$, $A_2 = 20$ and $A_3 = 10$, which is the initial basic feasible solution. The above information can be put in the form of a simple table or matrix.

Table 2.46

Objective function c_j		1	2	3	-1	$-M$	$-M$	$-M$	
e_i	variables in current solution	x_1	x_2	x_3	x_4	A_1	A_2	A_3	b
$-M$	A_1	1	2	3	0	1	0	0	15
$-M$	A_2	2	1	5	0	0	1	0	20
$-M$	A_3	1	2	1	1	0	0	1	10

Initial basic feasible solution

Step 3 : Perform optimality test

By performing optimality test we can find whether the current feasible solution can be improved or not. This is done by computing $c_j - E_j$ where $E_j = \Sigma e_i a_{ij}$. This is shown in table 2.47.

Table 2.47

	c_j	1	2	3	-1	$-M$	$-M$	$-M$			
e_i	current solution variables	x_1	x_2	x_3	x_4	A_1	A_2	A_3	b	θ	
$-M$	A_1	1	2	3	0	1	0	0	15	5	
$-M$	A_2	2	1	(5)	0	0	1	0	20	4	←key row
$-M$	A_3	1	2	1	1	0	0	1	10	10	
$E_j = \Sigma e_i a_{ij}$		$-4M$	$-5M$	$-9M$	$-M$	$-M$	$-M$	$-M$			
$c_j - E_j$		$1+4M$	$2+5M$	$3+9M$	$-1+M$	0	0	0			
				↑K							

As $c_j - E_j$ is positive, the current basic feasible solution is not optimal and hence has got to be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (i) : Mark the key column, key row and key element as shown in table 2.47 and explained in example 2.10-1. x_3 is the incoming non-basic variable while A_2 is the outgoing artificial variable. Make the key element unity as shown in table 2·48.

Table 2.48

c_i	Current solution variables	x_1	x_2	x_3	x_4	A_1	A_2	A_3	b
$-M$	A_1	1	2	3	0	1	0	0	15
$-M$	A_2	$\frac{2}{5}$	$\frac{1}{5}$	(1)	0	0	$\frac{1}{5}$	0	4
$-M$	A_3	1	2	1	1	0	0	1	10

Sub-step (ii) : Replace A_2 by x_3. Omit column 'A_2', This is shown in table 2.49.

Table 2.49

	c_j	1	2	3	-1	$-M$	$-M$			
c_i	C.S.V.	x_1	x_2	x_3	x_4	A_1	A_3	b	θ	
$-M$	A_1	$\frac{-1}{5}$	$\left(\frac{7}{5}\right)$	0	0	1	0	3	$\frac{17}{7}$	←key row
3	x_3	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	0	4	20	
$-M$	A_3	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	1	6	$\frac{10}{3}$	
$E_j=\Sigma c_i a_{ij}$		$\frac{6-2M}{5}$	$\frac{3-16M}{5}$	3	$-M$	$-M$	$-M$			
c_j-E_j		$\frac{-1+2M}{5}$	$\frac{7+16M}{5}$ ↑K	0	$-1+M$	0	0			*2nd basic feasible solution*

Step 5 :

Repeat step 3. The key column, key row and key element are shown in table 2.49. Make the key element unity. This is shown in table 2.50.

Table 2.50

c_i	C.S.V.	x_1	x_2	x_3	x_4	A_1	A_3	b
$-M$	A_1	$\frac{-1}{7}$	(1)	0	0	$\frac{5}{7}$	0	$\frac{15}{7}$
3	x_3	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	0	4
$-M$	A_3	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	1	6

Key element unity

Step 6 :

Repeat step 4. x_2 is the incoming non-basic variable while A_1 is the outgoing artificial variable. Replace A_1 by x_2. Omit column 'A_1'. This is shown in table 2.51.

Table 2.51

	c_j	1	2	3	-1	$-M$			
e_i	C.S.V.	x_1	x_2	x_3	x_4	A_3	b	θ	
2	x_2	$\frac{-1}{7}$	1	0	0	0	$\frac{15}{7}$	∞	
3	x_3	$\frac{3}{7}$	0	1	0	0	$\frac{25}{7}$	∞	
$-M$	A_3	$\frac{6}{7}$	0	0	(1)	1	$\frac{15}{7}$	$\frac{15}{7}$	←key row
$E_j=\Sigma e_i a_{ij}$		$\frac{1-6M}{7}$	2	3	$-M$	$-M$			
c_j-E_j		$\frac{6+6M}{7}$	0	0	$-1+M$	0			
					↑K			*3rd basic feasible solution*	

Step 7 :

Repeat step 4. x_4 is the incoming non-basic variable while A_3 is the outgoing artificial variable. Replace A_3 by x_4. Omit column 'A_3'. This is shown in table 2.52.

Table 2.52

	c_j	1	2	3	-1			
e_i	C.S.V.	x_1	x_2	x_3	x_4	b	θ	
2	x_2	$\frac{-1}{7}$	1	0	0	$\frac{15}{7}$	-15	
3	x_3	$\frac{3}{7}$	0	1	0	$\frac{25}{7}$	$\frac{25}{3}$	
-1	x_4	$\left(\frac{6}{7}\right)$	0	0	1	$\frac{15}{7}$	$\frac{5}{2}$	←key row
$E_j=\Sigma e_i a_{ij}$		$\frac{1}{7}$	2	3	-1			
c_j-E_j		$\frac{6}{7}$	0	0	0			
		↑K					*4th basic feasible solution*	

Step 8 :

Repeat step 3. Key column, key row and key element are shown in table 2·52. Make the key element unity as shown in table 2·53.

Table 2·53

e_i	*C.S.V.*	x_1	x_2	x_3	x_4	b
2	x_2	$-\frac{1}{7}$	1	0	0	$\frac{15}{7}$
3	x_3	$\frac{3}{7}$	0	1	0	$\frac{25}{7}$
-1	x_4	(1)	0	0	$\frac{7}{6}$	$\frac{5}{2}$

Step 9 : Repeat step 4. x_1 is incoming variable while x_4 is outgoing variable. Replace x_4 by x_1. This is shown in table 2·54.

Table 2·54

e_i	C_j / C.S.V.	1 x_1	2 x_2	3 x_3	−1 x_4	b
2	x_2	0	1	0	$\frac{1}{6}$	$\frac{5}{2}$
3	x_3	0	0	1	$-\frac{1}{2}$	$\frac{5}{2}$
1	x_1	1	0	0	$\frac{7}{6}$	$\frac{5}{2}$
$E_j=\Sigma c_i a_{ij}$		1	2	3	0	
c_j-E_j		0	0	0	−1	

Optimal basic feasible solution

$\because$ c_j-E_j is either zero or negative under all columns, the optimal basic feasible solution has been obtained. Optimal values are

$$x_1=\frac{5}{2},\quad x_2=\frac{5}{2},\quad x_3=\frac{5}{2},\ x_4=0.$$

Also $\quad A_1=A_2=A_3=0$ and $Z_{max}=15$.

EXAMPLE 2.11-4

Use Charnes big M method to

maximize $Z=3x_1+2x_2$

subject to constraints $2x_1+x_2 \leqslant 1$

$3x_1+4x_2 \geqslant 4$

$x_1, x_2 \geqslant 0$.

Solution

Step 1 : Make the problem as N+S co-ordinate problem

First of all we observe that all b_i are positive. Introducing slack and artificial variables, the above constraints can be written as

$$2x_1+x_2+s_1 = 1 \qquad ...(2\cdot52\ a)$$

$$3x_1+4x_2-s_2+A_1=4 \qquad ...(2\cdot52\ b)$$

where x_1, x_2, s_1, s_2, A_1, all $\geqslant 0$, ...(2·52 c)

and the objective function becomes

$$\text{maximize } Z=3x_1+2x_2+0s_1+0s_2-MA_1. \qquad ...(2\cdot53)$$

Thus the objective is to find the values of x_1 and x_2 which maximize equation (2·53), while satisfying constraints (2·52 a), (2·52 b) and meeting the non-negativity restriction (2·52 c).

Step 2 : Make N co-ordinates assume zero value

Putting $x_1=x_2=0$ in the equations for constraints, we get

$s_1=1$, $s_2=0$, $A_1=4$ as the initial basic feasible solution. The above information can be put in the form of a simple matrix or table.

Table 2·55

Objective function c_j		3	2	0	0	$-M$	
e_i	variables in current solution	x_1	x_2	s_1	s_2	A_1	b
0	s_1	2	1	1	0	0	1
$-M$	A_1	3	4	0	-1	1	4

Initial basic feasible solution

Step 3 : Perform optimality test

By performing optimality test we can find whether the current basic feasible solution is optimal or not. This is done by computing c_j-E_j, where $E_j=\Sigma\, e_i a_{ij}$. This is shown in table 2·56.

Table 2.56

c_j		3	2	0	0	$-M$		
e_i	current solution variables	x_1	x_2	s_1	s_2	A_1	b	θ
0	s_1	2	(1)	1	0	0	1	1 ← key row
$-M$	A_1	3	4	0	-1	1	4	1
$E_j=\Sigma\, e_i a_{ij}$		$-3M$	$-4M$	0	M	$-M$		
c_j-E_j		$3+3M$	$2+4M$ ↑ K	0	$-M$	0		

As c_j-E_j is positive under some columns, the current basic feasible solution is not optimal and needs to be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (i) : Mark the key column. As there is tie in case of rows, 1st row is selected as the key row as per the rules described in section 2·13. Key element is (1). x_2 is the incoming non-basic variable and s_1 is the outgoing basic variable.

Sub-step (ii) : Replace s_1 by x_2. This is shown in table 2·57.

Table 2.57

c_j		3	2	0	0	$-M$	
e_i	C.S.V.	x_1	x_2	s_1	s_2	A_1	b
2	x_2	2	1	1	0	0	1
$-M$	A_1	-5	0	-4	-1	1	0
$E_j=\Sigma\, e_i a_{ij}$		$4+5M$	2	$2+4M$	M	$-M$	
c_j-E_j		$-(1+5M)$	0	$-(2+4M)$	$-M$	0	

Optimal basic feasible solution

As $c_j - E_j$ is negative or zero under every column and the column 'variables in current solution' contains artificial variable A_1 at zero level (zero value under column 'b'), the second basic feasible solution is optimal. Optimal values are

$$x_1=0,\ x_2=1$$

Also $\quad s_1=0,\ s_2=0,\ A_1=0$ and $Z_{max}=2$.

EXAMPLE 2.11-5

Use penalty method to

minimize $Z=x_1+2x_2+x_3$

subject to $x_1+\frac{1}{2}x_2+\frac{1}{2}x_3 \leqslant 1$

$$\frac{3}{2}x_1+2x_2+x_3 \geqslant 8$$

$$x_1, x_2, x_3, \text{ all } \geqslant 0.$$

Solution

Step 1 : Make the problem as N+S co-ordinate problem

First of all we observe that all b_i are non-negative. At first thought it appears that we should introduce slack variables s_1 and s_2 (for converting inequalities into equalities) as shown below.

$$x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+s_1=1$$

$$\frac{3}{2}x_1+2x_2+x_3-s_1 \quad =8.$$

Step 2 : Make N co-ordinates assume zero value

Putting $x_1=x_2=x_3=0$, we get $s_1=1$ and $s_2=-8$ as the initial basic solution. However, it is not a feasible solution since s_2 is negative. Therefore we introduce artificial variable A_1 and the above constraints can be written as

$$x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+s_1=1 \qquad \text{...(2·54 } a)$$

$$\frac{3}{2}x_1+2x_2+x_3-s_2+A_1=8 \qquad \text{...(2·54 } b)$$

where $\quad x_1, x_2, x_3, s_1, s_2, A_1, \text{ all } \geqslant 0 \qquad$ ...(2·54 c)

Now addition of this artificial variable destroys the equality required by the L.P. model. Therefore A_1 must not appear in the final solution. To achieve this, it is assigned a very large penalty (+M, since Z is to be minimized) in the objective function which may be written as

$$\text{minimize } Z=x_1+2x_2+x_3+0s_1+0s_2+MA_1. \qquad \text{...(2·55)}$$

Thus the problem is to minimize equation (2·55) subject to the constraints (2·54 a) and (2·54 b), while satisfying the non-negativity condition (2·54 c).

The above information can be put in the form of a simple matrix or table.

Table 2·58

Objective function c_j		1	2	1	0	0	M	
e_i	variables in current solution	x_1	x_2	x_3	s_1	s_2	A_1	b
0	s_1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	1
M	A_1	$\frac{3}{2}$	2	1	0	-1	1	8

Initial basic feasible solution

Step 3 : Perform optimality test

By performing optimality test we can find whether the current basic feasible solution can be improved or not. This is done by computing c_j-E_j, where $E_j=\Sigma\, e_i\, a_{ij}$. This is shown in table 2.59.

Table 2.59

	c_j	1	2	1	0	0	M		
e_i	current solution variables	x_1	x_2	x_3	s_1	s_2	A_1	b	θ
0	s_1	1	$\left(\frac{1}{2}\right)$	$\frac{1}{2}$	1	0	0	1	2←key row
M	A_1	$\frac{3}{2}$	2	1	0	-1	1	8	4
$E_j=\Sigma\, e_i\, a_{ij}$		$\frac{3}{2}$M	2M	M	0	$-$M	M		
c_j-E_j		$1-\frac{3}{2}$M	$2-2$M	$1-$M	0	M	0		
			↑K						

As c_j-E_j is *negative* under some columns, the current basic feasible solution can be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (i) : Mark the key column, key row and key element as shown in table 2·59 and explained in example 2·10-1. x_2 is the incoming non-basic variable, while s_1 is the outgoing basic variable. Make the key element unity as shown in table 2·60.

Table 2·60

e_i	C.S.V.	x_1	x_2	x_3	s_1	s_2	A_1	b
0	s_1	2	(1)	1	2	0	0	2
M	A_1	$\frac{3}{2}$	2	1	0	-1	1	8

Sub-step (ii) : Replace s_1 by x_2. This is shown in table 2·61.

Table 2·61

c_j		1	2	1	0	0	M	
e_i	C.S.V.	x_1	x_2	x_3	s_1	s_2	A_1	b
2	x_2	2	1	1	2	0	0	2
M	A_1	$\frac{-5}{2}$	0	-1	-4	-1	1	4
$E_j=\Sigma e_i a_{ij}$		$4-\frac{5}{2}M$	2	$2-M$	$4-4M$	$-M$	M	
c_j-E_j		$-3+\frac{5}{2}M$	0	$-1+M$	$-4+4M$	M	0	

2nd basic feasible solution

$\because$ c_j-E_j is either *positive* or zero under all columns and the column 'variables in current solution' contains artificial variable A_1 not at zero level (value 4 under column 'b'), the second basic feasible solution is *not optimal but pseudo-optimal* with values

$$x_1=0,\ x_2=2,\ x_3=0$$

Also $$s_1=0,\ s_2=0,\ A_1=4 \quad \text{and} \quad Z_{min}=4+4M.$$

2·11-2. The Two-Phase Method

This method solves the L.P. problem in two phases.

PHASE I

It consists of the following steps :

Step 1 :

Ensure that all b_i (constant terms) are non-negative. If some of them are negative, make them non-negative by multiplying both sides of these inequations/equations by -1.

Step 2 :

Express the constraints in the standard form as discussed in section 2·7-2.

Step 3 :

Add non-negative variables (artificial variables) to the left hand sides of all the constraints of ($=$ and $\geqslant$) type.

Step 4 :

Formulate a new objective function w which consists of the sum of the artificial variables

$$w=A_1+A_2+\ldots+A_m.$$

The function w is known as the *infeasibility form.*

Step 5 :

Using simplex method *minimize* the function w subject to the above constraints of the original problem and obtain the optimum basic feasible solution. Three cases arise :

1. Min. $w>0$ and at least one artificial variable appears in column 'variables in current solution' at *positive level*. In such a case, no feasible solution exists for the original L.P.P. and the procedure is terminated.

2. Min. $w=0$ and at least one artificial variable appears in column 'variables in current solution' at *zero level*. In such a case, the optimum basic feasible solution to the infeasibility form (auxiliary L.P.P.) may or may not be a basic feasible solution to the given (original) L.P.P. To obtain a basic feasible solution, we continue phase I and try to drive all artificial variables out of the basis and then proceed to phase II.

3. Min. $w=0$ and no artificial variable appears in the column 'variables in current solution'. In such a case, a basic feasible solution to the original L.P.P. has been found. Proceed to phase II.

PHASE II

Use the optimum basic feasible solution of phase I as a starting solution for the original L.P.P. Using simplex method make iterations till an optimal basic feasible solution for it is obtained.

It may be noted that the new objective function w is always of minimization type regardless of whether the given (original) L.P.P. is of maximization or minimization type. The above three cases will now be explained with the help of examples.

EXAMPLE 2·11-6

Use two-phase simplex method to
maximize $Z=5x_1+3x_2$
subject to the constraints $2x_1+x_2 \leqslant 1$
$x_1+4x_2 \geqslant 6$
$x_1, x_2 \geqslant 0.$

Solution

PHASE I

It consists of the following steps :

Step 1 :

First of all we observe that all b_i are non-negative. So this step is not necessary in the present problem.

Step 2 :

Adding slack variables the given constraints take the form

$$2x_1+x_2+s_1=0$$
$$x_1+4x_2-s_2=6$$

where $x_1, x_2, s_1, s_2, \text{ all } \geqslant 0.$

Step 3 :

Putting $x_1=x_2=0$, we get $s_1=1$ and $s_2=-6$ as the initial basic solution. However it is not a feasible solution since s_2 is negative. Therefore we introduce artificial variable A_1 and the above constraints can be written as

$$2x_1+x_2+s_1=1 \qquad \text{...(2·56}a\text{)}$$

$$x_1+4x_2-s_2+A_1=6 \qquad \text{...(2.56}b\text{)}$$

where $\quad x_1, x_2, s_1, s_2, A_1, \text{ all} \geqslant 0 \qquad$ ...(2·56*c*)

Step 4 :

Here infeasibility form w is $=A_1$. ...(2.57)

We are, thus, to minimize equation (2·57) subject to constraint equations (2·56*a*), (2·56*b*), while satisfying the non-negativity condition (2·56*c*).

Step 5 :

Putting $x_1=x_2=0$ in the constraint equations we get $s_1=1$, $s_2=0$, $A_1=6$ as the initial basic feasible solution. The above information can be put in the form of a simple matrix or table.

Table 2·62

		body			identity		
	Objective function c_j variables in	0	0	0	0	1	
e_i	current solution	x_1	x_2	s_2	s_1	A_1	b
0	s_1	2	1	0	1	0	1
1	A_1	1	4	−1	0	1	6

Initial basic feasible solution

Step 6 : Perform optimality test

Compute c_j-E_j where $E_j=\Sigma e_i a_{ij}$. This is shown in table 2·63.

Table 2·63

	c_j	0	0	0	0	1		
e_i	current solution variables	x_1	x_2	s_2	s_1	A_1	b	θ
0	s_1	2	(1)	0	1	0	1	1 ← key row
1	A_1	1	4	−1	0	1	6	$\frac{3}{2}$
	$E_j=\Sigma e_i a_{ij}$	1	4	−1	0	1		
	c_j-E_j	−1	−4	1	0	0		
			↑ K					

As c_j-E_j is negative under some columns (minimization problem), the current basic feasible solution can be improved.

Step 7 : Iterate towards an optimal solution

Sub-step (*i*) : Mark the key column, key row and key element as shown in table 2·63 and explained in example 2·10-1. x_2 is the

incoming non-basic variable while s_1 is the outgoing basic variable. The key element is unity.

Sub-step (ii) : Replace s_1 by x_2. This is shown in table 2·64.

Table 2·64

	c_j	0	0	0	0	1	
e_i	C.S.V.	x_1	x_2	s_2	s_1	A_1	b
0	x_2	2	1	0	1	0	1
1	A_1	—7	0	—1	—4	1	2
$E_j=\Sigma e_i a_{ij}$		—7	0	—1	—4	1	
$c_j—E_j$		7	0	1	4	0	

Optimal basic feasible solution

$\because$ c_j-E_j is either positive or zero under all columns, an optimal basic feasible solution to the auxiliary L.P.P. has been obtained.

However, since $w=A_1=2$ (>0) and artificial variable A_1 appears in column 'current solution variables' at a positive level $(A_1=2)$, the given original L.P.P. does not possess any feasible solution and the procedure stops.

EXAMPLE 2·11·7

Use two-phase simplex method to

$$\text{maximize } Z=3x_1+2x_2+2x_3$$

$$\text{subject to} \quad 5x_1+7x_2+4x_3 \leqslant 7$$

$$-4x_1+7x_2+5x_3 \geqslant -2$$

$$3x_1+4x_2-6x_3 \geqslant \frac{29}{7}$$

$$x_1, x_2, x_3, \text{ all } \geqslant 0.$$

Solution

PHASE I

It consists of the following steps :

Step 1 :

First of all we observe that all b_i should be non-negative. Since for the second constraint, $b_2=-2$, we multiply its both sides by -1 transforming it to

$$4x_1-7x_2-5x_2 \leqslant 2$$

Step 2 :

Adding slack variables the given constraints take the form

$$5x_1+7x_2+4x_3+s_1=7$$

$$4x_1-7x_2-5x_3+s_2=2$$

$$3x_1+4x_2-6x_3-s_3=\frac{29}{7}$$

where $x_1, x_2, x_3, s_1, s_2, s_{31}$ all $\geqslant 0$.

Step 3 :

Putting $x_1=x_2=x_3=0$ in the constraints, we get, $s_1=7$,

$s_2=2$, $s_3=-\frac{29}{7}$ as the initial basic solution. However, it is not a feasible solution since s_3 is negative. Therefore we introduce artificial variable A_1 and the above constraints can be written as

$$5x_1+7x_2+4x_3+s_1=7 \qquad ...(2\cdot58\ a)$$

$$4x_1-7x_2-5x_3+s_2=2 \qquad ...(2\cdot58\ b)$$

$$3x_1+4x_2-6x_3-s_3+A_1=\frac{29}{7} \qquad ...(2\cdot58\ c)$$

where $x_1, x_2, x_3, s_1, s_2, s_3, A_1$ all $\geqslant 0$. ...(2·58 d)

Step 4 :

We introduce a new objective function $w=A_1$. ...(2·59)

We are, therefore, to minimize equation (2·59) subject to constraints (2·58 a), (2·58 b), (2·58 c), while satisfying the non-negativity condition (2·58 d).

Step 5 :

Putting $x_1=x_2=x_3=0$ in the constraints, we get $s_1=7$, $s_2=2$, $s_3=0$, $A_1=\frac{29}{7}$ as the initial basic feasible solution. The above information can be put in the form of a simplex matrix or table.

Table 2·65

		body				identity			
Objective function c_j		0	0	0	0	0	0	1	
e_i	variables in current solution	x_1	x_2	x_3	s_3	s_1	s_2	A_1	b
0	s_1	5	7	4	0	1	0	0	7
0	s_2	4	−7	−5	0	0	1	0	2
1	A_1	3	4	−6	−1	0	0	1	$\frac{29}{7}$

Initial basic feasible solution

Step 6 : Perform optimality test

Compute c_j-E_j, where $E_j=\Sigma e_i\ a_{ij}$. This is shown in table 2·66.

Table 2·66

	c_j	0	0	0	0	0	0	1		
e_i	current solution variables	x_1	x_2	x_3	s_3	s_1	s_2	A_1	b	θ
0	s_1	5	(7)	4	0	1	0	0	7	1←key row
0	s_2	4	−7	−5	0	0	1	0	2	$-\frac{2}{7}$
1	A_1	3	4	−6	−1	0	0	1	$\frac{29}{7}$	$\frac{29}{28}$
$E_j=\Sigma e_i a_{ij}$		3	4	−6	−1	0	0	1		
c_j-E_j		−3	−4	6	1	0	0	0		
			↑K							

As $c_j - E_j$ is negative under some columns (minimization problem) the current basic feasible solution can be improved.

Step 7 : Iterate towards an optimal solution

Sub-step (i) : Mark the key column, key row and key element as shown in table 2·66 and explained in example 2·10-1. x_2 is the incoming nonbasic variable while s_1 is the outgoing basic variable. Make the key element unity. This is shown in table 2·67.

Table 2·67

e_i	C.S.V.	x_1	x_2	x_3	s_3	s_1	s_2	A_1	b
0	s_1	$\frac{5}{7}$	(1)	$\frac{4}{7}$	0	$\frac{1}{7}$	0	0	1
0	s_2	4	-7	-5	0	0	1	0	2
1	A_1	3	4	-6	-1	0	0	1	$\frac{29}{7}$

Sub-step (ii) : Replace s_1 by x_2. This is shown in table 2·68.

Table 2·68

	c_j	0	0	0	0	0	0	1		
e_i	C.S.V.	x_1	x_2	x_3	s_3	s_1	s_2	A_1	b	θ
0	x_2	$\frac{5}{7}$	1	$\frac{4}{7}$	0	$\frac{1}{7}$	0	0	1	$\frac{7}{5}$
0	s_2	(9)	0	-1	0	1	1	0	9	1←key row
1	A_1	$\frac{1}{7}$	0	$-\frac{58}{7}$	-1	$-\frac{4}{7}$	0	1	$\frac{1}{7}$	
$E_j = \Sigma\, e_i a_{ij}$		$\frac{1}{7}$	0	$-\frac{58}{7}$	-1	$-\frac{4}{7}$	0	1		
$e_j - E_j$		$-\frac{1}{7}$	0	$\frac{58}{7}$	1	$\frac{4}{7}$	0	0		
		↑K								

2nd basic feasible solution

Step 8 :

Repeat step 6. The key column, key row and key element are shown in table 2·68. There is tie for the key row. However, the second row is marked as key row as per the rules described in section 2·13. Make the key element unity. This is shown in table 2·69.

subject to $y_1 - y_2 - 2y_3 + 2y_4 \leqslant 0$

$$2y_1 - 2y_2 - 3y_3 + 3y_4 \leqslant 6$$

where y_1, y_2, y_3, y_4, all $\geqslant 0$.

Introducing slack variables, the problem may be written as

minimize $Z = 2y_1 - 2y_2 + 3y_3 - 3y_4 + 0s_1 + 0s_2$ ...(2·60)

subject to $y_1 - y_2 - 2y_3 + 2y_4 + s_1 = 0$ ...(2·61 a)

$2y_1 - 2y_2 - 3y_3 + 3y_4 + s_2 = 6$ ...(2·61 b)

where $y_1, y_2, y_3, y_4, s_1, s_2$, all $\geqslant 0$. ...(2·61 c)

Thus the problem is to find the values of y_1, y_2, y_3 and y_4 which maximize equation (2·60) subject to constraints (2·61 a), (2·61b) and non-negativity condition (2·61 c).

Step 2 : Make N co-ordinates assume zero value

Putting $y_1 = y_2 = y_3 = y_4 = 0$ in the equations for constraints gives $s_1 = 0$, $s_2 = 6$ as the initial basic feasible solution. The above information can be put in the form of a simple matrix or table.

Table 2·72

Objective function c_j		2	−2	3	−3	0	0	
e_i	variables in current solution	y_1	y_2	y_3	y_4	s_1	s_2	b
0	s_1	1	−1	−2	2	1	0	0
0	s_2	2	−2	−3	3	0	1	6

Initial basic feasible solution

Step 3 : Perform optimality test

By performing optimality test we can find whether the current feasible solution can be improved or not. This is done by computing $c_j - E_j$, where $E_j = \Sigma e_i a_{ij}$. This is shown in table 2·73. Variable s_1 has zero value. We replace this zero value by a very small positive number ϵ to carry out iterations.

Table 2·73

	c_j	2	-2	3	-3	0	0		
e_i	current solution variables	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	-1	-2	(2)	1	0	ϵ	$\frac{\epsilon}{2}$ ← key row
0	s_2	2	-2	-3	3	0	1	6	2
	$E_j=\Sigma e_i a_{ij}$	0	0	0	0	0	0		
	c_j-E_j	2	-2	3	-3	0	0		
					↑K				

As c_j-E_j is negative under some columns. the current feasible solution is not optimal and is, therefore, to be improved.

Step 4 : Iterate towards an optimal solution

Sub-step (i) : Mark the key column, key row and key element as shown in table 2·73 and explained in example 2·10·1. y_4 is the incoming non-basic variable, while s_1 is the outgoing artificial variable. Make the key element unity. This is shown in table 2·74.

Table 2·74

e_i	C.S.V.	y_1	y_2	y_3	y_4	s_1	s_2	b
0	s_1	$\frac{1}{2}$	$-\frac{1}{2}$	-1	(1)	$\frac{1}{2}$	0	$\frac{\epsilon}{2}$
0	s_2	2	-2	-3	3	0	1	6

Sub-step (ii) : Replace s_1 by y_4. This is shown in table 2·75.

Table 2·75

	c_j	2	-2	3	-3	0	0		
e_i	C.S.V.	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
-3	y_4	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1	$\frac{1}{2}$	0	$\frac{\epsilon}{2}$	$-\epsilon$
0	s_2	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{3}{2}$	1	$6-\frac{3\epsilon}{2}$	$-6+\frac{3\epsilon}{2}$
	$E_j=\Sigma e_i a_{ij}$	$-\frac{3}{2}$	$\frac{3}{2}$	3	-3	$-\frac{3}{2}$	0		
	c_j-E_j	$\frac{7}{2}$	$-\frac{7}{2}$	0	0	$\frac{3}{2}$	0		
			↑K						

2nd basic feasible solution

Step 5 : Perform optimality test

Find $c_j - E_j$ where $E_j = \Sigma e_i a_{ij}$. Since $c_j - E_j$ is negative under column 'y_2' 2nd basic feasible solution is not optimal.

Step 6 : Iterate towards an optimal solution

Mark the key column ; this is column 'y_2'. To find key row we divide the elements in column 'b' by the corresponding elements of column 'k' and get column 'θ'. Since all the quotients in column 'θ' are negative, the value of incoming non-basic variable y_2 can be made as large as we like without violating the feasibility condition. The problem, therefore, has an unbounded solution and further iteration stops.

2·13. Degeneracy in Simplex Method

Degeneracy may become evident in the simplex method when the outgoing slack variable is being selected. In the iterative process, the minimum positive quotient under 'θ'-column determines the outgoing variable. If the minimum quotient is equal for two or more rows, arbitrary selection of one of these slack variables may result in one or more slack variables becoming zero in the next iteration and the problem is said to *degenerate*. There is no assurance that the value of objective function will improve, since the new solutions may remain degenerate. More serious, however, is the condition of *'cycling'* or *'circling'* which may result from a poor choice among the tied variables. Fortunately the problems in which cycling occurs are very rare. In fact it is difficult to find a practical problem in which cycling occurs.

Nevertheless, if such a situation occurs, this difficulty may be overcome by applying the following simple procedure, called *perturbation method* by A. Charnes :

1. Divide each element in the tied rows by the *positive coefficients* of the key-column in that row.

2. Compare the resulting ratios, column by column, first in the identity and then in the body, from left to right.

3. The row which first contains the smallest algebraic ratio contains the outgoing slack variable.

This procedure will be clear when we consider the following example :

Table 2·76

c_j		*body*			*identity*					
C.S.V.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	b	θ	
s_1	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0	tied rows
s_2	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0	
s_3	0	0	1	0	0	0	1	1		
	↑ K (key column)									

Now difficulty arises regarding the choice of s_1 or s_2 as the outgoing variable. For the identity of the first row, the ratio is $1/1/4=4$. Likewise, for the identity of the 2nd row, the ratio is $0/1/2=0$. Since the 2nd row yields the smaller ratio, s_2 is the outgoing slack variable and the simplex procedure can, then, be continued.

2·14. Some Additional Points

(1) *Restrictions* : As we have already said, any inequality with $\leqslant$ sign can be converted into an equality (equation) by the addition of slack variables which contribute zero to the objective function. For example,

$$\text{maximize } Z=3x_1+5x_2$$
$$\text{subject to } \quad 2x_1+3x_2 \leqslant 4$$

is written as

$$\text{maximize } Z=3x_1+5x_2+0s_1$$
$$\text{subject to } \quad 2x_1+3x_2+s_1=4.$$

Likewise, any inequality with $\geqslant$ sign can be converted into an equality (equation) by the subtraction of slack variables which again contribute zero to the objective function.

For instance, in the problem, minimize $Z=5x_1+4x_2$

subject to $3x_1+2x_2 \geqslant 6$,

restriction may be written as $3x_1+2x_2-s_1=6$.

But this does not yield an acceptable initial feasible solution ($\because$ $s_1=-6$ and negative values for slack variables are not acceptable). Therefore artificial variables are introduced and the above equation may be written as

$$3x_1+2x_2-s_1+A_1=6$$

and since the artificial variables should not appear in the final solution, a large penalty (a large value, $+M$) is assigned to these artificial variables in the objective function so that the problem becomes

$$\text{minimize } Z = 5x_1 + 4x_2 + 0s_1 + MA_1$$
$$\text{subject to } 3x_1 + ?x_2 - s_1 + A_1 = 6.$$

(2) *Equations.* If the restrictions are in the form of equations, they can be modified to fit into the simplex format by adding an artificial variable A exactly the same may as above.

For example, maximize $Z = 4x_1 + 3x_2$

subject to $3x_1 + 2x_2 = 6$

is modified to maximize $Z = 4x_1 + 3x_2 - MA_1$

subject to $3x_1 + 2x_2 + A_1 = 6.$

(3) *Approximations.* If the restrictions are of the form $\approx$, say, for example,

$$\text{maximise } Z = x + 6y$$
$$\text{subject to } 2x + 3y \approx 20,$$

they may be prepared for the simplex matrix by adding and subtracting slack variables, such as

$$2x + 3y - s_1 + s_2 = 20.$$

The two slack variables permit $2x+3y$ to be either slightly larger or slightly smaller than 20. The slack variable s_1 represents the amount by which $2x+3y$ exceeds 20 and the slack variable s_2 represents the amount by which $2x+3y$ is less than 20. The slack variable s_1 will appear in the body, and the slack variable s_2 will appear in the identity. We wish to minimize the values of the slack variables so that $2x+3y$ will be as close as possible to 20 and this may be accomplished by assigning the coefficient -1 to both slack variables in the objective function. Thus the objective function will be

$$\text{maximize } Z = x + 6y - s_1 - s_2.$$

EXAMPLE 2·14-1

Prepare the following data for simplex matrix :

$$\begin{aligned} \text{maximize } Z &= 30x + 25y \\ \text{subject to } 2x + y &\leqslant 40 \\ 2x + 3y &\geqslant 45 \\ 3x + 7y &= 55 \\ 4x + 3y &\approx 40. \end{aligned}$$

Solution. Adding and subtracting the appropriate slack and artificial variables, we have the following four equations for inclusion in the simplex matrix :

$$2x+y+s_1=40$$
$$2x+3y-s_2+A_1=45$$
$$3x+7y+A_2=55$$
$$4x+3y-s_3+s_4=40$$

and the objective function is modified to

$$\text{maximize } Z=30x+25y+0s_1+0s_2-MA_1-MA_2-s_3-s_4.$$

Rearranging the objective function so that the sequence of the variables is divided between the body and the identity, we can develop the initial simplex matrix for this problem as shown in the table below.

Table 2·77

		body				*identity*				
	c_j	30	25	0	-1	0	$-M$	$-M$	-1	
e_i	variables in current solution	x	y	s_2	s_3	s_1	A_1	A_2	s_4	b
0	s_1	2	1	0	0	1	0	0	0	40
$-M$	A_1	2	3	-1	0	0	1	0	0	45
$-M$	A_2	3	7	0	0	0	0	1	0	55
-1	s_4	4	3	0	-1	0	0	0	1	40

2·15. Advantages of Linear Programming Methods

Following are the main advantages of linear programming methods :

1. It helps in attaining the optimum use of productive factors. Linear programming indicates how a manager can utilize his productive factors most effectively by a better selection and distribution of these elements. For example, more efficient use of manpower and machines can be obtained by the use of linear programming.

2. It improves the quality of decisions. The individual who makes use of linear programming methods becomes more objective than subjective. The individual having a clear picture of the relationships within the basic equations, inequalities or constraints can have a better idea about the problem and its solution.

3. It can go a long way in improving the knowledge and skill of tomorrow's executives.

4. Although linear programming gives possible and practical solutions, there might be other constraints operating outside the problem which must be taken into account, for example, sales, demand, etc. Just because we can produce so many units does not mean that they can be sold. Linear programming method can handle such situations also because it allows modification of its mathematical solutions.

5. It highlights the bottlenecks in the production processes. When bottlenecks occur, some machines cannot meet demand while others remain idle, at least part of the time. Highlighting of bottlenecks is one of the most significant advantages of linear programming.

2·16 Limitations of Linear Programming Model

This model, though having a wide field has the following limitations :

1. For large problems having many limitations and constraints, the computational difficulties are enormous, even when assistance of large digital computers is available. It may be sometimes possible to get over this difficulty by splitting the main problem into smaller ones, deriving solutions for them and then combining the results.

2. According to the linear programming problem the solution variables can have any value, whereas sometimes it happens that some of the variables can have only integral values. For example, in finding how many lathes and how many milling machines to be produced, only integral values of decision variables x_1 and x_2 are meaningful. Except when the variables have large values, rounding the solution to the nearest integer will not yield an optimal solution. Such situations justify the use of special methods.

3. The model does not take into account the effect of time. The O.R. team must define the objective function and constraints, which can change overnight due to internal as well as external factors.

4. Many times, it is not possible to express both the objective function and constraints in linear form. For example, in production planning we often have non-linear constraints on production capacities like setup and take-down times which are often independent of the quantities produced. The misapplication of linear programming under non-linear conditions usually results in an incorrect solution.

When comparison is made between the advantages and disadvantages/limitations of linear programming, its advantages clearly overweigh its limitations. It must be clearly understood that linear programming techniques, like other mathematical tools only help the manager to take better decisions, they are in no way a substitute for the manager.

2·17 Further Developments of Linear Programming

The various maximization and minimization problems presented in this chapter represent the various fields in which linear programming technique may be used. Some more fields where it can be used are mentioned here.

An interesting example is the allocation problem in making frankfurters. They contain, among other things, beef, pork, cereal, grain and spices. There are upper and lower limits on the amount of each constituent, imposed by flavour, structural and legal considerations. The prices of the ingredients fluctuate daily, but the production rate of the line is constant. This is an allocation problem in which we are to allocate the mix of ingredients so as to give the least cost frankfurter, subject to the imposed constraints. This is an area that can be and has been explored by many meat packers for reducing daily costs.

Many more examples can be found for the minimization problem—such as the elimination of excessive amounts of high octane components in the blending of gasoline, finding the most suitable animal feeds, and finding the lowest cost mixture for a drug manufacturer. Oil refineries throughout the world have used linear programming with considerable success. Similar trends are developing in chemical industries, iron and steel industries, aluminium industry, wood products manufacture, food processing industry and banking. In fact linear programming may be used for any general situation where an objective function (expressed as a linear function) has to be maximized or minimized subject to certain restrictions (expressed as linear equations/inequations).

2·18. Bibliographic Notes

At present, linear programming is one of the best developed methods of operations research. There is, therefore, quite a lot of literature on this subject. All text books on operations research contain quite a lot of information on this subject. In addition, there are many research papers dealing exclusively with the theory and applications of linear programming.

The first good work published on the subject is by L. Kantorovich [4]. It deals with the applications of this method to a firm and gives comprehensive information of the theoretical basis. The algorithm proposed by Kantorovich for solving the linear programmes differs a bit from the simplex algorithm presented in this book.

Elementary details of linear programming are given in the book by R. Ferguson and L. Sargent [2] and also by R. Metzger [7]. One of the best presentations of linear programming is the book by A. Charnes, W. Cooper and A. Henderson [1]. This book discusses the theoretical basis along with applications of linear programming.

Book by S. Gass [3] gives a detailed information about the mathematical foundations of linear programming. It also contains a survey of principal applications. Book by S. Vajda [10] deals exclusively with the applications of linear programming.

The importance of linear programming in the theory of firms is dealt with in the book by R. Dorfman. Linear programming in connection with theory of games is dealt with in the book by S. Vajda [9]. A collective work edited by T. Koopmans [5] deals with linear programming, its applications, calculating techniques and mathematical basis.

A collective work edited by H. Kuhn and A. Tucker [6] is devoted solely to special mathematical problems associated with linear programming.

A comprehensive bibliography on linear programming is contained in the work by V. Riley and S. Gass [8].

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Exercises

1. What are the essential characteristics of a linear programming model ?

[Pb. Univ. Mech. Engg. 1976, 78, 79]

2. Explain the terms : key decision, objective, alternatives and restrictions in the context of linear optimization models by assuming a suitable industrial situation.

[Pb. Univ. Mech. Engg. 1975, 76]

3. Explain important characteristics of the industrial situations to which linear programming method can be successfully applied. Illustrate application of this technique with a suitable example.

[Pb. Univ. Mech. Engg. 1977, 78, 79]

Section 2·3.

4. A small manufacturer employs 5 skilled men and 10 semi-skilled men and makes an article in two qualities, a delux model and an ordinary model. The making of a delux model requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. The ordinary model requires 1 hour work by a skilled man and 3 hours work by a semi-skilled man. By union rules no man can work more than 8 hours per day. The manufacturer's clear profit of the delux model is Rs. 10 and of the ordinary model Rs. 8. Formulate the model of the problem.

[Baroda Univ. B.E., 1975]

(*Ans.* Maximize $Z=10x_1+8x_2$

subject to $2x_1+x_2 \leqslant 40$

$2x_1+3x_2 \leqslant 80$

$x_1, x_2 \geqslant 0$.)

5. Old hens can be bought for Rs. 2 each but young ones cost Rs. 5 each. The old hens lay 3 eggs per week and young ones 5 eggs per week, each egg being worth 30 paise. A hen costs Re. 1 per week to feed. If a person has only Rs. 80 to spend on the hens, how

many of each kind should he buy to give a profit of more than Rs. 6 per week assuming that he cannot house more than 20 hens ?

[*Meerut Univ. B.Sc. (Math.) 1969*]

(*Ans.* Maximize $Z=0{\cdot}3\ (3x_1+5x_2)(x_1+x_2)$

$=0{\cdot}5\ x_2-0{\cdot}1\ x_1$

subject to $x_1+x_2\leqslant 20$, $2x_1+5x_2\leqslant 80$ and $x_1, x_2\geqslant 0$) ;

$x_1=0$, $x_2=16$, $Z_{max}=$Rs. 8.

6. A firm manufactures three products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has two machines and the required processing time in minutes for each machine on each product is given below.

Table 2·78

		Product		
		A	B	C
Machine	C	4	3	5
	D	2	2	4

Machines C and D have 2,000 and 2,500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but no more than 150 A's. Set up an L.P. model to maximize it.

[*Roorkee M.E. (Mech.), 1977*]

Ans. Maximize $Z=3x_A+2x_B+4x_C$

subject to $4x_A+3x_B+5x_C\leqslant 2{,}000$

$2x_A+2x_B+4x_C\leqslant 2{,}500$

$x_A\geqslant 100$

$x_A\leqslant 150$

$x_B\geqslant 200$

$x_C\geqslant 50$.)

7. The manager of an oil refinery has to decide upon the optimal mix of two possible blending processes, of which the inputs and outputs per production run are as follows :

Table 2·79

	Input		*Output*	
Process	*Crude A*	*Crude B*	*Gasoline X*	*Gasoline Y*
1	5	3	5	8
2	4	5	4	4

The maximum amount available of crude A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 3 and Rs. 4 respectively. Formulate the problem as linear program-

ming broblem.

[*Pb. Univ. Mech. Engg. 1977*]

(*Ans.* Maximize $Z=3x_1+4x_2$

subject to $5x_1+4x_2 \leqslant 200$

$3x_1+5x_2 \leqslant 150$

$5x_1+4x_2 \geqslant 100$

$8x_1+4x_2 \geqslant 80$

$x_1, x_2 \geqslant 0$.)

8. A firm can produce three types of cloth, say, A, B and C. Three kinds of wool are required for it, say, red wool, green wool and blue wool. One unit length of type A cloth needs 2 yards of red wool and 3 yards of blue wool; one unit length of type B cloth needs 3 yards of red wool, 2 yards of green wool and 2 yards of blue wool ; and one unit length of type C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has a stock of only 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3, of type B cloth is Rs. 5 and that of type C cloth is Rs. 4. Formulate the problem as linear programming problem.

[*Meerut B. Sc. (Math.) 1971*]

(*Ans.* Maximize $Z=3x_A+5x_B+4x_C$

subject to $2x_A+3x_B \leqslant 8$

$2x_B+5x_C \leqslant 10$

$3x_A+2x_B+4x_C \leqslant 15$

$x_A, x_B, x_C \geqslant 0$.

9. A dairy feed company may purchase and mix one or more of the three types of grains containing different amounts of nutritional elements. The data is given in the table below.

The production manager specifies that any feed mix for his livestock must meet at least minimal nutritional requirements, and seeks the least costly among all such mixes.

Table 2·80

		One Unit Weight of			
	Item	*Grain 1*	*Grain 2*	*Grain 3*	*Minimal requirement*
	A	2	3	7	1,250
Nutritional	B	1	1	0	250
ingredients	C	5	3	0	900
	D	6	25	1	232·5
Cost/unit weight (Rs.)		41	35	96	

Analyse the situation to recognize the key decision, objective, alternatives and restrictions. Formulate linear programming model for the problem.

[*Pb. Univ. Mech. Engg. 1978*]

(*Ans.* Minimize $Z=41x_1+35x_2+96x_3$

subject to $2x_1+3x_2+7x_3 \geqslant 1{,}250$

$$x_1+x_2 \geqslant 250$$
$$5x_1+3x_2 \geqslant 900$$
$$6x_1+25x_2+x_3 \geqslant 232{\cdot}5$$
$$x_1, x_2, x_3 \geqslant 0.)$$

10. A farmer has a 100-acre farm. He can sell all the tomatos, lettuce or radishes he can raise. The price he can obtain is Re. 1 per kg for tomatos, Re. 0·75 a head for lettuce and Rs. 2 per kg for radishes. The average yield per acre is 2,000 kg of tomatos, 3,000 heads of lettuce and 1,000 kg of radishes. Fertilizer is available at Re. 0·50 per kg and the amount required per acre is 100 kg each for tomatos and lettuce and 50 kg for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatos and radishes and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20 per man-day.

Formulate an L.P. model for this problem in order to maximize the farmer's total profit.

[*Delhi M.B.A. 1976*]

(*Ans.* Maximize $Z=(2{,}000-50-100)\,x_1+(2{,}250-50-120)x_2$
$+(2{,}000-25-100)\,x_3$

$=1{,}850x_1+2{,}080x_3+1{,}875x_3$

subject to $x_1+x_2+x_3 \leqslant 100$

$$5x_1+6x_2+5x_3 \leqslant 400$$
$$x_1, x_2, x_3 \geqslant 0.)$$

11. A manufacturer of metal office equipment makes desks, chairs, cabinets and book cases. The work is carried out in three major departments : metal stamping, assembly and finishing. The exhibits A, B and C give requisite data of the problem.

Exhibit A

Time Required in Hours per Unit of Product

	Products				
Departments	*Desk*	*Chair*	*Cabinet*	*Book case*	*Hours available per week*
Stamping	4	2	3	3	800
assembly	10	6	8	7	1,200
Finishing	10	8	8	8	800

Exhibit B

Cost (Rs.) of Operation per Unit of Product

Department	Desk	Chair	Cabinet	Book case
		Products		
Stamping	15	8	12	12
Assembly	30	18	24	21
Finishing	35	28	25	21

Exhibit C

Selling Price (Rs.) per Unit of Product

Desk : 175	Chair : 95
Cabinet : 145	Book case : 130

In order to maximize weekly profits what should be the production programme ? Assume that the items produced can be sold. Which department needs to be expanded for increasing profits ?

[*Gujarat Univ. April, 1976*]

(*Ans.* maximize $Z=[175-(15+30+35)]x_1+[95-(8+18+28)]x_2$
$+[145-(12+24+125)]x_3+[130-(12+21+21)]x_4$
$=95x_1+41x_2+84x_3+76x_4$

subject to
$$4x_1+2x_2+3x_3+3x_4 \leqslant 800$$
$$10x_1+6x_2+8x_3+7x_4 \leqslant 1,200$$
$$10x_1+8x_2+8x_3+8x_4 \leqslant 00$$
$$x_1, x_2, x_3, x_4 \geqslant 0.)$$

12. A plant manufactures washing machines and dryers. The major manufacturing departments are the stamping deptt., motor and transmission deptt. and assembly deptt. The first two departments produce parts for both the products while the assembly lines are different for the two products. The monthly deptt. capacities are

Stamping deptt.	:	1,000 washers or 1,000 dryers
Motor and transmission deptt.	:	1,600 washers or 7,000 dryers
Washer assembly line	:	9,000 washers only
Dryer assembly line	:	5,000 washers only

Profits per piece of washers and dryers are Rs. 270 and Rs. 300 respectively. What number of each product should be produced to maximize profits ? [*Pb. Univ. Mech. Engg. Nov., 1976*]

13. A used car dealer wishes to stock up his lot to maximize his profit. He can select cars A, B and C which are valued wholesale at Rs. 5,000, Rs. 7,000 and Rs. 8,000 respectively. These can be sold at Rs. 6,000. Rs. 8,500 and Rs. 10,500 respectively. For each car type the probabilities of sale are

Type of car	Probability of sale in 90 days
A	0·7
B	0·8
C	0·6

For every two cars of type B, he should buy one car of type A or C. If he has Rs. 1,00,000 to invest, what should he buy to maximize his expected gain ? Formulate the linear programming problem.

[*Madurai B.E. Mech. 1976*]

14. A certain farming organization operates three forms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. Following are the data for the upcoming season :

Farm	Usable acreage	Water available in acre feet
1	400	1,500
2	600	2,000
3	300	900

The organization is considering three crops for planting which differ primarily in their expected profit per acre and in their consumption of water. Furthermore, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available.

Crop	Minimum acreage	Water consumption in acre feet per acre	Expected profit per acre
A	700	5	Rs. 400
B	800	4	Rs. 300
C	300	3	Rs. 100

In order to maintain a uniform work load among the farms, it is the policy of the organization that the percentage of the usable acreage planted must be the same at each farm. However, any combination of the crops may be grown at any of the farms. The organization wishes to know how much of each crop should be planted at the respective farms in order to maximize expected cost. Formulate this as a linear programming problem,

(*Delhi Univ. M.B.A. 1975*)

[**Hint** : Let x_{ij} ($i=1, 2, 3$; $j=$ A, B, C) represent the number of acres of ith farm allotted to the jth crop. Then the objective is to

$$\text{maximize } Z = 400 \sum_{i=1}^{3} x_{iA} + 300 \sum_{i=1}^{3} x_{iB} + 100 \sum_{i=1}^{3} x_{ic}$$

subject to the restrictions

$$5x_{1A}+4x_{1B}+3x_{1c} \leqslant 1,500$$

$$5x_{2A}+4x_{2B}+3x_{2c} \leqslant 2,000$$

$$5x_{3A}+4x_{3B}+3x_{3c} \leqslant 900\ ;$$

$$\sum_j x_{1j} \leqslant 400,\ \sum_j x_{2j} \leqslant 600,\ \sum_j x_{3j} \leqslant 300$$

$$\leqslant 300\ ;$$

$$\sum_i x_{iA} \leqslant 700\ \ \sum_i x_{iB} \leqslant 800.\ \sum_i x_{ic} \leqslant 300\ ;$$

$$3\sum_j x_{ij} = 2\sum_j x_{2j}\ ;$$

$$\sum_j x_{2j} = 2\sum_j x_{3j}\ ]$$

Section 2·4

15. Maximize, by using graphical method, the objective function

$$Z=2x_1+3x_2$$

subject to constraints $\quad x_1+x_2 \leqslant 1$

$$3x_1+x_2 \leqslant 4$$

$$x_1, x_2 \geqslant 0.$$

[*Meerut B. Sc.* (*Math.*) *1970*]

(*Ans.* $x_1=0, x_2=1, Z_{max}=3.$)

16. Maximize $Z=5x_1+3x_2$

subject to $\quad 3x_1+5x_2 \leqslant 15$

$$5x_1+2x_2 \leqslant 10$$

$$x_1, x_2 \geqslant 0.$$

[*Delhi M. Sc.* (*Math.*) *1969*]

$$\left(Ans.\ x_1=\frac{20}{19},\ x_2=\frac{45}{19},\ Z_{max}=\frac{235}{19}.\right)$$

17. Maximize $Z=5x_1+3x_2$

subject to $\quad x_1+x_2 \leqslant 6$

$$0 \leqslant x_1 \geqslant 3$$

$$0 \leqslant x_2 \geqslant 3$$

$$2x_1+3x_2 \geqslant 3.$$

(*Ans.* $x_1=3, x_2=3, Z_{max}=24.$)

18. Minimize $Z=20x_1+10x_2$

$$\begin{aligned}\text{subject to}\quad x_1+2x_2 &\leqslant 40\\ 3x_1+x_2 &\geqslant 30\\ 4x_1+3x_2 &\geqslant 60\\ x_1, x_2 &\geqslant 0.\end{aligned}$$

[*Meerut M. Com. 1977*]

(*Ans.* $x_1=6$, $x_2=12$, $Z_{min}=240$.)

19. Determine x_1 and x_2 so as to

$$\text{maximize } Z=2x_1+3x_2$$

$$\begin{aligned}\text{subject to the constraints}\quad x_1+x_2 &\leqslant 30\\ x_1-x_2 &\geqslant 0\\ x_1 &\leqslant 20\\ x_2 &\geqslant 3\\ x_2 &\leqslant 12\\ \text{and } x_1, x_2 &\geqslant 0.\end{aligned}$$

(*Ans.* $x_1=18$, $x_2=12$, $Z_{max}=72$)

20. Find the maximum as well as minimum value of the objective function

$$Z=4x+5y$$

$$\begin{aligned}\text{subject to}\quad 2x+y &\leqslant 6\\ x+2y &\leqslant 5\\ x-2y &\leqslant 2\\ -x+y &\leqslant 2\\ x+y &\geqslant 1\\ \text{and}\quad x, y &\geqslant 0.\end{aligned}$$

$$\left(\textit{Ans.}\quad x=\frac{7}{3},\ y=\frac{4}{3},\ Z_{max}=16\ ;\quad x=1,\ y=0,\ Z_{min}=4.\right)$$

21. Find the maximum as well as minimum value of the objective function

$$Z=-3x+6y$$

$$\begin{aligned}\text{subject to}\quad x+2y+1 &\geqslant 0\\ -4x+y+23 &\geqslant 0\\ 2x+y-4 &\geqslant 0\\ x-4y+13 &\geqslant 0\\ x-y+1 &\geqslant 0\\ \text{and } x\geqslant 0, y &\geqslant 0.\end{aligned}$$

(*Ans.* $x=3$, $y=4$, $Z_{max}=15$;
$x=5$, $y=-3$, $Z_{min}=33$.)

22. Two products A and B are to be manufactured. One single unit of product A requires 2·4 minutes of punch press time and 5 minutes of assembly time. The profit for product A is Rs. 0·60 per unit. One single unit of product B requires 3 minutes of punch press time and 2·5 minutes of welding time. The profit for product B is Rs. 0·70 per unit. The capacity of the punch press deptt. available for these products is 1,200 minutes/week. The welding deptt. has an idle capacity of 600 minutes/week and assembly deptt. has 1,500 minutes/week.

(*i*) Formulate the problem as linear programming problem.

(*ii*) Determine the quantities of products A and B so that total profit is maximized.

[*Pb. Univ. Mech. Engg. April, 1976*]

(*Ans.* $x_A=200$, $x_B=240$ units.)

23. A feed mixing operation can be described in terms of two activities. The required mixture must contain four kinds of ingredients w, x, y and z. Two basic feeds A and B, which contain the required ingredients are available in the market. 1 kg of A contains 0·1 kg of w, 0·1 kg of y and 0·2 kg of z. Likewise, 1 kg of feed B contains 0·1 kg of x, 0·2 kg of y and 0·1 kg of z. The daily per head requirement is of, at least, 0·4 kg of w, 0·6 kg of x, 2 kg of y and 1·6 kg of z. Feed A can be bought for £ 0·07 per kg and B for £ 0·05 per kg. Determine the quantity of feeds A and B in the mixture in order that the total cost be minimum.

[*Pb.Univ. Mech. Engg. Nov., 1976*]

$$\left(Ans.\ x_A=\frac{16}{3},\ x_B=\frac{22}{3}.\right)$$

24. A firm manufactures two items. It purchases castings which are then machined, bored and polished. Castings for items A and B cost Rs. 3 and Rs. 4 respectively and are sold at Rs. 6 and Rs. 7 each respectively. Running costs of the three machines are Rs. 20, Rs. 14 and Rs. 17·50 per hour respectively. What product mix maximizes the profit ? Capacities of the machines are

	Part A	*Part B*
Machining capacity	25/hr	40/hr
Boring capacity	28/hr	35/hr
Polishing capacity	35/hr	25/hr

[*Pb. Univ. Mech. Engg. 1978, 79*]

$$\left(Ans.\ x_1=16\frac{29}{31},\ x_2=12\frac{28}{31}.\right)$$

[**Hint** :

Cost and Profit per Part

	Part A (Rs.)	Part B (Rs.)
Machining cost	$\frac{20}{25}=0{\cdot}80$	$\frac{20}{40}=0{\cdot}50$
Boring cost	$\frac{14}{28}=0{\cdot}50$	$\frac{14}{35}=0{\cdot}40$
Polishing cost	$\frac{17{\cdot}5}{35}=0{\cdot}50$	$\frac{17{\cdot}5}{25}=0{\cdot}70$
Casting cost	2·00	3·00
∴ Total cost	3·80	4·60
Selling price	5·00	6·00
∴ Profit	1·20	1·40

∴ Objective is to maximize $Z=1{\cdot}2x_1+1{\cdot}4x_2$

Constraints are due to the capacities of the three machines. They are

$$\frac{1}{25}x_1+\frac{1}{40}x_2 \leqslant 1 \quad i.e., \quad 40x_1+25x_2 \leqslant 1{,}000$$

$$\frac{1}{28}x_1+\frac{1}{35}x_2 \leqslant 1 \quad i.e., \quad 35x_1+20x_2 \leqslant 980$$

$$\frac{1}{35}x_1+\frac{1}{25}x_2 \leqslant 1 \quad i.e., \quad 25x_1+35x_2 \leqslant 875$$

where $x_1, x_2 \geqslant 0$.]

25. A company is manufacturing products Y and Z. One unit of product Y requires 4·8 minutes of machining and 10 minutes of assembly time. The profit for product Y is Rs. 0·70 per unit. Product Z requires 6 minutes of machining time and 5 minutes of welding time for manufacturing one unit. Profit for Z is Rs. 0·90 per unit. The capacity of the machining deptt. available for these products is 1,400 minutes per week. The welding deptt. has an idle capacity of 800 minutes/week and assembly deptt. has 1,800 minutes/week. Determine the quantities of Y and Z so that total profit is maximized.

[*Pb. Univ. Mech. Engg. April, 1979*

$\left(Ans.\quad Y=\frac{275}{3},\ Z=160 \text{ units.}\right)$

26. A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicines can be put. Furthermore, it takes 3 hours to prepare enough material to fill 1,000 bottles of A, it takes 1 hour to prepare enough material to prepare 1,000 bottles of B and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for A and Rs. 7 per bottle for B.

(a) Formulate this problem as linear programming problem.

(b) How should the manufacturer schedule the production in order to maximize his profit ?

[*Meerut B.Sc. (Math.) 1973*]

(*Ans.* (a) Maximize $Z = x_A + 7x_B$

subject to $x_A \leqslant 20{,}000$

$x_B \leqslant 40{,}000$

$x_A + x_B \leqslant 45{,}000$

$$\frac{3}{1{,}000}x_A + \frac{1}{1{,}000}x_B \leqslant 66$$

or $3x_A + x_B \leqslant 66{,}000$

$x_A, x_B \geqslant 0.$

(b) $x_A = 10{,}500$, $x_B = 34{,}500$, $Z_{max} =$ Rs. 3,25,500.)

27. Two grades of paper X and Y are produced on a paper machine. Because of raw material restrictions, not more than 400 tonnes of grade X and 300 tonnes of grade Y can be produced in a week. There are 160 production hours in a week. It requires 0·2 and 0·4 hour to produce one tonne of products X and Y respectively with corresponding profits of Rs. 20 and Rs. 50 per tonne. Find the optimum product mix using the graphic method.

[*Bangalore Univ. July, 1978*]

(*Ans.* X = 200 tonnes, Y = 300 tonnes, Z_{max} = Rs. 19,000.)

28. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. Because of the need to ensure certain nutrient constituents, it is necessary to buy additionally one or two products, which we shall call A and B. The nutrient constituents (vitamins and proteins) in each unit of the products are given below.

Table 2·81

Nutrient	*Nutrient contents in the products*		*Minimum amount of nutrient*
	A	*B*	
1	36	6	108
2	3	12	36
3	20	10	100

Product A costs Rs. 20 per unit and product B costs Rs. 40 per unit. How much of products A and B be purchased at the lowest

possible cost so as to provide the pigs, nutrients not less than that given in the table ?

[*Delhi M.B.A. 1973*]

(*Ans.* $x_A=4$ units, $x_B=2$ units $Z_{max}=$ Rs. 160.)

[**Hint.** Minimize $Z=20x_A+10x_B$

subject to
$$36x_A+6x_B \geqslant 108$$
$$3x_A+12x_B \geqslant 36$$
$$20x_A+10x_B \geqslant 100$$
$$x_A, x_B \geqslant 0.]$$

29. The ABC Electric Appliance Company produces two products : refrigerators and ranges. Production takes place in two separate departments. Refrigerators are produced in deptt. I and ranges are produced in deptt. II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators in deptt. I and 35 ranges in deptt. II, because of limited available facilities in the two deptts. The company regularly employs a total of 60 workers in the two deptts. A refrigerator requires 2 man-weeks of labour, while a range requires 1 man-week of labour. A refrigerator contributes a profit of Rs. 60 and a range contributes a profit of Rs. 40. Formulate the problem as L.P. problem. How many units of refrigerators and ranges should the company produce to realise a maximum profit ?

[*Delhi M.B.A. 1975, 77*]

[*Ans.* (*a*) Maximize $Z=60x_1+40x_2$

subject to
$$x_1 \leqslant 25$$
$$x_2 \leqslant 35$$
$$2x_1+x_2 \leqslant 60$$
$$x_1, x_2 \geqslant 0]$$

(*b*) 12·5 refrigerators and 35 ranges, $Z_{max}=$ Rs. 2,150.)

30. A company produces two types of leather belts, say types A and B. Belt A is a superior quality and belt B is of a lower quality. Profits on the two types of belts are 40 and 30 paise per belt respectively. Each belt of type A requires twice as much time as required by a belt of type B. If all belts were of type B, the company could produce 1,000 belts per day. The supply of leather, however, is sufficient only for 800 belts per day. Belt A requires a fancy buckle and only 400 fancy buckles are available for this per day. For belt of type B, only 700 buckles are available per day. How should the company manufacture the two types of belts in order to have a maximum overall profit ?

[*Delhi M. Com. 1975*])

(*Ans.* $x_A=400$ belts, $x_B=400$ belts, $Z_{max}=$ Rs. 280

[**Hint.** Maximize $Z=0\cdot4x_A+0\cdot3x_B$

subject to
$$x_A+x_B \leqslant 800$$
$$x_A \leqslant 400$$
$$x_B \leqslant 700$$
$$x_A, x_B \geqslant 0.]$$

31. The ABC company wishes to plan its advertising strategy. There are two media under consideration, call them magazine I and II respectively. Magazine I has a reach of 2,000 potential customers and magazine II has reach of 3,000 potential customers. The cost per page of advertising is Rs. 400 and Rs. 600 in magazines I and II respectively. The firm has a monthly budget of Rs. 6,000. There is an important requirement that the total reach for the income group under Rs. 20,000 per annum should not exceed 4,000 potential customers. The reach in magazines I and II for this income group is 400 and 200 potential customers. How many pages should be bought in the two magazines to maximize the total reach ?

[*Delhi Dip. in Mkt. and Salesman, 1975*]

(*Ans.* Maximize $Z=2,000\ x_{I}+3,000\ x_{II}$

subject to $400x_I+600x_{II} \leqslant 6,000$

$400x_I+200x_{II} \leqslant 4,000$

$x_I, x_{II} \geqslant 0.$)

Section 2·5

32. Maximize $Z=2x_1+x_2$

subject to
$$\frac{3}{2}x_1+x_2 \leqslant 6$$
$$x_1 \leqslant 2$$
$$x_1+x_2 \geqslant 7$$
$$-x_1+x_2 \geqslant 4$$

and $x_1, x_2 \geqslant 0.$

(*Ans.* Solution does not exist.)

33. Maximize $Z=3x_1+4x_2$

subject to
$$x_1-x_2 \leqslant -1$$
$$-x_1+x_2 \leqslant 0$$
$$x_1, x_2 \geqslant 0.$$

[*Meerut M.Sc.* (***Math.***) ***1974***]

(*Ans.* Solution does not exist.)

34. Maximize $Z=8x_1+x_2$

subject to
$$8x_1+x_2 \leqslant 8$$
$$2x_1+x_2 \leqslant 6$$
$$3x_1+x_2 \leqslant 6$$
$$x_1+6x_2 \leqslant 8$$
$$x_1, x_2 \geqslant 0.$$

(*Ans.* Infinite number of optimal solutions exist.)

35. Maximize $Z=4x_1+5x_2$

subject to
$$x_1+x_2 \geqslant 1$$
$$-2x_1+x_2 \leqslant 1$$
$$4x_1-x_2 \leqslant 1$$
$$x_1, x_2 \geqslant 0.$$

(*Ans.* Unbounded solution)

36. A plant manufactures two products A and B. The profit contribution of each product has been estimated as Rs. 20 for product A and Rs. 24 for product B. Each product passes through three departments of the plant. The time required for each product and total time available in each department are as follows :

Table 2·82

Department	*Hours required* Product A	Product B	*Available hours during the month*
1	2	3	1,500
2	3	2	1,500
3	1	1	600

The company has a contract to supply at least 250 units of product B per month. Formulate the problem as a linear programming model and solve by graphical method.

[*Pb. Univ. Mech. Engg. 1978*]

(*Ans.* maximize $Z=20x_A+23x_B$

subject to $2x_A+3x_B \leqslant 1,500$
$$3x_A+2x_B \leqslant 1,500$$
$$x_A+x_B \leqslant 600$$
$$x_B \leqslant 250$$
$$x_A, x_B \geqslant 0.)$$

37. Solve exercise 4 by using graphical method.

[*Baroda Univ. B.E. May, 1975*]

(*Ans.* $x_1=10$, $x_2=20$ units, $Z_{max}=$Rs. 360.)

38. The sales manager of a company has budgeted Rs. 120,000 for an advertising programme for one of the firm's products. The selected advertising program consists of running advertisements in two different magazines. The advertisement for magazine 1 costs Rs. 2,000 per run while the advertisement for magazine 2 costs Rs. 5,000 per run. Past experience has indicated that at least 20 runs in magazine 1, and at least 10 runs in magazine 2 are necessary to penetrate the market with any appreciable effect. Also, experience has indicated that there is no reason to make more than 50 runs in either of the two magazines. How many runs in maga-

zine 1 and how many in magazine 2 should be made ?

(*Ans.* Magazine 1 : 20 to 35 runs ; Magazine 2 : 10 to 16 runs.)

39. A factory is to produce two products P_1 and P_2. The products require machining on two critical machines M_1 and M_2. Product P_1 requires 5 hrs. on machine M_1 and 3 hrs. on machine M_2. Product P_2 requires 4 hrs. on machine M_1 and 6 hrs. on M_2. Machine M_1 is available for 120 hrs. per week during regular working hours and 50 hrs. on overtime. Weekly machine hours on M_2 are limited to 150 hrs. on regular working hours and 40 hrs. on overtime. Product P_1 earns a unit profit of Rs. 8 if produced on regular time, Rs. 6 if produced on overtime on one machine and Rs. 4 if produced on overtime on both machines. Product P_2 earns a unit profit of Rs. 10 if produced on regular time, Rs. 8 if produced on overtime on both machines.

Formulate an L.P. model for designing an optimum production schedule for maximizing the profit and solve by graphical method.

[*Pb. Univ. B.Sc.* (*Prod.*) *Engg.* 1977]

Section 2·7

40. Reduce the following linear programming problem to the standard form :

Determine $x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0$

so as to maximize $A = 5x_1 + 3x_2 + 4x_2$

subject to the constraints
$$2x_1 - 5x_2 \leqslant 6$$
$$2x_1 + 3x_2 + x_3 \geqslant 5$$
$$3x_1 \quad + 4x_3 \leqslant 3.$$

41. Reduce the following linear programming problem to the standard form :

Determine $x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3$

so as to minimize $G = 2x_1 + x_2 + 3x_3$

subject to constraints
$$-5x_1 + 2x_2 \leqslant 5$$
$$3x_1 + 2x_2 + 4x_3 \geqslant 7$$
$$2x_1 \quad + 5x_3 \leqslant 3.$$

42. Put the following problem in the standard form :

Maximize $Z = 4x_1 + x_2 - 3x_3$

subject to
$$x_1 + 5x_2 - 3x_3 \leqslant 20$$
$$2x_1 + 7x_2 + 2x_3 \leqslant 10$$
$$x_1 - 5x_2 - 3x_3 \geqslant 3$$
$$x_1 \geqslant 0$$

43. Express the following problem in the standard form :

Minimize $Z = 2x_1 + 5x_2$

subject to
$$x_1 + 2x_2 - x_3 = -4$$
$$3x_1 + 4x_2 - x_4 = -7$$

$$x_1+2x_2 \quad +x_5 \quad =9$$
$$5x_1-2x_2 \quad +x_6 \quad =17$$
$$x_3, x_4, x_5, x_6 \geqslant 0.$$

Section 2·8

44. Obtain all the basic solutions to the following system of linear equations :

$$x_1+2x_2+x_3=4$$
$$2x_1+x_2+5x_3=5.$$

Which of them are basic feasible solutions and which are non-degenerate basic solutions ? Is the non-degenerate solution feasible ?

[*Meerut M.Sc. (Math.) 1974*]

(*Ans.* (*i*) $x_1=1$, $x_2=0$; $x_3=0$, $x_1=0$, $x_2=\frac{5}{3}$, $x_3=-\frac{1}{3}$; $x_1=1$, $x_2=0$, $x_3=0$.

(*ii*) $x_1=1$, $x_2=0$, $x_3=0$; $x_1=1$, $x_2=0$, $x_3=0$.

(*iii*) $x_1=0$, $x_2=\frac{5}{3}$, $x_3=-\frac{1}{3}$.

(*iv*) *No.*)

45. What do you mean by an optimal basic feasible solution to a linear programming problem ? Is the solution $x_1=1$, $x_2=\frac{1}{2}$, $x_3=x_4=x_5=0$ a basic solution of the equations

$$x_1+2x_2+x_3+x_4=2$$
and
$$x_1+2x_2+\tfrac{1}{2}x_3+x_5=2 \;?$$

[*Delhi B.Sc. (Math.) 1975*]

(*Ans.* No.)

46. Compute all the basic feasible solutions to the L. P. problem :

Maximize $Z=2x_1+3x_2+4x_3+7x_4$

subject to the constraints $2x_1+3x_2-x_3+4x_4=8$

$$x_1-2x_2+6x_3-7x_4=-3$$
$$x_1, x_2, x_3, x_4 \geqslant 0$$

and choose that one which maximizes Z.

[*Meerut M.Sc. (Math.) 1972, 77*]

(*Ans.* (*i*) $x_1=1$, $x_2=2$, $x_3=0$, $x_4=0$

(*ii*) $x_1=\frac{22}{9}$, $x_2=0$, $x_3=0$, $x_4=\frac{7}{9}$

(*iii*) $x_1=0$, $x_2=\frac{45}{16}$, $x_3=\frac{7}{16}$, $x_4=0$

(*iv*) $x_1=0$, $x_2=0$, $x_3=\frac{44}{17}$, $x_4=\frac{45}{17}$; $Z_{max}=\frac{491}{17}$.)

47. The following table gives the calorie values and the protein contents of five types of foods along with their costs. Find the amount of each type of food to be purchased in order to meet exactly the daily requirements of a person at minimum cost. Assume that a person, on the average, requires 3,000 calories and 100 grams of proteins.

Table 2·83

	Bread	*Meat*	*Potatoes*	*Cabbage*	*Milk*
Calories	2,500	3,000	600	100	600
Proteins (gms.)	80	150	20	10	40
Cost/kg (Rs.)	3	10	1	2	3

Also find

(*i*) basic feasible solutions

(*ii*) non-degenerate basic feasible solutions

(*iii*) optimal basic feasible solution.

(*Ans.* 1. $x_1=\frac{10}{9}$, $x_2=\frac{2}{27}$; $Z=4\frac{2}{27}$, other non-basic variables zero.)

2. $x_1=0$, $x_3=5$; $Z=5$, —do—

3. $x_1=\frac{20}{27}$, $x_4=\frac{10}{17}$; $Z=4\frac{12}{17}$, —do—

4. $x_1=\frac{15}{13}$, $x_5=\frac{5}{26}$; $Z=4\frac{1}{26}$, —do—

5. $x_2=0$, $x_3=5$; $Z=5$, —do—

6. $x_2=\frac{4}{3}$, $x_4=-10$; $Z=-6\frac{2}{3}$, —do—

7. $x_2=2$, $x_5=-5$; $Z=5$, —do—

8. $x_3=5$, $x_4=0$; $Z=5$, —do—

9. $x_3=5$, $x_5=0$; $Z=5$, —do—

10. $x_4=-30$, $x_5=10$; $Z=-30$, —do—

(*i*) No. 1, 2, 3, 4, 5, 8, 9.

(*ii*) No. 1, 3, 4.

(*iii*) No. 4; $x_1=\frac{15}{13}$, $x_2=0$, $x_3=0$, $x_4=0$, $x_5=\frac{5}{26}$; $Z_{max}=4\frac{1}{26}$.)

Solve the following problems by the simplex method.

Section 2·10

48. Maximize $Z=2x_1+x_2$

$$\text{subject to } x_1+2x_2 \leqslant 10$$
$$x_1+x_2 \leqslant 6$$
$$x_1-x_2 \leqslant 2$$
$$x_1-2x_2 \leqslant 1$$
$$x_1, x_2 \geqslant 0.$$

[*Meerut M.Sc. (Math.) 1971*]

(*Ans.* $x_1=4, x_2=2$; $Z_{max}=10.$)

49. Maximize $Z=3x_1+2x_2+5x_3$

$$\text{subject to } x_1+x_2+x_3 \leqslant 9$$
$$2x_1+3x_2+5x_3 \leqslant 30$$
$$2x_1-x_2-x_3 \leqslant 8$$
$$x_1, x_2, x_3 \geqslant 0.$$

[*Pb. Univ. (Prod.) Engg. April, 1977*]

50. Maximize $Z=5x_1+3x_2+7x_3$

$$\text{subject to } x_1+x_2+2x_3 \leqslant 22$$
$$3x_1+2x_2+x_3 \leqslant 26$$
$$x_1+x_2+x_3 \leqslant 18$$
$$x_1, x_2, x_3 \geqslant 0.$$

What will be the solution if the first restriction changes to $x_1+x_2+2x_2 \leqslant 26$. [*Pb. Univ. (Prod.) Engg. 1977*]

51. Maximize $Z=2x_1+x_2-3x_3+5x_4$

$$\text{subject to } x_1+7x_2+3x_3+7x_4 \leqslant 46$$
$$3x_1-x_2+x_3+2x_4 \leqslant 8$$
$$2x_1+3x_2-x_3+x_4 \leqslant 10$$
$$x_1, x_2, x_3, x_4 \geqslant 0.$$

$\left(\textit{Ans.} \quad x_1=0, x_2=\frac{12}{7}, x_3=0, x_4=\frac{34}{7} ; \; Z_{max}=26. \right)$

52. Maximize $Z=2x_1+4x_2+x_3+x_4$

$$\text{subject to } x_1+3x_2+x_4 \leqslant 4$$
$$2x_1+x_2 \leqslant 3$$
$$x_2+4x_3+x_4 \leqslant 3$$
$$x_j \geqslant 0 \; (j=1, 2, 3, 4).$$

(*Ans.* $Z_{max}=6{\cdot}5$)

53. Minimize $x_0 = x_1 - 3x_2 - 2x_3$

subject to
$$3x_1 - x_2 + 2x_3 \leqslant 7$$
$$-2x_1 + 4x_2 \leqslant 12$$
$$-4x_1 + 3x_2 + 8x_3 \leqslant 10$$
$$x_1, x_2, x_3 \geqslant 0.$$

[*Meerut M. Sc. (Math.)* 1972, 74

$\left(Ans. \quad \frac{78}{25}, \frac{114}{25}, \frac{11}{10}; x_{0\ min} = -\frac{319}{25}. \right)$

54. Maximize $Z = 3x_1 + 4x_2 + x_3 + 5x_4$

subject to
$$8x_1 + 3x_2 + 2x_3 + 2x_4 \leqslant 10$$
$$2x_1 + 5x_2 + x_3 + 4x_4 \leqslant 5$$
$$x_1 + 2x_2 + 5x_3 + x_4 \leqslant 6$$
$$x_1, x_2, x_3, x_4 \geqslant 0.$$

[*Meerut B. Sc. (Math.)* 1970]

$\left(Ans. \quad x_1 = \frac{15}{14}, x_2 = 0, x_3 = 0, x_4 = \frac{5}{7}; Z_{max} = \frac{95}{14}. \right)$

55. Maximize $Z = 2x_1 + 3x_2 + x_3 + 7x_4$

subject to
$$8x_1 + 3x_2 + 4x_3 + x_4 \leqslant 6$$
$$2x_1 + 6x_2 + x_3 + 5x_4 \leqslant 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 \leqslant 7$$
$$x_1, x_2, x_3, x_4 \geqslant 0.$$

[*Ans.* $Z_{max} = 4{\cdot}2$.)

56. Give an outline of the complex method for solving an L.P.P. and solve :

Maximize $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to constraints
$$x_1 + x_2 + x_3 + x_4 \leqslant 15$$
$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leqslant 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leqslant 100$$
$$x_j \geqslant 0 \ (j = 1, 2, 3, 4.)$$

[*Kuru. M. Sc. (Math.)* 1976]

$\left(Ans. \quad x_1 = \frac{50}{7}, x_2 = 0, x_3 = \frac{55}{7}, x_4 = 0; Z_{max} = \frac{695}{7}. \right)$

57. Show by simplex method that the following problem has an infinite number of solutions :

Minimize $Z = -40x_1 - 100x_2$

subject to
$$10x_1 + 5x_2 \leqslant 250$$
$$2x_1 + 5x_2 \leqslant 100$$
$$2x_1 + 3x_2 \leqslant 90$$
$$x_1, x_2 \geqslant 0.$$

(*Ans.* $Z_{min} = -2{,}000$.)

58. Show by simplex method that the following problem has an unbounded optimal solution :

Maximize $x_0=2x_1+x_2$

subject to
$$x_1-x_2 \leqslant 10$$
$$2x_1-x_2 \leqslant 40$$
$$x_1, x_2 \geqslant 0.$$

59. Show that the L.P.P.

maximize $Z=2x_1+3x_2+4x_3+x_4$

subject to
$$-x_1-5x_2-9x_3+6x_4 \leqslant 2$$
$$3x_1-x_2+x_3+3x_4 \leqslant 10$$
$$2x_1+3x_2-7x_3+8x_4 \leqslant 0$$
$$x_1, x_2, x_3, x_4 \geqslant 0,$$

has an unbounded solution.

Section 2·11·1

60. Use Charnes penalty method to

maximize $Z=3x_1-x_2$

subject to constraints
$$2x_1+x_2 \geqslant 2$$
$$x_1+3x_2 \leqslant 3$$
$$x_2 \leqslant 4$$
$$x_1, x_2 \geqslant 0.$$

[*Punjab M. Sc. (Math.) 1974*]

(*Ans.* $x_1=3, x_2=0$; $Z_{max}=9$)

61. Use M-technique to

minimize $Z=4x_1+x_2$

subject to
$$3x_1+x_2=3$$
$$4x_1+3x_2 \geqslant 6$$
$$x_1+2x_2 \leqslant 3$$
$$x_1, x_2 \geqslant 0.$$

$$\left(\textit{Ans.} \quad x_1=\frac{3}{5},\ x_2=\frac{6}{5}\ ;\ Z_{min}=\frac{18}{5}\ . \right)$$

62. Solve the following problem using big M method :

Maximize $Z=6x_1-3x_2+2x_3$

subject to
$$2x_1+x_2+x_3 \leqslant 16$$
$$3x_1+2x_2+x_3 \leqslant 18$$
$$x_2-2x_3 \geqslant 8$$
$$x_1, x_2, x_3 \geqslant 0.$$

[*Pb. Univ. Prod. Engg. Dec., 1978*]

63. Use Charnes penalty method to solve the problem :

Maximize $Z=4x_1+5x_2+2x_3$

subject to
$$2x_1+x_2+x_3 \leqslant 10$$
$$x_1+3x_2+x_3 \leqslant 12$$
$$x_1+x_2+x_3 = 6$$
$$x_1, x_2, x_3 \geqslant 0.$$

[*Pb. Univ. Prod. Engg. April, 1979*]

64. Solve the following problem by using x_4, x_5 and x_6 for the starting basic feasible solution :

Maximize $Z=3x_1+x_2+2x_3$

subject to
$$12x_1+3x_2+6x_3+3x_4=9$$
$$8x_1+x_2-4x_3+2x_5=10$$
$$3x_1-x_6=0$$
$$x_1, x_2, \ldots, x_6 \geqslant 0.$$

$$\left(\textit{Ans.} \quad x_1=x_2=x_4=x_6=0,\ x_3=\frac{3}{2},\ x_5=8;\ Z_{max}=3. \right)$$

[**Hint.** Divide first constraint equation by 3, second by 2 and third by -1].

65. Minimize $Z=2x_1+9x_2+x_3$

subject to the constraints
$$x_1+4x_2+2x_3 \geqslant 5$$
$$3x_1+x_2+2x_3 \geqslant 4$$
$$x_1, x_2, x_3 \geqslant 0.$$

[*Meerut M.Sc. (Stat.) 1972*]

$$\left(\textit{Ans.} \quad x_1=0,\ x_2=0,\ x_3=\frac{5}{2};\ Z_{max}=\frac{5}{2}. \right)$$

Section 2·11-2.

66. Use two phase method to minimize $Z=x_1+x_2$

subject to
$$2x_1+x_2 \geqslant 4$$
$$x_1+7x_2 \geqslant 7$$
$$x_1, x_2 \geqslant 0.$$

[*Kuru. M.Sc. (Math.) 1975, 77*]

$$\left(\textit{Ans.} \quad x_1=\frac{21}{13},\ x_2=\frac{10}{13},\ Z_{max}=\frac{31}{13}. \right)$$

67. Solve the following linear programming problem, using the two phases of the simplex method :

Minimize $Z=2x_1+x_2$

subject to
$$5x_1+10x_2-x_3=8$$
$$x_1+x_2+x_4=1$$

and
$$x_1, x_2, x_3, x_4, \text{ all} \geqslant 0.$$

$$\left(\textit{Ans.} \quad x_1=0,\ x_2=\frac{4}{5},\ x_5=\frac{1}{5};\ Z_{min}=\frac{4}{5}. \right)$$

68. Using two phase method,
maximize $Z=5x-2y+3z$
subject to
$$2x+2y-z \geqslant 2$$
$$3x-4y \leqslant 3$$
$$y+3z \leqslant 5$$
where
$$x, y, z \geqslant 0$$
$$\left(Ans. \ x=\frac{23}{3}, y=5, z=0 \ ; \ Z_{max}=\frac{85}{3}. \right)$$

69. Minimize $Z=\frac{15}{2}x_1-3x_2$

subject to the constraints
$$3x_1-x_2-x_3 \geqslant 3$$
$$x_1-x_2+x_3 \geqslant 2$$
$$x_1, x_2, x_3 \geqslant 0.$$
Use the two-phase method.

[*Roorkee M.Sc. (Math.) 1974*]
$$\left(Ans. \ x_1=\frac{5}{4}, x_2=0, x_3=\frac{3}{4}; \ Z_{min}=\frac{75}{8}. \right)$$

70. Minimize $Z=2x_1+3x_2+2x_3-x_4+x_5$
subject to the constraints
$$3x_1-3x_2+4x_3+2x_4-x_5=0$$
$$x_1+x_2+x_3+3x_4+x_5=2$$
and
$$x_j \geqslant 0, j=1 \text{ to } 5.$$
$$\left(Ans. \ x_1=x_2=x_3=0, x_4=\frac{2}{5}, x_5=\frac{4}{5}; \ Z_{min}=\frac{2}{5}. \right)$$

71. A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below.

Table 2·84

Machine	*Time per unit (minutes)*			*Machine capacity (minutes/day)*
	Product 1	*Product 2*	*Product 3*	
M_1	2	3	2	440
M_2	4	—	3	470
M_3	2	5	—	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for products 1, 2 and 3 is Rs. 4, Rs. 3 and Rs. 6 respectively. It is assumed that all the amounts produced are consumed in the market.

[*Pb. Univ. Mech. Engg. 1977, 79*]

$$\left(\textit{Ans.} \quad x_1=0,\ x_2=\frac{380}{7},\ x_3=\frac{470}{3}\ ;\ Z_{max}=\text{Rs. } \frac{3,200}{3}\ . \right)$$

72. A company has the option of using one or more of different types of production processes. The first and second processes result in product P_1, while third and fourth result in product P_2. For each process, input (resources) are

(*i*) labour measured in man-weeks

(*ii*) kg. of materials, K

(*iii*) boxes of materials, B

As each process varies in its input requirements, the profits obtained for each process are different even for processes producing the same item. The amounts of manpower and both kinds of materials are limited. The data is given below in table 2·85.

Table 2·85

Item	*One unit of product P_1*		*One unit of product P_2*		*Total available resources*
	Process 1	*Process 2*	*Process 3*	*Process 4*	
Manweeks	1	1	1	1	15 (max.)
Kg. of materials, K	7	5	3	3	120 (max.)
Boxes of materials, B	3	5	10	15	100 (max.)
Unit profit (Rs.)	4	5	9	11	

What is the optimum production level of each process to get maximum profit ?

$$\left(\textit{Ans.} \quad x_1=\frac{50}{7},\ x_2=0,\ x_3=\frac{55}{7},\ x_4=0;\ Z_{max}=\text{Rs. } \frac{695}{7}. \right)$$

73. A firm manufactures three products P_1, P_2 and P_3. The minimum number of units of P_1, P_2 and P_3 that must be produced are 100, 200 and 150 respectively. These products require two types of raw materials M_1 and M_2 which the firm can purchase upto a maximum of 500 and 400 units respectively. Design a production plan so as to maximize the profit if the respective individual profits of P_1, P_2 and P_3 are Rs. 2, Rs. 5 and Rs. 4 respectively. Consumption of raw materials is shown below.

Table 2·86

Raw material	*Consumption of raw material per unit product*		
	P_1	P_2	P_3
M_1	$\frac{1}{2}$	1	1
M_2	2	$\frac{1}{2}$	$\frac{1}{5}$

(*Ans.* $x_1=100, x_2=300, x_3=150.$)

[**Hint.** Maximize $Z=2x_1+5x_2+4x_3$

subject to $$\frac{x_1}{2}+x_2+x_3 \leqslant 500$$

$$2x_1+\frac{x_2}{2}+\frac{x_3}{5} \geqslant 400$$

$$x_1 \geqslant 100$$

$$x_2 \geqslant 200$$

$$x_3 \geqslant 150.]$$

74. A small scale industrialist produces four types of machine components M_1, M_2, M_3 and M_4 made of steel and brass. The amounts of steel and brass required for each component and the number of man-weeks of labour required to manufacture and assemble one unit of each component are as follows :

Table 2·87

	M_1	M_2	M_3	M_4	*Availability*
Steel	6	5	3	2	100 kg
Brass	3	4	9	2	75 kg
Man-weeks	1	2	1	2	20

The labour is restricted to 20 man-weeks, steel is restricted to 100 kg per week and brass to 75 kg per week. The industrialist's profit on each unit of M_1, M_2, M_3 and M_4 is Rs. 6, Rs. 4, Rs. 7 and Rs. 5 respectively. How many of each type of machine components should he produce to maximize his profit and how much is his profit ?

[*Pb. Univ. Mech. Engg. 1978*]

$$\left(\textit{Ans.}\quad M_1=14,\ M_2=0,\ M_3=\frac{10}{3},\ M_4=0\ ;\quad Z_{max}=\text{Rs. } \frac{340}{3} \text{ per week.}\right)$$

75. A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M_1, M_2 and M_3. Belt A requires 2 hours on machine M_1 and 3 hours on machine M_3. Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 and belt C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2 and 15 hours of time per day available on machine M_3. The profit gained from belt A is Rs. 3·00 per unit, from belt B is Rs. 5·00 per unit and from belt C is Rs. 4·00 per unit. What should be the daily production of each type of belt so that the profit is maximum ?

[*Pb. Univ. Mech. Engg. 1979*]

$$\left(\textit{Ans.}\quad A=\frac{89}{41},\ B=\frac{50}{41} \text{ and } C=\frac{62}{41}.\right)$$

76. A company produces four products A, B, C and D. Raw material requirements, storage space needed, production rates and profits are given in the table below. The total amount of raw material available per day for all four products is 180 kg. Total space available for storage is 230 square metre and 7 hours/day is used for production.

Table 2·88

	A	*B*	*C*	*D*
Raw material (kg/piece)	2	2	1·5	4
Space (metre^2/piece)	2	2·5	2	1·5
Production rate (pieces/hr.)	15	30	10	15
Profit (Rs./piece)	5	6·5	5	5·5

How many units of each product should be produced to maximize total profit ?

[*Pb. Univ. Mech. Engg. 1977*]

77. A food products company is considering to market three products A, B and C. The firm has three basic manufacturing departments : mixing, frying and packing. The time required for each product and the total available hours are as given below.

Table 2·89

Product	*Mixing deptt. (minutes)*	*Frying deptt. (minutes)*	*Packing deptt. (minutes)*
A	6	12	6
B	12	24	6
C	24	12	6
Available monthly hours	5,000	5,500	4,500

The profit contribution of products A, B and C is expected to be Rs. 3, Rs. 4 and Rs. 5 respectively. Using simplex method determine the optimum quantity for each product and the total profits per month.

[*Pb. Univ. Mech. Engg. 1978*]

78. A factory manufactures three products, which are processed through three different stages. The time required to manufacture one unit of the three products and the daily capacity of the stages are given by the following table :

Table 2·90

Stage	Time/unit (minutes)			Stage capacity (mintues/day)
	Product 1	Product 2	Product 3	
1	1	2	1	430
2	3	—	2	460
3	1	4	—	420

The profit/unit for product 1, 2 and 3 are 3, 2 and 5 monetary units respectively. Find the daily number of units to be manufactured of each product.

[*Pb. Univ. Mech. Engg. 1978*]

79. A manufacturer produces three products A, B and C. Each product can be produced on either one of the two machines I and II. The time required to produce 1 unit of each product on a machine is

Table 2·91

Product	Time to produce one unit (hrs.)	
	Machine I	Machine II
A	0·5	0·6
B	0·7	0·8
C	0·9	1·05

There are 85 hours available on each machine. The operating cost is Rs. 5/hr for machine I and Rs. 4/hr for machine II. The market requirements are at least 90 units of A, at least 80 units of B and at least 60 units of C. The manufacturer wishes to meet the requirements at minimum cost. Solve the problem by simplex method.

[*Delhi M.B.A. 1975*]

[**Hint** : Minimize $Z=(0\cdot5\times5+0\cdot6\times4)x_A+(0\cdot7\times5+0\cdot8\times4)x_B+(0\cdot9\times5+1\cdot05\times4)x_C$

$$=4\cdot9x_A+6\cdot7x_B+8\cdot7x_C$$

subject to
$$0\cdot5x_A+0\cdot7x_B+0\cdot9x_C \leqslant 85$$
$$0\cdot6x_A+0\cdot8x_B+1\cdot05\,x_C \leqslant 85$$
$$x_A \geqslant 90$$
$$x_B \geqslant 80$$
$$x_C \geqslant 60.]$$

80. A factory has decided to diversify its activities. The data collected by sales and production departments is summarised below.

Potential demand exists for three products A, B and C. Market can take any amount of A and C whereas the share of B for this organisation is expected to be not more than 400 units a month.

For every three units of C produced, there will be one unit of a by-product which sells at a contribution of Rs. 3 a unit and only 100 units of this by-product can be sold per month. Contribution per unit of products A, B and C is expected to be Rs. 6, Rs. 8 and Rs. 4 respectively.

These products require three different processes and the time required per unit product is given in the table below.

Table 2·92

Process	*Hours/unit*			*Available hours*
	Product A	*Product B*	*Product C*	
I	2	3	1	900
II	—	1	2	600
III	3	2	2	1,200

Determine the optimum product mix for maximizing the contribution.

[*Bombay Dip. Ind. Man. 1974*]

[**Hint :** Maximize

$$Z=6x_A+8x_B+4x_C+\frac{x_C}{3}\times 3$$

$$=6x_A+8x_B+5x_C$$

subject to

$$2x_A+3x_B+x_C \leqslant 900$$

$$x_B+2x_C \leqslant 600$$

$$3x_A+2x_B+2x_C \leqslant 1{,}200$$

$$x_B \leqslant 400$$

$$\frac{x_C}{3} \leqslant 100$$

$$\text{or } x_C \leqslant 300$$

$$x_A, x_B, x_C \geqslant 0.]$$

81. A company manufactures products A, B, C and D which are processed by planner, milling, drilling and assembly departments. The requirements per unit of product in hours and contribution are as follows :

Table 2·93

	Department				*Contribution/unit*
	Planner	*Milling*	*Drilliny*	*Assembly*	
Product A	0·5	2·0	0·5	3·0	Rs. 8
Product B	1·0	1·0	0·5	1·0	Rs. 9
Product C	1·0	1·0	1·0	2·0	Rs. 7
Product D	0·5	1·0	1·0	3·0	Rs. 6

Capacities of various departments and minimum sales requirements are

Table 2·94

Department	*Capacity (hours)*	*Minimum sales requirements*	
Planner	1,800	Product A	100 units
Milling	2,800	Product B	600 units
Drilling	3,000	Product C	500 units
Assembly	6,000	Product D	400 units

(*a*) Determine the number of products A, B, C and D to be manufactured to maximize production.

(*b*) Determine the total maximum contribution for products A, B, C and D.

(*c*) Determine the slack time in each department.

[*Pb. Univ. M.Sc. (Math.) 1975*]

[**Hint** : Maximize

$$Z=8x_1+9x_2+7x_3+6x_4$$

subject to

$$0{\cdot}5x_1+x_2+x_3+0{\cdot}5x_4 \leqslant 1{,}800$$

$$2x_1+x_2+x_3+x_4 \leqslant 2{,}800$$

$$0{\cdot}5x_1+0{\cdot}5x_2+x_3+x_4 \leqslant 3{,}000$$

$$3x_1+x_2+2x_3+3x_4 \leqslant 6{,}000$$

$$x_1 \geqslant 100$$

$$x_2 \geqslant 600$$

$$x_3 \geqslant 500$$

$$x_4 \geqslant 400.]$$

Section 2·12

82. Maximize $Z=8x_2$

subject to $x_1-x_2 \geqslant 0$

$$2x_1+3x_2 \leqslant -6$$

x_1, x_2 unrestricted.

83. Maximize $Z = 2x_1 + x_2 + 4x_3$

subject to
$$-2x_1 + 4x_2 \leqslant 4$$
$$x_1 + 2x_2 + x_3 \geqslant 5$$
$$2x_1 \quad + 3x_3 \leqslant 2$$
$$x_1, x_2 \geqslant 0, \; x_3 \text{ unrestricted.}$$

84. Maximize $Z = 2x_1 - 2x_2 + 3x_3$

subject to
$$2x_1 + 3x_2 - x_3 \leqslant 30$$
$$3x_1 - 2x_2 + x_3 \leqslant 24$$
$$x_1 - 4x_2 - 6x_3 \geqslant 2$$
$$x_1 \geqslant 0.$$

3

Transportation Model

As stated in previous sections, simplex algorithm can be used to solve any linear programming model. But this algorithm is laborious. For this reason, wherever possible, we try to simplify the calculations. One such model requiring simplified calculations is called *transportation model*. The model is applicable to that subclass of linear programming problems in which resources and requirements are expressed in terms of only one kind of unit. The name of this model is derived from transport to which it was first applied. The name "transportation model" is, however, misleading. This model can be used for machine assignment, plant location, product mix problems and many others, so that the model is really not confined to transportation or distribution. We shall present now a few industrial situation which will establish the relevance and utility of this model.

3·1. Examples on the Applications of Transportation Model

EXAMPLE 3.1-1 (Transportation Problem) :

A dairy firm has three plants located throughout a state. Daily milk production at each plant is as follows :

plant 1......6 million litres

plant 2......1 million litres

plant 3......10 million litres

Each day the firm must fulfil the needs of its four distribution centres. Minimum requirement at each centre is as follows :

distribution centre 1......7 million litres

distribution centre 2......5 million litres
distribution centre 3......3 million litres
distribution centre 4......2 million litres

Cost of shipping one million litres of milk from each plant to each distribution centre is given in the following table in hundreds of rupees :

Table 3·1

		Distribution Centres			
		1	2	3	4
	1	2	3	11	7
Plants	2	1	0	6	1
	3	5	8	15	9

The dairy firm wishes to decide as to how much should be the shipment from which plant to which distribution centre so that the cost of shipment may be minimum.

EXAMPLE 3·1-2: (Transportation Problem with Degeneracy):

A company has four warehouses and six stores. The warehouses altogether have a surplus of 22 units of a given commodity, divided among them as follows :

Warehouses	1	2	3	4
Surplus	5	6	2	9

The six stores altogether need 22 units of commodity. Individual requirements at stores 1, 2, 3, 4, 5 and 6 are 4, 4, 6, 2, 4 and 2 units respectively. Cost of shipping one unit of commodity from warehouse i to store j in rupees is given in the matrix below.

Table 3·2

		Stores					
		1	2	3	4	5	6
	1	9	12	9	6	9	10
Warehouses	2	7	3	7	7	5	5
	3	6	5	9	11	3	11
	4	6	8	11	2	2	10

How should the products be shipped from the warehouses to the stores so that the transportation cost is minimum ?

EXAMPLE 3.1-3. (Unbalanced Supply and Demand) :

A product is produced by four factories A, B, C and D. The unit production costs in them are Rs. 2, Rs. 3, Re. 1 and Rs. 5 respectively. Their production capacities are, phase A—50 units, B—70 units, C—30 units and D—50 units. These factories supply the product to four stores, demands of which are 25, 35, 105 and 20 units respectively. Unit transport cost in rupees from each factory to each store is given in the table below.

Table 3·3

		Stores 1	2	3	4
Factories	A	2	4	6	11
	B	10	8	7	5
	C	13	3	9	12
	D	4	6	8	3

Determine the extent of deliveries from each of the factories to each of the stores so that the total production and transportation cost is minimum.

[*Pb. Univ. Mech. Engg. April, 1977*]

EXAMPLE 3.1-4 (Profit Maximization Problem) :

A company manufacturing air-coolers has two plants located at Bombay and Calcutta with a capacity of 200 units and 100 units per week respectively. The company supplies the air-coolers to its four show rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a maximum demand of 75, 100, 100 and 30 units respectively. Due to the differences in raw material cost and transportation cost, the *profit* per unit in rupees differs which is shown in the table below.

Table 3·4

	Ranchi	*Delhi*	*Lucknow*	*Kanpur*
Bombay	90	90	100	100
Calcutta	50	70	130	85

Plan the production programme so as to ***maximize the profit.*** The company may have its production capacity at both plants partly or wholly unused.

EXAMPLE 3.1-5 (Least-time Transportation Model) :

A military equipment is to be transported from three origins to four destinations. The supply at the origins, the demand at the destinations and *time* of shipment is shown in the table below. The units to be shipped as obtained by North-West corner rule are given in parentheses. Work out a transportation plan so that the total *time* required for shipment is minimum.

Table 3·5

Destinations

Origins	1	2	3	4	a_i
1	10 (12)	0 (3)	20	11	15
2	1	7 (5)	9 (15)	20 (5)	25
3	12	14	16	18 (5)	5
b_j	12	8	15	10	45 (*Total*)

EXAMPLE 3.1-6 (Problem on Placement of Orders on Machines) :

Three machines can produce four different products. The machines, however, differ in type and degree of automation. The time required to produce these four products differs for each machine. The total available times per month are

machine 1 ...320 hrs.

machine 2 ...390 hrs.

and machine 3 ...375 hrs.

The products to be manufactured are

product A—1,500 units, product B—1,800 units, product C—2,100 units and product D—2,250 units. Some machines cannot produce certain products due to their technical features. The table below shows the number of units produced in one hour by each machine.

Table 3·6

Machines

Units	1	2	3
A	7·5	10·0	8·0
B	9·0	12·0	9·6
C		6·0	
D		9·0	7·2

The selling price for each unit is $ 2.45 for A, $ 2.40 for B, $ 2.25 for C and $ 9.10 for D. The production costs are

product A : $ 0.83, $ 0.91 and $ 0.87 for machines 1, 2 and 3 respectively.

product B : $ 0.79, $ 0.93 and $ 0.91 for machines 1, 2 and 3 respectively.

product C : $ 0.60 for machine 2.

product D : $ 0.81 and $ 0.82 for machines 2 and 3 respectively.

Schedule and assign the appropriate work load to the three machines so as to maximize the total contribution (profit).

3.2. Introduction to the Model

The origin of transportation models dates back to 1941 when F. L. Hitchcock presented a study entitled 'The Distribution of a Product from Several Sources to Numerous Localities.' The presentation is

regarded as the first important contribution to the solution of transportation problems. In 1947, T.C. Koopmans presented a study called 'Optimum Utilization of the Transportation System'. These two contributions are mainly responsible for the development of transportation models which involve a number of shipping sources and a number of destinations. Each shipping source has a certain capacity and each destination has certain requirements associated with a certain cost of shipping from the sources to the destinations. The object is to minimize the cost of transportation while meeting the requirements at the destinations.

3.3. Matrix Terminology

The matrix used in the transportation models consists of squares called 'cells', which when stacked form 'columns' vertically and 'rows' horizontally.

Table 3.7

		Warehouses			
	1	2	3	4	*Output*
A	2	3	11	4	15
Plant B	5	6	8	7	20
Demand	10	5	12	8	35 (*Total*)

The cell located at the intersection of a row and a column is designated by its row and column headings. Thus the cell located at the intersection of row A and column 3 is called cell (A, 3). Unit costs are placed in each cell.

3.4 Definition of Transportation Model

Transportation models deal with problems concerning as to what happens to the effectiveness function when we associate each of number of origins (sources) with each of a possibly different number of destinations (jobs). The total movement from each origin and the total movement to each destination is given and it is desired to find how the associations be made subject to the limitations on totals. In such problems, sources can be divided among the jobs or jobs may

be done with a combination of sources. The distinct feature of transportation problems is that sources and jobs must be expressed in terms of only one kind of unit.

Suppose that there are m sources and n destinations. Let a_i be the number of supply units available at source $i (i=1, 2, 3, \ldots, m)$ and let b_j be the number of demand units required at destination j ($j=1, 2, 3 \ldots, n$). Let c_{ij} represent the per unit transportation cost for transporting the units from source i to destination j. The objective is to determine the number of units to be transported from source i to destination j so that the total transportation cost is minimum. In addition, the supply limits at the sources and the demand requirements at the destinations must be satisfied exactly.

If x_{ij} ($x_{ij} \geqslant 0$) is the number of units shipped from source i to destination j, then the equivalent linear programming model will be

Find x_{ij} ($i=1, 2, 3, \ldots, m$; $j=1, 2, 3, \ldots, n$) in order to

$$\text{minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} = a_i, \qquad i=1, 2, 3, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j=1, 2, 3, \ldots, n.$$

where $x_{ij} \geqslant 0$

The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

This restriction causes one of the constraints to be redundant (and hence it can be deleted) so that the problem will have $(m+n-1)$ constraints and $(m \times n)$ unknowns. The above information can be put in the form of a general matrix shown below.

Table 3·8

Jobs to be done

		j_1	$j_2 \ldots j_j \ldots j_n$	*Supply*
	R_1	c_{11}	$c_{12} \ldots c_{1j} \ldots c_{1n}$	a_1
	R_2	c_{21}	c_{22} c_{2j} c_{2n}	a_2
Sources	$\vdots$	$\vdots$		$\vdots$
	R_i	c_{i1}	c_{i2} c_{ij} c_{in}	a_i
	$\vdots$	$\vdots$		$\vdots$
	R_m	c_{m1}	c_{m2} c_{mj} c_{mn}	a_m
Demand		b_1	$b_2 \ldots b_j \ldots \ldots b_n$	

3.5. Formulation and Solution of Transportation Models

In this section we shall consider a few examples which will make clear the technique of formulation and solution of transportation models.

EXAMPLE 3.5-1

For example 3.1-1 show that it represents a network situation. Also find the optimal shipment policy.

Solution. Let us represent the example graphically :

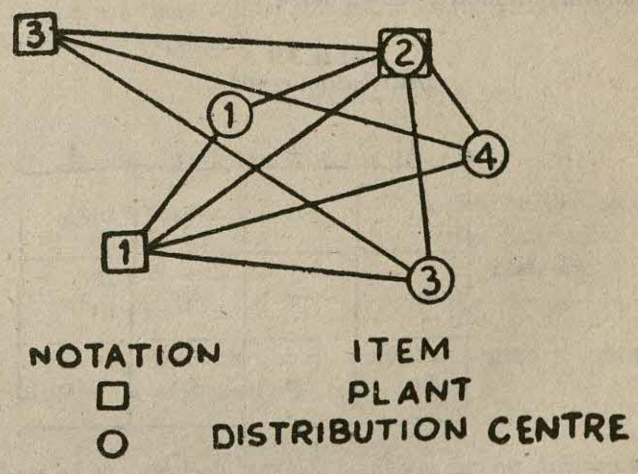

Figure 3.1.

We find that the above situation takes the shape of a network.

Similar situations occur whenever raw materials, semi-finished materials and finished products have to be transported. For example, three blade manufacturing plants owned by a company may come across a problem of finding the minimum transportation cost when transporting blades from these plants to a large number of distribution centres in the country. Again, a mining company may have to make a similar decision while transporting ore from mines to process-

ing plants (say six) and then to ship steel to distribution centres (say 20). Shipping crude oil from oil fields to refineries and then shipping petroleum products to, say, 200 distribution centres is another similar complex situation.

FORMULATION OF MODEL

Step 1 :

Key decision to be made is to find how much quantity of milk from which plant to which distribution centre be shipped so as to satisfy the constraints and minimize the cost. Thus the variables in the situation are : x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33} and x_{34}. These variables represent the quantities of milk to be shipped from different plants to different distribution centres and can be represented in the form of a matrix shown below.

Table 3.9
Distribution centres

Plants	1	2	3	4
1	x_{11}	x_{12}	x_{13}	x_{14}
2	x_{21}	x_{22}	x_{23}	x_{24}
3	x_{31}	x_{32}	x_{33}	x_{34}

In general, we can say that the key decision to be made is to find the quantity of units from each origin to each destination. Thus if there are m origins and n destinations, then x_{ij} are the decision variables (quantities to be found), where

$$i=1, 2, 3, \ldots, m$$

and $$j=1, 2, 3, \ldots, n.$$

Step 2 :

Feasible alternatives are sets of values of x_{ij} where $x_{ij} \geqslant 0$.

Step 3 :

Objective is to minimize the cost of transportation.

i.e., minimize $2x_{11}+3x_{12}+11x_{13}+7x_{14}$

$$+\ x_{21}+0x_{22}+6x_{23}+x_{24}$$
$$+5x_{31}+8x_{32}+15x_{33}+9x_{34}$$

In general, we can say that if c_{ij} is the unit cost of shipping from ith source to jth destination, the objective is

$$\text{minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} \left(= \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}\, x_{ij} \right)$$

Step 4 : *Constraints* are

(*i*) because of availability or supply

$x_{11}+x_{12}+x_{13}+x_{14}=6$	**(for milk plant 1)**
$x_{21}+x_{22}+x_{23}+x_{24}=1$	**(for milk plant 2)**
$x_{31}+x_{32}+x_{33}+x_{34}=10$	**(for milk plant 3)**

Thus, in all, there are 3 constraints (equal to the number of plants).

In general, there will be m constraints if number of origins is m, which can be expressed as

$$\sum_{j=1}^{n} x_{ij}=s_i,\ i=1, 2, 3 \ldots m.$$

(*ii*) because of requirements or demand

$x_{11}+x_{21}+x_{31}=7$	**(for distribution centre 1)**
$x_{12}+x_{22}+x_{32}=5$	**(for distribution centre 2)**
$x_{13}+x_{23}+x_{33}=3$	**(for distribution centre 3)**
$x_{14}+x_{24}+x_{34}=2$	**(for distribution centre 4)**

In general, there are n constraints if the number of destinations is n, which can be expressed as

$$\sum_{i=1}^{m} x_{ij}=D_j,\ j=1, 2, 3, \ldots, n.$$

Thus we find that the given situation involves $(3\times 4=12)$ variables and $(3+4=7)$ constraints. *In general,* such a solution will involve $(m \times n)$ variables and $(m+n)$ constraints.

It can be easily seen that in this model the objective function as well as the constraints are linear functions of the variables and therefore the model can be solved by simplex method. However, as a large number of variables are involved, many times computation will be required which may even exceed the capacity of an electronic computer. Again in the transportation situations, the general requirement is minimization of the objective function whereas simplex method was more suitable for maximization problems.

Note that the coefficients of x_{ij} in the constraints are all unity. Such a model called transportation model can be solved by transportation technique which is easier and shorter than simplex technique.

SOLUTION OF THE TRANSPORTATION MODEL

The solution involves making a transportation model (in the form of a matrix), finding a feasible solution, performing optimality test and iterating towards optimal solution if required.

Step 1 : Make a Transportation Model

This consists in expressing supply from origins, requirements at destinations and cost of shipping from origins to destinations in the form of a matrix shown below. A check is made to balance the supply and requirements.

Table 3·10

Distribution centres (Destinations)

	1	2	3	4	Supply
1	2	3	11	7	6
(*Plants Origins*) 2	1	0	6	1	1
3	5	8	15	9	10
Requirements	7	5	3	2	17 (Total)

Total supply $=6+1+10=17$

Total requirement $=7+5+3+2=17$

Such a problem is said to be "self contained" problem. Thus in the given example, supply and requirements are balanced. Whenever it is not so, a dummy origin or destination (as the case may be) is created to balance the supply and requirements. For example, observe the following situations.

	1	2	3	4	Supply
1	2	3	11	7	6
(A) 2	1	0	6	1	1
3	5	8	15	9	10
Requirement	7	5	5	2	

Total supply $=17$

Total requirement $=19$

(requirement is more)

(B)

	1	2	3	4	Supply
1	2	3	11	7	8
2	1	0	6	1	1
3	5	8	5	9	10
Requirement	7	5	3	2	

Total supply = 19

Total requirement = 17
(supply is more)

For case (A), a dummy origin can be created as shown below.

	1	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10 = 19
D	0	0	0	0	2
Requirement	7	5	5	2	

= 19

Since nothing can be produced at the dummy origin, nothing can be transported. This is represented in the matrix by associating zero transportation cost from such dummy origin to the different destinations.

For case (B), we can creat a fictitious or dummy destination as shown below.

	1	2	3	4	*d*	Supply
1	2	3	11	7	0	8
2	1	0	6	1	0	1 = 19
3	5	8	15	9	0	10
Requirement	7	5	3	2	2	

= 19

In this case the cost coefficients are zero, since the surplus quantity remains lying in the respective plants and is not shipped at all.

Step II : Find a Basic Feasible Solution

A *feasible solution*, in transportation technique, is said to exist if sum of origin supplies equals the sum of destination requirements *i.e.*, if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. A feasible solution is said to be *basic* if it satisfies $m+n-1$ constraints *i.e.*, if it contains at most $m+n-1$ positive allocations, otherwise the solution will *degenerate*. If a feasible solution (not necessarily basic) also minimizes the total cost, it is said to be an *optimal feasible*, in this case, a *minimal feasible* solution. This can be easily obtained by applying a technique which has been developed by Dantzig and which Charnes and Cooper refer to as "the north-west corner rule". Other methods for finding the initial feasible solutions are also described. In all these techniques it is assumed at the beginning that the transportation table is blank *i.e.*, initially all $x_{ij}=0$.

(1) North-West Corner Rule or Stepping Stone Method

This rule may be stated as follows :

(*i*) Start in the north-west (upper left) corner of the requirements table *i.e.*, the transportation matrix framed in step I and compare the supply of plant 1 (call it S_1) with the requirement of distribution centre 1 (call it D_1).

(*a*) If $D_1 < S_1$ *i.e.*, if the amount required at D_1 is less than the number of units available at S_1, set x_{11} equal to D_1 and proceed to cell (1, 2) (i.e., proceed horizontally).

Table 3.11

Distribution centres

	1	2	3	4	
1	2 (6)	3	11	7	6
Plants 2	1 (1)	0	6	1	1
3	5	8 (5)	15 (3)	9 (2)	10
	7	5	3	2	

(*b*) If $D_1 = S_1$, set x_{11} equal to D_1 and proceed to cell (2, 2) (i.e., proceed diagonally).

(*c*) If $D_1 > S_1$, set x_{11} equal to S_1 and proceed to cell (2, 1). (i.e., proceed vertically).

(*ii*) Continue in this manner, step by step, away from the northwest corner until, finally, a value is reached in the south-east corner.

Thus in the present example (see table 3.11), one proceeds as follows :

(*i*) set x_{11} equal to 6, namely, the smaller of the amount available at S_1 (6) and that needed at D_1 (7) and

(*ii*) proceed to cell (2, 1) (rule *a*). Compare the number of units available at S_2 (namely 1) with the amount required at D_1 (1) and accordingly set $x_{21} = 1$.

(*iii*) proceed to cell (3, 2) (rule *b*). Now supply from plant S_3 is 10 units while the demand for D_2 is 5 units ; accordingly set x_{32} equal to 5.

(*iv*) proceed to cell (3, 3) (rule *a*) and allocate 3 there.

(*v*) proceed to cell (3, 4) (rule *a*) and allocate 2 there.

It can be easily seen that the proposed solution is a feasible solution since all the supply and requirement constraints are fully satisfied. In this method, allocations have been made to various cells without any consideration of the cost of transportation associated with them. Hence the solution obtained may not be the best (most economical) solution. The transportation cost associated with this solution is

$$Z = \text{Rs. } (2 \times 6 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2) \times 100$$
$$= \text{Rs. } (12 + 1 + 40 + 45 + 18) \times 100$$
$$= \text{Rs. } 11{,}60.$$

Note that for any cell in which no allocation is made, the corresponding x_{ij} is equal to zero. A cell that gets an allocation is called a *basic cell.*

(2) Row Minima Method

This method consists in allocating as much as possible in the lowest cost cell of the first row so that either the capacity of the first plant is exhausted or the requirement at *j*th distribution centre is satisfied or both. Three cases arise :

(*i*) if the capacity of the first plant is completely exhausted, cross off the first row and proceed to the second row.

(*ii*) if the requirement at *j*th distribution centre is satisfied. cross off the *j*th column and reconsider the first row with the remaining capacity.

(*iii*) if the capacity of the first plant as well as the requirement at *j*th distribution centre are completely satisfied, make a zero allocation in the second lowest cost cell of the first row. Cross off the row and move down to the second row.

Continue the process for the resulting reduced transportation table until all the *rim conditions* (supply and requirement conditions) are satisfied.

Table 3·12

Distribution centres

		1	2	3	4	Supply
	1	2 (6)	3	11	7	~~6~~
Plants	2	1	0 (1)	6	1	~~1~~
	3	5 (1)	8 (4)	15 (3)	9 (2)	~~10~~
		~~7~~ ~~1~~	~~5~~ ~~4~~	~~3~~	~~2~~	

In this problem, we first allocate to cell (1, 1) in the first row as it contains the minimum cost 2. We allocate min. (6, 7)=(6) in this cell. This exhausts the supply capacity of plant 1 and thus the first row is crossed off. The next allocation, in the resulting reduced matrix is made in cell (2, 2) of row 2 as it contains the minimum cost 0 in that row. We allocate min. (1, 5)=(1) in this cell. This exhausts the supply capacity of plant 2 and thus the second row is crossed off. The next allocation, in the resulting reduced matrix is made in cell (3, 1) of row 3 as it contains the minimum cost 5 in that row. We allocate min. (1, 10)=(1) in this cell. This exhausts the requirement condition of distribution centre 1 and hence the first column is crossed off.

Proceeding in this way we allocate (4), (2) and (3) units to cells (3, 2), (3, 4) and (3, 3) till all the rim conditions are met with. The resulting matrix is shown in table 3·12.

The transportation cost associated with this solution is

$$Z = \text{Rs. } [2\times6+0\times1+5\times1+8\times4+15\times3+9\times2]\times100$$

$$= \text{Rs. } 11{,}200,$$

which is less than the cost associated with solution obtained by N-W corner method.

(3) Column Minima Method

This method consists in allocating as much as possible in the lowest cost cell of the first column so that either the demand of the first distribution centre is satisfied or the capacity of the *i*th plant is exhausted or both. Three cases arise :

(*i*) if the requirement of the first distribution centre is satisfied, cross off the first column and move right to the second column.

(*ii*) if the capacity of *i*th plant is satisfied, cross off *i*th row and reconsider the first column with the remaining requirement.

(*iii*) if the requirement of the first distribution centre as well as the capacity of the *i*th plant are completely satisfied, make a zero allocation in the second lowest cost cell of the first column. Cross off the column and move right to the second column.

Continue the process for the resulting reduced transportation table until all the rim conditions are satisfied.

Table 3·13

Distribution centres

		1	2	3	4	Supply
	1	2 (6)	3	11	7	6
Plants	2	1 (1)	0	6	1	1
	3	5 (0)	8 (5)	15 (3)	9 (2)	10 5 2
Requirement		7 6	5	3	2	

In the given problem, we allocate first to cell (2, 1) in the first column as it contains the minimum cost 1. We allocate min. (1, 7) = (1) in this cell. This exhausts the supply capacity of plant 2 and thus the second row is crossed off. The next allocation in the resulting reduced matrix is made in cell (1, 1) of column 1 as it contains the second lowest cost 2 in that column. We allocate min. (6, 6) = (6) in this cell. This exhausts the supply capacity of plant 1 as well as the requirement of distribution centre 1. Therefore, we allocate zero in cell (3, 1) of the first column. cross off first row and first column and move on to the second column. Proceeding in this way we allocate (5), (3) and (2) to cells (3, 2), (3, 3) and (3, 4) till all the rim conditions are met with. The resulting matrix is shown in table 3·13.

The transportation cost associated with this solution is

$$Z = \text{Rs. } [2 \times 6 + 1 \times 1 + 5 \times 0 + 8 \times 5 + 15 \times 3 + 9 \times 2] \times 100$$
$$= \text{Rs. } 11{,}600$$

which is same as the cost associated with solution obtained by N-W corner method.

(4) Least Cost Method

This method consists in allocating as much as possible in the lowest cost cell/cells and then further allocation is done in the cell/cells with second lowest cost and so on. Consider the matrix for the problem under study.

Here, the lowest cost cell is (2, 2) and maximum possible allocation (meeting supply and requirement positions) is made here. Evidently, maximum feasible allocation in cell (2, 2) is (1). This meets the supply position of plant 2. Therefore, row 2 is crossed out, indicating that no allocations are to be made in cells (2, 1), (2, 3) and (2, 4).

Table 3·14

Distribution centres

		1	2	3	4	supply
	1	2 (6)	3	11	7	6
Plants	2	1	0 (1)	6	1	1
	3	5 (1)	8 (4)	15 (3)	9 (2)	~~10~~ ~~8~~ ~~5~~ ~~2~~
Requirement		~~7~~ ~~1~~	~~5~~ ~~4~~	~~3~~	~~2~~	

The next lowest cost cell (excluding the cells in row 2) is (1, 1), max. possible allocation of (6) is made here and row 1 is crossed out. Next lowest cost cell in row 3 is (3, 1) and allocation of (1) is done here. Likewise, allocations of (4), (2) and (3) are done in cells (3, 2), (3, 4) and (3, 3) respectively. The transportation cost associated with this solution is

$$\begin{aligned} Z &= \text{Rs. } (2\times6+0\times1+5\times1+8\times4+15\times3+9\times2)\times100 \\ &= \text{Rs. } (12+0+5+32+45+18)\times100 \\ &= \text{Rs. } 11{,}200, \end{aligned}$$

which is less than the cost associated with the solution obtained by N-W corner method.

(5) Vogel's Approximation Method (VAM)

Vogel's approximation method yields a very good initial solution, which, sometimes may be the optimal solution. This method takes into account not only the least cost c_{ij} but also the costs that just exceed c_{ij}. The technique is simple and considerably reduces the number of iterations required to arrive at the optimal solution. This methed consists of the following substeps :

Substep 1 : Write down the cost matrix as shown below.

Table 3·15

Distribution centres

		1	2	3	4	supply
	1	2	3	11	7	6 [1]
Plants	2	1	0	6	1 (1)	1 [1]
Requirement	3	5	8	15	9	10 [3]
		7	5	3	1	
		[1]	[3]	[5]	[6]	

Enter the difference between the smallest and second smallest elements in each column below the corresponding column and the difference between the smallest and second smallest elements in each row to the right of the row. Put these numbers in parentheses as shown. For example, in column 1, the two lowest elements are 1 and 2 and their difference is 1 which is entered as [1] below column 1. Similarly, the two smallest elements in row 2 are 0 and 1 and their difference 1 is entered as [1] to the right of row 2. A row or column "difference" indicates the unit penalty incurred by failing to make an allocation to the smallest cost cell in that row or column.

Substep 2 : Select the row or column with the greatest difference and allocate as much as possible within the restrictions of the *rim conditions* to the lowest cost cell in the row or column selected. In case a tie occurs, use any arbitrary tie breaking choice.

Thus since [6] is the lowest number in parentheses we choose column 4 and allocate as much as possible to the cell (2, 4) as it has the lowest cost 1 in column 4. Since suppy is 1 while the requirement is 2, maximum possible allocation is (1).

Substep 3 : Cross out the row or column completely satisfied by the allocation just made. For the assignment just made at (2, 4), supply of plant 2 is completely satisfied. So, row 2 is crossed out and the shrunken matrix is written below.

Table 3·16

	1	2	3	4	
1	2	3 (5)	11	7	~~6~~ 1 [1]
3	5	8	15	9	10 [3]
	7	5	3	1	
	[3]	[5] ↑	[4]	[2]	

This matrix consists of the rows and columns where allocations have not yet been made, including revised row and column totals which take the already made allocation into account.

Substep 4 : Repeat steps 1 to 3 until all assignments have been made.

(*a*) Column 2 exhibits the greatest difference of [5]. Therefore, we allocate (5) units to cell (1, 2), since it has the smallest transportation cost in column 2. Since requirements of column 2 are completely satisfied, this column is crossed out and the reduced matrix is written again as table 3·17.

Table 3·17

	1	3	4		
1	2 (1)	11	7	1	[5]←
2	5	15	9	10	[4]
	~~7~~ 6	3	1		
	[3]	[4]	[2]		

(*b*) Differences are recalculated. The maximum difference is [5]. Therefore, we allocate (1) to the cell (1, 1) since it has the lowest cost in row 1. Since requirements of row 1 are fully satisfied, it is crossed out and the reduced matrix is written below.

Table 3·18

	1	3	4	
3	5 (6)	15 (3)	9 (1)	10
	6	3	1	

(*c*) As cell (3, 1) has the lowest cost 5, maximum possible allocation of (6) is made here. Likewise, next allocation of (1) is made in cell (3, 4) and (3) in cell (3, 3) as shown.

All allocations made during the above procedure are shown below in the allocation matrix.

Table 3·19

Distribution centres

		1	2	3	4	supply
	1	2 (1)	3 (5)	11	7	6
Plants	2	1	0	6	1 (1)	1
	3	5 (6)	8	15 (3)	9 (1)	10
requirement		7	5	3	2	

The above repetitions can be made in a single matrix as shown below

Table 3·20

Distribution centres

		1	2	3	4	supply
	1	2 (1)	3 (5)	11	7	~~6~~ 1 [1] [1] [5]←
Plants	2	1	0	6	1 (1)	~~1~~ [1]
	3	5 (6)	8	15 (3)	9 (1)	10 [3] [3] [4]
requirement		~~7~~ 6	5	3	~~2~~ 1	
		[1]	[3]	[5]	[6]↑	
		[3]	[5]↑	[4]	[2]	
		[3]		[4]	[2]	

The cost of transportation associated with the above solution is

$$Z = \text{Rs. } (2\times1+3\times5+1\times1+5\times6+15\times3+9\times1)\times100$$
$$= \text{Rs. } (2+15+1+30+45+9)\times100$$
$$= \text{Rs. } 10{,}200,$$

which is evidently the least of all the values of transportation cost found by different methods. Since Vogel's approximation method results in the most economical initial feasible solution, we shall use

this method for finding such a solution for all transportation problems henceforth.

Step III : Perform Optimality Test

Make an optimality test to find whether the obtained feasible solution is optimal or not. An optimality test can, of course, be performed only on that feasible solution in which

(*a*) number of allocations is $m+n-1$, where m is the number of rows and n is the number of columns. In the given situation, $m=3$ and $n=4$ and number of allocations is 6 which is equal to $(m+n-1)$ ($\because$ $3+4-1=6$). Hence optimality test can be performed.

(*b*) these $(m+n-1)$ allocations should be in independent positions.

A look at the feasible solution of the situation under consideration indicates that all the allocations are in independent positions as it is impossible to increase or decrease any allocation without either changing the position of the allocations or violating the row and column restrictions. For example, if the allocation in cell (1, 1) is changed from (1) to (3), the allocation in cell (1, 2) must be changed from (5) to (3) in order to satisfy the row restriction. Similarly, the allocation in cell (3, 1) must be changed from (6) to (4) in order to meet the column restriction. This will, in turn, require changes in the allocations of cells (3, 3) and/or cell (3, 4).

Table 3·21

Distribution centres

		1	2	3	4	5	6	Supply
	1	(3)						3
	2		(8)...	...(6)				14
Plants	3		(4)...	...(5)		(7)		16
	4				(9)		(4)	13
Demand		3	12	11	9	7	4	

A simple rule for allocations to be in independent positions is that it is impossible to travel from any allocation, back to itself, by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of route. For instance, the occupied cells in table 3.21 are not in independent positions because the cells (2, 2), (2, 3), (3, 3) and (3, 2) form a closed loop.

Let us now understand the motivation of the optimality test. Consider the matrix giving the first feasible solution for the problem under consideration. Let us start with any *arbitrary empty* cell (a cell without allocation), say (2, 2) and allocate +1 unit to this cell. As already discussed, in order to keep up the column 2 restriction, −1 must be allocated to cell (1, 2) and to keep up the row 1 restriction, +1 must be allocated to cell (1, 1) and consequently −1 must be allocated to cell (2, 1); this is shown in the matrix below.

Table 3.22

		2	3	4	
1	2 (1) +1	3 −1(5)	11	7	6
2	1 −1	0 +1	6	1 (1)	1
3	5 (6)	8	15 (3)	9 (1)	10
	7	5	3	2	

The net change in transportation cost as a result of this perturbation is called the *evaluation* of the empty cell in question.

$$\therefore \quad \text{Evaluation of cell } (2, 2) = \text{Rs. } 100 \times (0 \times 1 - 3 \times 1 + 2 \times 1 - 1 \times 1)$$

$$= \text{Rs. } (-2 \times 100)$$

$$= -\text{Rs. } 200.$$

Naturally, as a result of this perturbation, the transportation cost decreases by Rs. 200. Now the total number of empty cells will be $m.n-(m+n-1)=(m-1)\,(n-1)$.

Therefore, there are $(m-1)(n-1)$ such cell evaluations which must be calculated. If any cell evaluation is negative, the cost can be reduced so that the solution under consideration can be improved i.e., it is not optimal. On the other hand, if all cell evaluations are positive or zero, the solution in question must be optimal. Actually, it is not necessary to compute each cell evaluation individually, the optimality test described below calculates all of them simultaneously. This test consists of the following substeps :

Substep 1 : Set up a cost matrix containing the costs associated with the cells for which allocations have been made. This matrix for the present example is

Table 3.23

2	3			
			1	*cost matrix*
5		15	9	

Substep 2 : Enter a set of numbers v_j across the top of the matrix and a set of numbers u_i across the left side so that their sums equal the costs entered in substep 1.

Thus $\quad u_1+v_1=2 \qquad u_3+v_1=5$

$u_1+v_2=3 \qquad u_3+v_3=15$

$u_2+v_4=1$ and $u_3+v_4=9$

Let $v_1=0$, then $u_1=2 \qquad u_3=5$

$v_2=1 \qquad v_3=10$

$u_2=-3 \qquad v_4=4$

Therefore, the matrix may be written as

Table 3.24

u_i \ v_j	0	1	10	4
2	2	3		
-3				1
5	5		15	9

Substep 3 : Fill the vacant cells with the sums of u_i and v_j. This is shown in table 3.25.

Table 3.25

u_i \ v_j	0	1	10	4
2	.	.	12	6
–3	–3	–2	7	.
5	.	6	.	.

Substep 4 : Subtract the cell values of the matrix of substep 3 from the original cost matrix.

Table 3.26

.	.	11–12	7–6
1+3	0+2	6–7	.
.	8–6	.	.

→

Table 3.27

.	.	–1	1
4	2	–1	.
.	2	.	.

Cell evaluation matrix

The resulting matrix is called *cell evaluation matrix.*

Substep 5 : If any of the cell evaluations are negative, the basic feasible solution is not optimal. In the present example, since two cell evaluations are negative, steps should be taken to obtain an optimal solution.

Step IV : Iterate towards an Optimal Solution

This involves the following substeps :

Substep 1 : From the cell evaluation matrix, identify the cell with the most negative entry. Since, in our case, two cells have the same negative entry (—1), we can identify either of them. Let us choose the cell (1, 3).

Substep 2 : Write down again the initial feasible solution.

Table 3·28

	1	2	3	4	
1	− 1	5	+ √		6
2				1	1
3	6 +		3 −	1	10
	7	5	3	2	

Check mark (√) the empty cell for which the cell evaluation is most negative. This is the cell chosen in substep 1 and is called *identified cell.*

Substep 3. Trace a path in this matrix consisting of a series of alternately horizontal and vertical lines. The path begins and terminates in the identified cell. All corners of the path lie in the cells for which allocations have been made.

Substep 4 : Mark the identified cell as positive and each occupied cell at the corners of the path alternately *—ve, +ve, —ve* and so on.

Substep 5 : Make a new allocation in the identified cell by

Table 3·29

1−1	5	+1		6
			1	1
6+1		3−1	1	10
7	5	3	2	

→

Table 3·30

	5	1		6
			1	1
7		2	1	10
7	5	3	2	

2nd feasible solution

entering the smallest allocation on the path that has been given a $-ve$ sign. Add and subtract this new allocation from the cells at the corners of the path, maintaining the row and column requirements. This causes one basic variable to become zero and other variables remain non-negative. The basic cell whose allocation has been made zero, leaves the solution.

The total cost of transportation for this 2nd feasible solution is

$$=\text{Rs. } (3\times5+11\times1+1\times1+7\times5+2\times15+1\times9)\times100$$

$$=\text{Rs. } (15+11+1+35+30+9)\times100=\text{Rs. } 10{,}100,$$

which is less than for the first (starting) feasible solution.

Step V : Check for Optimality

Let us check whether the solution obtained above is optimal or not. This shall be checked by repeating the steps under 'check for optimality' already made. In the above feasible solution,

(*a*) number of allocations is $(m+n-1)$ *i.e.*, 6,

(*b*) these $(m+n-1)$ allocations are in independent positions.

Above conditions being satisfied, an optimality test can be performed as follows :

Substep 1 : Set up the cost matrix containing the costs associated with the cells for which allocations have been made.

Substep 2 : Enter a set of numbers v_j along the top of the matrix and a set of numbers u_i at the left side so that their sum is equal to costs entered in matrix of substep 1, shown below.

Table 3·31

v_j / u_i	0	2	10	4
1		3	11	
−3				1
5	5		15	9

Thus $\quad u_1+v_2=3 \qquad u_3+v_1=5$

$u_1+v_3=11 \qquad u_3+v_3=15$

$u_2+v_4=1$ and $u_3+v_4=9$

Let $v_1=0$,

Then $\quad u_3=5 \qquad u_2=-3$

$v_3=10 \qquad u_1=1$

$v_4=4 \qquad v_2=2$

These values are shown entered in matrix 3·31.

Substep 3 : Fill the vacant cells with the sums of u_i and v_j.

Table 3·32

u_i \ v_j	0	2	10	4
1	1	.	.	5
−3	−3	−1	7	.
5	.	7	.	.

Substep 4 : Subtract the cell values of this matrix from the original cost matrix.

Table 3·33

2−1	.	.	7−5
1+3	0+1	6−7	.
.	8−7	.	.

→

Table 3·34

1	.	.	2
4	1	−1	.
.	1	.	.

Cell evaluation matrix

This matrix 3.34 is called cell evaluation matrix.

Substep 5 : Since one cell value is *—ve*, the 2nd feasible solution is not optimal.

Step VI : **Iterate Towards an Optimal Solution**

This involves the following substeps :

Substep 1 : In the cell evaluation matrix, identify the cell with the most negative entry. It is the cell (2, 3).

Substep 2 : Write down again the feasible solution in question.

Table 3·35

	5	1	
		+ √	1 −
7		2 −	1 +

Mark the empty cell (✓) for which the evaluation is negative. This is called identified cell.

Substep 3 : Trace the path shown in the matrix.

Substep 4 : Mark the identified cell as $+ve$ and others alternately $-ve$ and $+ve$.

Substep 5 : Make the new allocation in the identified cell by entering the smallest allocation on the path which has been given a negative sign. Subtract and add this amount from other cells. Tables 3·36 and 3·37 result.

Table 3·36

	5	1	
		+1	1−1
7		2−1	1+1

→

Table 3·37

	5	1	
		1	
7		1	2

Third feasible solution

For this allocation matrix the transportation cost is

$$Z = \text{Rs. } (5\times3+1\times11+1\times6+1\times15+2\times9+7\times5)\times100$$
$$= \text{Rs. } 10{,}000.$$

Thus it is a better solution. Let us use if it is an optimal solution.

Step VII : Test for Optimality

In the above feasible solution

(*a*) number of allocations is $m+n-1$ i.e., 6.

(*b*) these $m+n-1$ allocations are in independent positions. Hence repeat the following substeps :

Substep 1 : Set up the cost matrix containing costs associated with cells for which allocations have been made. This is table 3·38.

Table 3·38

	3	11	
		6	
5		15	9

Substep 2 : Enter a set of numbers v_j and u_i such that

$$u_1+v_2=3 \qquad u_3+v_1=5$$

$$u_1+v_3=11 \qquad u_3+v_3=15$$

$$u_2+v_3=6 \quad \text{and} \quad v_3+v_4=9$$

Let $v_1=0$

Then $u_3=5$ $\qquad u_2=-4$

$v_3=10$ $\qquad u_1=1$

$v_4=4$ $\qquad v_2=2$

The resulting matrix is shown in table 3·39.

Table 3·39

u_i \ v_j	0	2	10	4
1	1	.	.	5
−4	−4	−2	.	0
5	.	7	.	.

Substep 3 : Fill up the vacant cells also as shown above.

Substep 4 : Subtract the cell values of the above matrix from the original cost matrix. Tables 3·40 and 3·41 result.

Table 3·40

2−1	.	.	7−5
1+4	0+2	.	1−0
	8−7	.	.

→

Table 3·41

1	.	.	2
5	2	.	1
.	1	.	

Cell evaluation matrix

Substep 5 : Since all the cell values are positive, the third feasible solution given by the following matrix is the optimal solution :

Table 3·42

	Distribution centres 1	2	3	4	Supply
Plants 1	2	3 (5)	11 (1)	6	6
2	1	0	6 (1)	1	1
3	5 (7)	8	15 (1)	9 (2)	10
Requirement	7	5	3	2	

and the optimal (minimum) transportation cost = Rs. 10,000.

The above iterative procedure of determining an optimal solution of a transportation problem (T.P.) is called *Modi Method.*

EXAMPLE 3.5-2 :

Solve example 3.1-2. Also explain *degeneracy in transportation technique* in the context of this example.

Formulation of the Model

Step 1

Key decision to be made is to find how many units of the commodity be shipped from which warehouse to which store so that cost of transportation is minimum. Let x_{ij} represent the decision variables i.e., number of units to be shipped.

Here $i=1, 2, 3, 4$

$j=1, 2, 3, 4, 5, 6.$

Step 2

Feasible alternatives are sets of values of x_{ij}, where $x_{ij} \geqslant 0$.

Step 3

Objective is to minimize the transportation cost i.e., minimize

$$Z=\sum_{i=1}^{4} \sum_{j=1}^{6} c_{ij}x_{ij}$$

where c_{ij} is the unit cost of shipping from ith warehouse to j_{th} store.

Step 4

Constraints are

(*i*) because of surplus or supply :

$$x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16}=5 \text{ i.e., } \sum_{j=1}^{6} x_{ij}=5$$

$$\text{Similarly } \sum_{j=1}^{6} x_{2j}=6, \ \sum_{j=1}^{6} x_{3j}=2 \ \& \ \sum_{j=1}^{6} x_{4j}=9.$$

(*ii*) because of requirement or demand :

$$x_{11}+x_{21}+x_{31}+x_{41}=4 \text{ i.e., } \sum_{i=1}^{4} x_{i1}=4$$

$$\text{Similarly } \sum_{i=1}^{4} x_{i2}=4, \ \sum_{i=1}^{4} x_{i3}=6, \ \sum_{i=1}^{4} x_{i4}=2,$$

$$\sum_{i=1}^{4} x_{i5}=4 \ \& \ \sum_{i=1}^{4} x_{i6}=2.$$

Thus the given situation involves $4 \times 6(=24)$ variables and $4+6(=10)$ constraints.

Solution of the Model

The solution involves making a transportation model (in the form of a matrix), finding a feasible solution, performing optimality test and iterating towards optimal solution if required.

Step I : Make a Transportation Model

This consists in expressing supply from origins, requirements at destinations and the unit cost of shipping from each origin to each destination in the form of a matrix shown below. A check is made to balance supply and demand

Table 3.43

Stores (destinations)

		1	2	3	4	5	6	Surplus (supply)	
	1	9	12	9	6	9	10	5	
Ware-houses (ori-gins)	2	7	3	7	7	5	5	6	Total surplus = 22
	3	6	5	9	11	3	11	2	
	4	5	8	11	2	2	10	9	Total requirement = 22
Requirement (demand)		4	4	6	2	4	2		

Thus supply and demand are balanced.

Step II : Find a Feasible Solution

Vogel's approximation method will be used to find the initial feasible solution. This method consists of the following substeps :

Substep 1 : Write down the cost matrix. Enter the difference between the smallest and second smallest element in each column, below the corresponding column and the difference between the smallest and second smallest element in each row to the right of the row. Put these numbers in parentheses.

Substep 2 : Select the row or column with the greatest difference and allocate as much as possible within the restrictions of rim conditions to the lowest cost cell in the row or column selected.

Substep 3 : Cross out the row or column completely satisfied by the allocation made and revise the row and column totals which take into account the allocations already made.

Substep 4 : Repeat the substeps 1 to 3 until all allocations have been made.

All allocations made during the above procedure are shown below in a single matrix.

Table 3.44

Stores

	1	2	3	4	5	6	supply	
1	9	12	9 (5)	6	9	10	~~5~~	[3] [3] [0] [0] [0] [0]
2	7	3 (4)	7	7	5	5 (2)	~~6~~ ~~4~~	[2] [2] [2] [4]←
3	6 (1)	5	9 (1)	11	3	11	~~2~~ ~~1~~	[2] [2] [2] [1] [3] [3]←
4	6 (3)	8	11	2 (2)	2 (4)	10	~~9~~ ~~7~~ ~~3~~	[0] [0] [4]←[2] [5]←
demand	~~4~~ ~~1~~	~~4~~	~~6~~ ~~1~~	~~2~~	~~4~~	~~2~~		
	[0]	[2]	[2]	[4]	[1]	[5]		
	[0]	[2]	[2]	[4]	[1]	↑		
	[0]	[2]	[2]	↑	[1]			
	[0]	[2]	[2]					
	[0]		[0]					
	[3]		[0]					

These allocations are entered below in the allocation matrix.

Table 3.45

Stores

		1	2	3	4	5	6	Supply	
	1	9	12	9 (5)	6	9	10	5	
	2	7	3 (4)	7	7	5	5 (2)	6	
Ware-house	3	6 (1)	5	9 (1)	11	3	11	2	
	4	6 (3)	8	11	2 (2)	2 (4)	10	9	*Frist feasible solution*
Demand		4	4	6	2	4	2		

The cost of transportation associated with the above solution is

$$Z = \text{Rs. } (9\times5+3\times4+5\times2+6\times1+9\times1+6\times3+2\times2+2\times4)$$
$$= \text{Rs. } (45+12+10+6+9+18+4+8)$$
$$= \text{Rs. } 112.$$

Step III : Perform Optimality Test

Make an optimality test to find whether the obtained feasible solution is optimal or not. An optimality test can however be performed on that feasible solution in which

(*a*) number of allocations is $m+n-1$.

In the given situation $m=4$, $n=6$

$\therefore\ m+n-1=4+6-1=9.$

Now the number of allocations$=8(<9)$. Therefore optimality test cannot be performed as such. Such a solution is called a *degenerate solution.*

(*b*) these $m+n-1$ allocations must be in independent positions.

In the present example, these $m+n-1$ allocations are in independent positions as it is impossible to increase or decrease any allocation without either changing the positions of allocations or violating the row and column restrictions. The degeneracy will occur

whenever, while finding the initial feasible solution, the supply and demand are satisfied *simultaneously*. Normally, any allocation made satisfies either row or column. The above *degenerate* solution is made permissible for optimality test in the following manner :

First of all the requisite number of vacant cells is chosen so that

(*i*) these cells plus the existing number of allocations are equal to $m+n-1$.

(*ii*) these $m+n-1$ cells are in independent positions. (This can always be done if the solution we start with contains loaded cells in independent positions.)

Now allocate an infinitesimally small but positive value ϵ (Greek letter epsilon) to each of the chosen cells. Subscripts are used when more than one such letter is required (e.g., ϵ_1, ϵ_2, etc.). These ϵ's are then treated like any other positive basic variable and are kept in the transportation array (matrix) until temporary degeneracy is removed or until the optimal solution is reached, whichever occurs first. At that point we set each $\epsilon=0$. Notice that ϵ is infinitesimally small and hence its effect can be neglected when it is added to or subtracted from a positive value. Consequently, they do not appreciably alter the physical nature of the original set of allocations but do help in carrying out further iterations.

In the present example, there is need for making one infinitesimal allocation. Out of the unoccupied cells, cell (3, 5) has the least cost of Rs. 3. The infinitesimal allocation should be made in this cell. However, allocating ϵ to this cell forms a closed loop among the cells (3, 1), (3, 5), (4, 5) and (4, 1) so that allocations in these cells do not remain in independent positions. Therefore, no allocation is made in cell (3, 5). There are two next higher cost cells, viz. cell (2, 5) and (3, 2) each with a cost of Rs. 5. Allocation in either of these cells does not result in closed loop. Hence the infinitesimal allocation can be made in either of these two cells. Let us make the allocation in cell (2, 5). Table 3.46 formed thereby is shown below.

Table 3·46

	1	2	3	4	5	6
1			5			
2		4			ϵ	2
3	1		1			
4	3			2	4	

Now optimality test can be applied. It consists of the following substeps :

Substep 1. Set up the cost matrix containing the costs associated with the cells for which allocations have been made. This matrix for the present example is

Table 3·47

	1	2	3	4	5	6
1			9			
2		3			5	5
3	6		9			
4	6			2	2	

Substep 2. Enter a set of numbers v_j along the top of the matrix and a set of numbers u_i down the left side so that their sums are equal to the costs entered in substep 1.

Thus $u_1+v_3=9$ $\qquad$ $u_3+v_1=6$

$u_2+v_2=3$ $\qquad$ $u_3+v_3=9$

$u_2+v_5=5$ $\qquad$ $u_4+v_1=6$

$u_2+v_6=5$ $\qquad$ $u_4+v_4=2$

and $u_4+v_5=2$

Let $v_1=0$ $\qquad$ $\therefore$ $u_3=6$ $\qquad$ $u_1=6$

$u_4=6$ $\qquad$ $u_2=9$

$v_3=3$ $\qquad$ $v_2=-6$

$v_4=-4$ $\qquad$ $v_6=-4$

$v_5=-4$

Table 3·48.

v_j / u	0	-6	3	-4	-4	-4
6			9			
9		3			5	5
6	6		9			
6	6			2	2	

Substep 3. Fill up the vacant cells with the sums of u_i and v_i. The resulting array is shown below.

Table 3·49

u_i \ v_j	0	-6	-3	-4	-4	-4
6	6	0	•	2	2	2
9	9	•	12	5	•	•
6	•	0	•	2	2	2
6		0	9	•	•	2

Substep 4. Subtract the cell values of the above matrix from the original cost matrix. The resulting matrix called cell evaluation matrix is shown below.

Table 3·50

3	12	.	4	7	8
−2	.	−5	2	.	.
.	5	.	9	1	9
.	8	2	.	.	8

Cell evaluation matrix

Substep 5. Since cells (2, 1) and (2, 3) have negative values, the current feasible solution is not optimal.

Step IV. Iterate Towards an Optimal Solution.

This involves the following substeps :

Substep 1. In the cell evaluation matrix, identify the cell with the most negative entry. This is cell (2, 3).

Substep 2. Write down again the initial feasible solution.

Table 3·51

	1	2	3	4	5	6
1			5			
2		4	+ ✓		6	2
3	+ 1		− 1			
4	− 3			2	4 +	

Check mark (✓) the empty cell for which the cell evaluation is most negative. This is the cell chosen in substep 1 and is called *identified cell.*

Substep 3. Trace a path in this matrix consisting of a series of alternately horizontal and vertical lines. The path begins and terminates in the identified cell. All corners of the path lie in the cells for which allocations have been made.

Substep 4. Mark the identified cell as positive and each occupied cell at the corners of the path alternately $-ve$, $+ve$, $-ve$ and so on.

Substep 5. Make a new allocation in the identified cell by entering the smallest allocation on the path that has been assigned a $-ve$ sign. Add and subtract this new allocation from the cells at the corners of the path, maintaining the row and column requirements.

Table 3·52

	1	2	3	4	5	6
1			5			
2		4	$+\epsilon$		$\epsilon-\epsilon$	2
3	$1+\epsilon$		$1-\epsilon$			
4	$3-\epsilon$			2	$4+\epsilon$	

Table 3·53

	1	2	3	4	5	6
1			5			
2		4	ϵ			2
3	1		1			
4	3			2	4	

2nd feasible solution

As can be seen, this new allocation gives the same cost of transportation (Rs. 112) as the old one. But this places us in a position to carry further iterations.

Step V. Test for Optimality

In the above solution

(*a*) number of allocations are $m+n-1$ $(=9)$

(*b*) these $m+n-1$ allocations are in independent positions.

Hence optimality test can be performed as follows :

Substep 1 Set up a cost matrix containing the costs associated with the cells for which allocations have been made.

Table 3·54

u_i \ v_j	0	−1	3	−4	−4	1
6			9			
4		3	7			5
6	6		9			
6	6			2	2	

Substep 2. Enter a set of numbers v_j along the top of the matrix and u_i along the left side so that their sum is equal to the costs entered in matrix of substep 1.

Table 3·55

u_i \ v_j	0	-1	3	-4	-4	1
6	6	5	.	2	2	7
4	4	.	.	0	0	.
6	.	5	.	2	2	7
6	,	5	9		.	7

$u_1+v_3=9$ Let $v_1=0$

$u_2+v_2=3$ $\therefore$ $u_3=6$

$u_2+v_3=7$ $u_4=6$

$u_2+v_6=5$ $u_1=6$

$u_3+v_1=6$ $u_2=4$

u_3+v_3+9 $v_3=3$

$u_4+v_1=6$ $v_4=-4$

$u_4+v_4=2$ $v_5=-4$

$u_4+v_5=2$ $v_2=-1$

$v_6=$ 1.

Substep 3. Fill up the vacant cells as shown in the above matrix.

Substep 4. Subtract the cell values of this matrix from the original cost matrix. Tables 3·56 and 3·57 result.

Table 3·56

9−6	12−5	.	6−2	9−2	10−7
7−4	.	.	7−0	5−0	.
.	5−5	.	11−2	3−2	11−7
.	8−5	11−9	.	.	10−7

↓

Table 3·57

3	7	.	4	7	3
3	.	.	7	5	.
.	0	.	9	1	4
.	3	2	.	.	3

Since all the cell values are positive, the 2nd feasible solution is an optimal solution. Since the above matrix contains zero entries, there exists alternative optimal solution Thus the optimal solution for our problem is

Table 3·58

Stores

		1	2	3	4	5	6	*Supply*
	1	9	12	9 (5)	6	9	10	5
Warehouses	2	7	3 (4)	7	7	5 (2)	5	6
	3	6 (1)	5	9 (1)	11	3	11	2
	4	6 (3)	8	11	2 (2)	2 (4)	10	9
Demand		4	4	6	2	4	2	

Total cost of transportation = Rs. 112.

In solving this problem, infinitesimal allocation was made in

cell (2, 5). If this allocation is made in cell (3, 2), the same optimal solution (as above) is obtained without having to make any iteration.

Remark. In this problem, the initial (starting) feasible solution, obtained by Vogel's approximation method was a degenerate one. However, sometimes, even if the starting feasible solution is non-degenerate, degeneracy may develop later at some iteration stage. This happens when the selection of the entering variable (least value in the closed path which has been assigned a negative sign) causes two or more current basic variables to become zero. We allocate ϵ_1, ϵ_2, etc. to such cells so that there are exactly $m+n-1$ cells in independent positions and the procedure is then continued in the usual manner. Example 3·5-4 makes this point clear.

EXAMPLE 3.5-3

Find the optimum solution to the following transportation problem in which the cells contain the transportation cost in rupees.

Table 3·59

	W_1	W_2	W_3	W_4	W_5	*Available*
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Required	30	30	15	20	5	100 (Total)

Solution

Step I : Make the Transportation Matrix

This step is not necessary in the current problem.

Step II : Find a Basic Feasible Solution

We shall use Vogel's approximation method to find initial basic feasible solution. The method consists of substeps 1, 2, 3 and 4 already explained in example 3.5-2.

Table 3·60

	W_1	W_2	W_3	W_4	W_5	Available				
F_1	7 (5)	6	4 (15)	5 (20)	9	~~40~~ ~~25~~ ~~5~~	[1]	[1]	[2]←	[2]←
F_2	8	5 (30)	6	7	8	~~30~~	[1]	[2]←		
F_3	6 (15)	8	9	6	5 (5)	~~20~~ ~~15~~	[1]	[1]	[1]	[1]
F_4	5 (10)	7	7	8	6	~~10~~	[1]	[1]	[1]	[1]
Required	~~30~~ ~~25~~	~~30~~	~~15~~	~~20~~	~~5~~					
	[1]	[1]	[2]↑	[1]	[1]					
	[1]	[1]		[1]	[1]					
	[1]			[1]	[1]					
	[1]				[1]					

The degenerate basic feasible solution is given by table 3·61.

Table 3·61

	W_1	W_2	W_3	W_4	W_5	Available
F_1	7 (5)	6	4 (15)	5 (20)	9	40
F_2	8	5 (30)	6 (ϵ)	7	8	30
F_3	6 (15)	8	9	6	5 (5)	20
F_4	5 (10)	7	7	8	6	10
Required	30	30	15	20	5	

Initial basic feasible solution

Step III : Perform Optimality Test

From the above matrix we find that

(*a*) required number of allocations $= m+n-1 = 4+5-1 = 8$

Actual number of allocations $= 7$.

$\therefore$ We shall allocate a very small positive value ϵ to one of cells (F_1, W_2), (F_2, W_3), (F_3, W_4) and (F_4, W_5), each of which has the same minimum transportation cost of Rs. 6 (out of the unoccupied cells). Allocations to either of cells (F_3, W_4) and (F_4, W_5) results in closed loops and hence no allocations will be made in these cells. Thus ϵ can be allocated to either cell (F_1, W_2) or (F_2, W_3). Let us

allocate it to cell (F_2, W_3) so that the number of allocated cells becomes 8. This is shown in Table 3·61.

(*b*) these 8 allocations are in independent positions. Therefore optimality test can be performed. This consists of the following substeps :

Substeps 1, 2, 3, 4 and 5, details of which are given in example 3.5-2.

Table 3·62

u_i \ v_j	0	−4	−3	−2	−1
7	7		4	5	
9		5	6		
6	6				5
5	5				

Matrix of u_i+v_j

Table 3·63

u_i \ v_j	0	−4	−3	−2	−1
7	.	3	.	.	6
9	9	.	.	7	8
6	.	2	3	4	.
5	.	1	2	3	4

Matrix with cell values of u_i+v_j for empty cells

Table 3·64

	3	.		3
−1		.	0	0
.	6	6	2	.
.	6	5	5	2

Cell evaluation matrix

As cell value in cell (F_2, W_1) is negative, the initial basic feasible solution given by table 3·61 is not optimal.

Step IV : Iterate Towards an Optimal Solution

This involves substeps 1, 2, 3, 4 and 5, details of which are given in example 3.5-2.

Table 3·65

5		15	20	
✓	30	ε		
15				5
10				

Initial feasible solution with closed path

↓

Table 3·66

5−ε		15+ε	20	
ε	30			
15				5
10				

Table 3·67

5		15	20	
ε	30			
15				5
10				

2nd feasible solution

Step V : Test for Optimality

Repeating step III we get the following tables :

Table 3·68

u_i \ v_j	0	−3	−3	−2	−1
7	7		4	5	
8	8	5			
6	6				5
5	5				

Matrix of u_i+v_j

→

Table 3·69

u_i \ v_j	0	−3	−3	−2	−1
7	.	4	.	.	6
8	.	.	5	6	7
6	.	3	3	4	.
5	.	2	2	3	4

Matrtx with cell values u_i+v_j for empty cells

Table 3·70

.	2	:	.	3
.	.	1	1	1
.	5	6	2	.
	5	5	5	2

Cell evaluation matrix

As all cell values are positive, the second feasible solution is optimal. Therefore optimal transportation policy is

Table 3·71

	W_1	W_2	W_3	W_4	W_5	*Available*
F_1	7 (5)	6	4 (15)	5 (20)	9	40
F_2	8	5 (30)	6	7	8	30
F_3	6 (15)	8	9	6	5 (5)	20
F_4	5 (10)	7	7	8	6	10
Required	30	30	15	20	5	

Total transportation cost

$=$Rs. $[7\times5+4\times15+5\times20+5\times30+6\times15+5\times5+5\times10]$

$=$Rs. $[35+60+100+150+90+25+50]$

$=$Rs. 510.

EXAMPLE 3·5·4

Find the basic feasible solution of the following transportation problem by north-west corner rule. Also find the optimal transportation plan.

Table 3·72

	1	2	3	4	5	*Available*
A	4	3	1	2	6	80
B	5	2	3	4	5	60
C	3	5	6	3	2	40
D	2	4	4	5	3	20
Required	60	60	30	40	10	200 (Total)

Solution

Step I : Make a Transportation Table

This step is not necessary in the current problem.

Step II : Find Basic Feasible Solution

By following the north-west corner rule (explained in example 3·5-1), we get the non-degenerate initial basic feasible solution shown below.

Table 3·73

	1	2	3	4	5	*Avaible*
A	4 (60)	3 (20)	1	2	6	~~80~~ ~~20~~
B	5	2 (40)	3 (20)	4	5	~~60~~ ~~20~~
C	3	5	6 (10)	3 (30)	2	~~40~~ ~~30~~
D	2	4	4	5 (10)	3 (10)	~~20~~ ~~10~~
Required	~~60~~	~~60~~ ~~40~~	~~30~~ ~~10~~	~~40~~ ~~10~~	~~10~~	*Initial basic feasible solution*

Step III : Test for Optimality

Required number of allocations $= m+n-1 = 4+5-1 = 8$

Actual number of allocations $= 8$

These 8 allocations are in independent positions. Therefore optimality test can be performed. This step consists of substeps 1, 2, 3, 4 and 5, details of which are given in example 3·5-2.

Table 3·74

u_i \ v_j	0	−1	0	−3	−5
4	4	3			
3		2	3		
6			6	3	
8				5	3

Matrix of u_i+v_j

→

Table 3·75

	0	−1	0	−3	−5
4	.	.	4	1	−1
3	3	.	.	0	−2
6	6	5	.	.	1
8	8	7	8	.	.

Matrix of u_i+v_j for empty cells

Table 3·76

.	.	−3	1	7
2	.	.	4	7
−3	0	.	.	1
−6	−3	−4	.	.

Cell evaluation matrix

As many cell values are negative, initial basic feasible solution is not optimal.

Step IV : Iterate Towards an Optimal Solution

This involves substeps 1, 2, 3, 4 and 5, details of which are given in example 3·5-2.

Table 3·77

	1	2	3	4	5
A	60	20 +			
B		40	20 +		
C			10	30 +	
D	+ ✓			10	10

Initial feasible solution with closed path

Table 3·78

	1	2	3	4	5
A	50	30			
B		30	30		
C			*	40	
D	10			*	10

2nd feasible solution

In the second feasible solution the number of occupied cells (allocations) becomes less than $m+n-1$ $(=8)$ on account of simultaneous vacation of two cells (C, 3) and (D, 4) as indicated by*. Hence this is a degenerate solution.

This degeneracy can be overcome by allocating ϵ to one of these two cells (C, 3) and (D, 4). Since it is a minimization problem, ϵ should be allocated to cell (D, 4) which has the lower cost 5 out of the two cells. This is shown in able 3·79.

Table 3·79

50	30			
	30	30		
			40	
10			ϵ	10

2nd feasible solution

The rest of the procedure is exactly the same as explained in example 3·5-2. The optimal solution is given by table 3·80.

Table 3.80

	1	2	3	4	5	Available
A	4 (10)	3	1 (30)	2 (40)	6	80
B	5	2 (60)	3	4	5	60
C	3 (30)	5	6	3	2 (10)	40 *Optimal solution*
D	2 (20)	4	4	5	3	20
Required	60	60	30	40	10	200 (Total)

EXAMPLE 3.5-5

Solve example 3.1.3.

Solution : First of all we shall compile a new table of unit costs which consists of both production and transportation costs. The new table or matrix is given below.

Table 3.81

Stores

		1	2	3	4	*Supply*
	A	2+2	4+2	6+2	11+2	50
Factories	B	10+3	8+3	7+3	5+3	70
	C	13+1	3+1	9+1	12+1	30
	D	4+5	6+5	8+5	3+5	50
Demand		25	35	105	20	

↓

Table 3.82

	1	2	3	4	Supply
A	4	6	8	13	50
B	13	11	10	8	70
C	14	4	10	13	30
D	9	11	13	8	50
Demand	25	35	105	20	

The cell values of the first row of this matrix are obtained by adding unit production costs of factory A i.e., Rs. 2 to each value in the first row of Table 3.81. The cell values in the other rows are obtained in a similar fashion. The above table, therefore, gives unit production plus transportation costs from each of the factories A, B, C and D to each of the stores 1, 2, 3 and 4.

Step I : Make the Transportation matrix

Let us again write down the above matrix (table 3.82).

		Stores 1	2	3	4	Supply	
	A	4	6	8	13	50	Total supply = 200
	B	13	11	10	8	70	Total demand = 185
Factories	C	14	4	10	13	30	Surplus supply = 15
	D	9	11	13	8	50	
Demand		25	35	105	20		

Supply and demand are not balanced in this case and we have a surplus of 15 units of the product. Therefore, we create a fictitious (dummy) destination (store). The associated cost coefficients are taken as zero, as the surplus quantity remains lying in the respective

factories and is, in fact, not shipped at all. Therefore our starting cost matrix becomes

Table 3.83

Stores

		1	2	3	4	d	Supply
Factories	A	4	6	8	13	0	50
	B	13	11	10	8	0	70
	C	14	4	10	13	0	30
	D	9	11	13	8	0	50
Demand		25	35	105	20	15	

Total supply = 200

Total demand = 200

Step II : Find a Basic Feasible Solution

We shall use Vogel's approximation method to find the initial feasible solution. The method consists of substeps 1, 2, 3 and 4 already explained in example 3.5.2.

Table 3.84

Stores

		1	2	3	4	d	supply
Factories	A	4 (25)	6 (5)	8 (20)	13	0	~~50~~ ~~25~~ ~~20~~ [2] [2] [2] [5]←
	B	13	11	10 (70)	8	0	~~70~~ [2] [2] [2] [2] [2]
	C	14	4 (30)	10	13	0	~~30~~ [6]←
	D	9	11	13 (15)	8 (20)	0 (15)	~~50~~ ~~30~~ [1] [1] [3] [5] [6]←
demand		~~25~~	~~35~~ ~~5~~	~~105~~ 85	~~20~~	~~15~~	
		[5]	[2]	[2]	[0]	[0]	
		[5]	[5]	[2]	[0]	[0]	
		↑	[5]	[2]	[0]	[0]	
			↑	[2]	[0]	[0]	
				[3]	[0]	[0]	

The initial feasible solution is given by the following matrix :

Table 3.85

Stores

Factories	1	2	3	4	d	Supply
A	4 (25)	6 (5)	8 (20)	13	0	50
B	13	11	10 (70)	8	0	70
C	14	4 (30)	10	13	0	30
D	9	11	13 (15)	8 (20)	0 (15)	50
Demand	25	35	105	20	15	

Initial feasible solution

Step III : Perform Optimality Test

From the above matrix we find that

(*a*) number of allocations $= m+n-1 = 4+5-1 = 8$.

(*b*) these $m+n-1$ allocations are in independent positions. Therefore optimality test can be performed. This consists of the following substeps :

Substeps 1, 2, 3, 4 and 5, details of which are explained in example 3.5-2.

Table 3.86

u_i \ v_j	0	2	4	−1	−9
4	4	6	8		
6			10		
2		4			
9			13	8	0

Matrix of u_i+v_j

Table 3.87

u_i \ v_j	0	2	4	−1	−9
4	.	.	.	3	−5
6	6	8	.	5	−3
2	2	.	6	1	−7
9	9	11	.	.	.

Matrix with cell values of u_i+v_j

Table 3·88

.	.	.	13−3	0+5
13−6	11−8	.	8−5	0+3
14−2	.	10−6	13−1	0+7
9−9	11−11	.	.	.

↓

Table 3·89

.	.	.	10	5
7	3	.	3	3
12	.	4	12	7
0	0	.	.	.

Cell evaluation matrix

Since all the cell values are $+ve$, the first feasible solution is an optimal solution. Now, though the 15 extra units of the product are not transported, they are definitely manufactured in factory D, thus involving production cost only.

The optimum (minimum) transportation plus production cost

$$Z = \text{Rs. } (4 \times 25 + 6 \times 5 + 8 \times 20 + 10 \times 70 + 4 \times 30 + 13 \times 15 + 8 \times 20 + 0 \times 15 + 5 \times 15)$$

$$= \text{Rs. } (100 + 30 + 160 + 700 + 120 + 195 + 160 + 0 + 75)$$

$$= \text{Rs. } 1{,}465.$$

EXAMPLE 3.5-6

Solve example 3.1-4.

Solution

It consists of the following steps :

Step I : Make the Transportation Matrix

For the given data, the transportation matrix is as shown below :

Table 3.90

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Bombay	90	90	100	110	200
Calcutta	50	70	130	85	100
Demand	75	100	100	30	

Total supply = 300 units
Total demand = 305 units

Thus supply and demand are not balanced. As the demand is more than supply, a dummy source is introduced to meet the extra demand and zero profit coefficients are introduced since nothing is produced at the dummy origin and therefore nothing can be sold. The resulting matrix is shown in table 3.91.

Table 3.91

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Bombay	90	90	100	110	200
Calcutta	50	70	130	85	100
Dummy source	0	0	0	0	5
Demand	75	100	100	30	

Step II : Find Initial Basic Feasible Solution

We shall use Vogel's approximation method to find the initial feasible solution. This method consists of substeps 1, 2, 3, 4 and 5 already explained in example 3.5-2.

Note that we are dealing with *maximization* problem. Hence while making allocations, we select the cell with *largest* entry in the profit table 3.92. Thus the first maximum possible allocation of 75 units is made in cell (1, 1) and not in cell (2, 1). Same is true for other allocations.

Table 3.92

	Ranchi	*Delhi*	*Lucknow*	*Kanpur*	Supply
Bombay	90 (75)	90 (100)	100	110 (25)	~~200~~ [0] [10] ~~125~~ ~~25~~
Calcutta	50	70	130 (100)	85	~~100~~ [20] [15]
Dummy source	0	0	0	0 (5)	~~5~~ [0] [0]
Demand	~~75~~	~~100~~	~~100~~	~~30~~ ~~5~~	
	[40] ↑	[20] [20]	[30] [30] ↑	[25] [25]	

Step III : **Perform Optimality Test**

Required number of allocations $= m+n-1 = 3+4-1 = 6$

Actual number of allocations $= 5$

Therefore we allocate very small positive number ϵ to cell (1, 3) [cell having maximum profit out of vacant cells] so that the number of allocations becomes 6. This is shown in table 3.93. These 6 allocations are in independent positions. Therefore optimality test can be performed. This consists of substeps 1, 2, 3, 4 and 5, details of which are given in example 3.5-2.

Table 3.93

	Ranchi	*Delhi*	*Lucknow*	*Kanpur*	Supply
Bombay	90 (75)	90 (100)	100	110 (25)	200
Calcutta	50	70	130 (100)	85	100
Dummy source	0	0	0	0 (5)	5
Demand	75	100	100	30	

Initial basic feasible solution

Table 3.94

v_j / u_i	0	0	10	20
90	90	90	100	110
120			130	
−20				0

Matrix of u_i+v_j

Table 3.95

v_j / u_i	0	0	10	20
90	.	.	.	.
→120	120	120	.	140
−20	−20	−20	−10	.

Matrix of u_i+v_j *for vacant cells*

Table 3·96

.	.	.	.
−70	−50	.	−55
20	20	10	.

Cell evaluation matrix

As some of the cell values are *positive* (maximization problem), the initial basic feasible solution is not optimal.

Step IV : Iterate Towards an Optimal Solution

This involves substeps 1, 2, 3, 4 and 5, details of which are given in example 3·5-2.

Table 3·97

175 −	100	ϵ	25 +
		100	
√ +			5 −

First feasible solution with closed path

→

Table 3·98

70	100	ϵ	30
		100	
5			

2nd feasible solution

Table 3·98 gives the optimal solution. Its verification is left as an exercise for the reader. The demand at Ranchi is left unsatisfied by 5 units. The profit corresponding to the above scheme is

$$Z_{max} = \text{Rs. } (90 \times 70 + 90 \times 100 + 110 \times 30 + 130 \times 100)$$

$$= \text{Rs. } 31{,}600.$$

EXAMPLE 3·5·7 :

Solve example 3·1·5.

Solution

These are some transportation problems where the objective is to minimize *time* rather than transportation cost. Such problems are usually encountered in hospital management, military services, fire services, etc. The present example deals with the transportation of military equipment and speed of delivery is more important than the transportation cost. Therefore, in this example, the objective is to minimize the *transportation time* rather than transportation cost. The transportation matrix is shown in table 3·99.

Table 3·99

Destinations

Origins	1	2	3	4	a_i
1	10 (12)	0 (3)	20	11	15
2	1	7 (5)	9 (15)	(20) (5)	25
3	12	14	16	18 (5)	5
b_j	12	8	15	10	

Now, while solving problems where the objective is to minimize time for each route, the cost per unit is replaced by the total time required to ship the quantity x_{ij} from origin i to destination j, where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. The corresponding transportation matrix is given below.

Table 3·100

	1	2	3	...	n	a_i
1	t_{11}	t_{12}	t_{13}	...	t_{1n}	a_1
2	t_{21}	t_{22}	t_{23}	...	t_{2n}	a_2
3	t_{31}	t_{32}	t_{33}	...	t_{3n}	a_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮
m	t_{m1}	t_{m2}	t_{m3}	...	t_{mn}	a_m
b_j	b_1	b_2	b_3	...	b_n	

and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Note that the shipment of a feasible plan will be complete when the shipment with the largest time will be complete. Let T_k be the largest time associated with kth feasible plan. Our objective is, therefore, to find out a plan for which T_k is minimum for all values of k. The procedure for getting minimum T_k consists of the following steps :

Step I

Find an initial basic feasible solution. This is obtained by using the same method as for the normal transportation technique.

Step II

Find T_k corresponding to the current feasible solution and cross out all the non-basic cells for which $t_{ij} \geqslant T_k$.

Step III

Draw a closed path (as in the normal transportation technique) for the basic variable associated with T_k such that when the values at the corner elements are shifted around this basic variable reduces to zero and no variable becomes negative. This procedure ends if no such closed path can be traced out, otherwise repeat step II.

Thus in the given matrix

$x_{11}=12$, $x_{12}=3$, $x_{22}=5$, $x_{23}=15$, $x_{24}=5$ and $x_{34}=5$ and all other variables are zero.

The shipping times are $t_{11}=10$, $t_{12}=0$, $t_{22}=7$, $t_{23}=9$, $t_{24}=20$ and $t_{34}=18$. Therefore, all the shipments of this plan will be complete after

$$T_1=\max(t_{11}, t_{12}, t_{22}, t_{23}, t_{24}, t_{34})$$
$$=t_{24}=20 \text{ time units.}$$

Therefore, the cell (1, 3) is crossed out according to step II, since it has $t_{13}=20$ $(=T_1)$. The closed path for x_{24} is shown in the table below.

Table 3·101

	1	2	3	4
1	10 (12)	0 (3) −	20	11 +
2	1	7 (5) +	9 (15)	20 (5) −
3	12	14	16	18 (5)

It is clear from the closed path that x_{24} can be decreased by only three units, for, otherwise x_{12} will become negative. The new solution is shown in table 3·102.

Table 3·102

	1	2	3	4
1	10 (12)	0	20	11 (3)
2	1	7 (8)	9 (15)	20 (2)
3	12	14	16	18 (5)

For table 3·102, T_2 is still equal to 20. The next closed path is shown in table 3·103, where x_{24} is decreased further by 2 units, thereby reducing it to zero. This is shown in table 3·104.

Table 3·103

	1	2	3	4
1	10 (12) −	0	20	11 (3) +
2	1 +	7 (8)	9 (15)	20 (2) −
3	12	14	16	18 (5)

Table 3·104

	1	2	3	4
1	10 (10) −	0	20	11 (5) +
2	1 (2)	7 (8)	9 (15)	20
3	12 +	14	16	18 (5) −

In table 3·104, $T_3 = t_{34} = 18$, corresponding to x_{34}. The closed path for x_{34} is shown in table below.

Table 3·105

	1	2	3	4
1	10 (10)	0	~~20~~	11 (5)
2	1 (2)	7 (8)	9 (15)	~~20~~
3	12	14	16	18 (5)

Table 3·106

	1	2	3	4
1	10 (5)	0	~~20~~	11 (10)
2	1 (2)	7 (8)	9 (15)	~~20~~
3	12 (5)	14	16	~~18~~

x_{34} can be reduced by 5 units, thus reducing it to zero. So it is crossed out and the resulting solution is shown in table 3·106.

Now $T_4 = t_{31} = 12$. Hence cells (3, 2) and (3, 3) are crossed out since t_{32} and t_{33} are both $> t_{31}$. This is shown in table 3·107.

Table 3·107

	1	2	3	4
1	10 (5)	0	~~20~~	11 (10)
2	1 (2)	7 (8)	9 (15)	~~20~~
3	12 (5)	~~14~~	~~16~~	~~18~~

Since no closed path can be traced for x_{31}, the iterative procedure ends here. Thus the above plan is optimal and the total shipment time is 12 units. Details of the plan are

$x_{11} = 5$ with $t_{11} = 10$ $\quad$ $x_{22} = 8$ with $t_{22} = 7$

$x_{14} = 10$ with $t_{14} = 11$ $\quad$ $x_{23} = 15$ with $t_{23} = 9$

$x_{21} = 2$ with $t_{21} = 1$ $\quad$ $x_{31} = 5$ with $t_{31} = 12$.

3·6. Excessive Number of Filled Cells

Sometimes instead of arriving at a degenerate solution we may arrive at a solution having filled cells larger than given by the formula $m+n-1$. In such cases, a reduction in the number of used (filled or allocated) cell routes must be made. This can be done by finding four (six, eight,...) used cell routes which form a loop or a closed system. Consider the following example.

Table 3·108

Warehouse

Factory	1	2	3	4	Supply
A	+ 100	− 25			125
B	− 50	+ 200			250
C		125	50		175
D			40	60	100
Demand	150	350	90	60	

To decrease the number of filled cells, let us shift 25 units from cell (A, 2) as shown by the closed path above. The resulting matrix with reduced number of filled cells in shown below.

Table 3·109

Warehouse

Factory	1	2	3	4	Supply
A	125				125
B	25	225			250
C		125	50		175
D			40	60	100
Demand	150	350	90	60	

3·7. Alternate Optimal Solutions

A look at the cell evaluation matrices of the optimal solutions of examples 3·5·2 and 3·5·5 indicates that there are cells with zero

evaluations. In example 3·5-2, cell (3, 2) and in example 3·5-5, cells (4, 1) and (4, 2) have zero evaluations. In such cases, alternate optimal solutions exist. Alternate optimal solutions or programs are useful since they provide the programmer with a wider selection of 'best'

Destinations

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	8 +7	8 (1)	10 +3	6 (16)	10 0	17
S_2	0 (10)	8 (9)	6 (6)	12 +5	14 +3	25 (A)
S_3	12 +9	10 (1)	8 0	14 +5	12 (23)	24
	10	11	6	16	23	66 (Total)

choices and offer him the opportunity to consider secondary objectives as well. These alternate optimal solutions are obtained the same way as that used for normal optimal solutions. The only difference is that zeros appearing in the optimal cost evaluation matrix are treated exactly the same way as negative entries.

As an example, let us consider the following optimal solutions :

Destinations

		D_1	D_2	D_3	D_4	D_5	Supply
	S_1	8 +7	8 (1)	10 +3	6 (16)	10 0	17
Origins	S_2	0 (10)	8 (9)	6 (6)	12 +5	14 +3	25 (A)
	S_3	12 +9	1 (10)	8 (0)	14 +5	12 (23)	24
Demand		10	11	6	16	23	(66)

As the cell S_3D_3 in table (A) has a zero evaluation we may make the shifts in allocations indicated in table (A') by its closed path and we arrive at the following alternate optimal solution [table (B)] :

	D_1	D_2	D_3	D_4	D_5	
S_1	8 / +7	8 / (1)	10 / +3	6 / (16)	10 / 0	17
S_2	0 / (10)	8 / (10)	6 / (5)	12 / +5	14 / +3	25 (B)
S_3	12 / +9	10 / (1)	8 / (0)	14 / +5	12 / (23)	24
	10	11	6	16	23	

Similarly, as the cell S_1D_5 in table (A) has a zero evaluation, we may make the shifts in allocations indicated by its closed path in table (A′) and we arrive at another alternate optimal solution [table (C)].

	D_1	D_2	D_3	D_4	D_5	
S_1	8 / +7	8 / 0	10 / +3	6 / (16)	10 / (1)	17
S_2	0 / (10)	8 / (9)	6 / (6)	12 / +5	4 1 / +3	25 (C)
S_3	12 / +9	10 / (2)	8 / 0	14 / +5	12 / (22)	24
	10	11	6	16	23	

In addition to the above two alternate optimal solutions, we can derive two more optimal solutions, firstly utilizing the zero value in cell S_1D_5 of table (B) and secondly utilizing the zero value of the cell S_3D_3 in table (C). And that is not all; we can, in fact, find infinite number of optimal solutions. While deriving the alternate optimal solution matrix (B) from (A), we shifted one unit from S_3D_2 to S_3D_3. But instead of shifting one complete unit, we can shift 1/2 or 1/4 or 1/8th...units and we shall again arrive at an optimal solution.

For example, the alternate optimal solution obtained by shifting 1/2 unit from cell S_3D_2 in table (A) is shown below.

	D_1	D_2	D_3	D_4	D_5	
S_1	8 +7	8 (1)	10 +3	6 (16)	10 0	17
S_2	0 (10)	8 $(9\frac{1}{2})$	6 $(5\frac{1}{2})$	12 +5	14 +3	25 (D)
S_3	12 +9	10 $(\frac{1}{2})$	8 $(\frac{1}{2})$	14 +5	12 (23)	24
	10	11	6	16	23	

Thus we can get an infinite number of alternate optimal solutions. Thus whenever optimal solution contains vacant cells with zero evaluations, we have great flexibility in distribution at minimum cost.

3.8. Techniques for Simplifying Problem Solution

The arithmetic complexity of a problem can be considerably reduced by three ways :

(*i*) a little consideration over the examples discussed in this chapter will convince that it is the unit cost differences rather than their absolute values which determine the optimal solution. Therefore, if all the costs are reduced by a fixed amount (e.g. minimum cost in any cell), the resulting allocation will remain unaltered. This considerably reduces the calculation work.

(*ii*) the rim conditions (rim requirements) may be expressed in the simplest possible terms. For example, instead of writing down the output from plants as 70,000, 18,000, 25,000 units; it may be expressed as simply 70, 18 and 25 in terms of thousands of units. This will enable us to work with two digit numbers only instead of five digit numbers, which is undoubtedly easier.

(*iii*) north-west corner rule, row minima method, column minima method and minimum cost method may not be used for deriving initial feasible solution as they yield rather poor solutions, requiring many iterations. Instead, Vogel's approximation method may be used which yields a better initial feasible solution requiring lesser iterations to reach optimal solution.

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Exercises

Section 3·5

1. Determine an initial basic feasible solution to the following transportation problem using north-west corner rule :

Table 3.110

	To					Available
From	3	4	6	8	9	20
	2	10	1	5	8	30
	7	11	20	40	3	15
	2	1	9	14	16	13
Demand	40	6	8	18	6	

[*Meerut M.Sc. (Stat.) 1975*]

(*Ans.* $x_{11}=20$, $x_{21}=20$, $x_{22}=6$, $x_{23}=4$, $x_{33}=4$, $x_{34}=11$, $x_{44}=7$ and $x_{45}=6$.)

2. Find the feasible solution of the following transportation problem using north-west corner method :

Table 3·111

		Warehouse W$_1$	W$_2$	W$_3$	W$_4$	Supply
Factory	F$_1$	14	25	45	5	6
	F$_2$	65	25	35	55	8
	F$_3$	35	3	65	15	16
Requirement		4	7	6	13	30 (Total)

(*Ans.* $x_{11}=4$, $x_{12}=2$, $x_{22}=5$, $x_{23}=3$, $x_{33}=3$, $x_{34}=13$.)

3. Determine an initial basic feasible solution to the following transportation problem using row minima method :

Table 3·112

	To				Availability
From	5	2	4	3	22
	4	8	1	6	15
	4	6	7	5	8
Demand	7	12	17	9	

[*Delhi M.Sc.* (*Math.*) *1972*]

(*Ans.* $x_{12}=12$, $x_{13}=1$, $x_{14}=9$, $x_{23}=15$, $x_{31}=7$, $x_{33}=1$.)

4. Determine an initial basic feasible solution to the following transportation problem using the row minima method :

Table 3·113

	To			Available
From	3	8	5	5
	5	5	3	8
	7	6	9	7
	4	9	5	14
Required	7	9	18	

(*Ans.* $x_{11}=5$, $x_{23}=8$, $x_{32}=7$, $x_{41}=2$, $x_{42}=2$, $x_{43}=10$).

5. Determine an initial basic feasible solution to the following transportation problem using column minima method :

Table 3·114

		To			Availability
From	10	13	6		10
	16	7	13		12
	8	22	2		8
Requirement	6	11	13		30 (Total)

(*Ans.* $x_{13}=10, x_{22}=11, x_{23}=1, x_{31}=6, x_{33}=2.$)

6. Determine an initial basic feasible solution to the following transportation problem using column minima method :

Table 3·115

		Destination				
		D_1	D_2	D_3	D_4	
	O_1	1	2	1	4	20
Origin	O_2	3	3	2	1	40 Availability
	O_3	4	2	5	9	20
	O_4	5	3	6	10	20
Requirement		20	40	30	10	

(*Ans.* $x_{11}=20, x_{23}=30, x_{24}=10, x_{32}=20, x_{42}=20.$)

7. Find the initial basic feasible solution to the following transportation problem by :

(*a*) minimum cost method

(*b*) north-west corner rule.

State which of the methods is better.

Table 3·116

To

	2	7	4	5	
From	3	3	1	8	Supply
	5	4	7	7	
	1	6	2	14	
	7	9	18		

Demand

[*Meerut M.Sc.* (*Math.*) *1970*, *77*]

(*Ans.* (*a*) $x_{12}=2$, $x_{13}=4$, $x_{23}=8$. $x_{32}=7$, $x_{41}=7$, $x_{43}=7$.

(*b*) $x_{11}=5$, $x_{21}=2$, $x_{22}=6$, $x_{33}=3$, $x_{34}=4$, $x_{43}=14$; First.)

8. Find the initial basic feasible solution of the following transportation problem by Vogel's approximation method.

Table 3·117

Warehouse

		W_1	W_2	W_3	W_4	Capacity
	F_1	19	30	50	10	7
Factory	F_2	70	30	40	60	9
	F_3	40	8	70	20	18
Requirement		5	8	7	14	34 (Total)

[*Delhi M.Sc.* (*Math.*) *1971*]

(*Ans.* $x_{11}=5$, $x_{14}=2$, $x_{23}=7$, $x_{24}=2$, $x_{32}=8$, $x_{34}=10$.)

9. Determine an initial basic feasible solution to the following T. P. using :

(*a*) north-west corner rule

(*b*) Vogel's approximation method.

Table 3·118

Destination

Origin	A_1	B_1	C_1	D_1	E_1	Supply
A	2	11	10	3	7	4
B	1	4	7	2	1	8
C	3	9	4	8	12	9
Demand	3	3	4	5	6	

[*Delhi M.Sc.* (*Stat.*) *1973*]

(*Ans.* (*a*) $x_{11}=3$, $x_{12}=1$, $x_{22}=2$, $x_{23}=4$, $x_{24}=2$, $x_{34}=3$, $x_{35}=6$

(*b*) $x_{14}=4$, $x_{22}=2$, $x_{25}=6$, $x_{31}=3$, $x_{32}=1$, $x_{33}=4$, $x_{34}=1$.)

10. Find an initial basic feasible solution to the following T. P. using Vogel's approximation method :

Table 3·119

Destination

Origin	1	2	3	4	Availability
A	7	2	5	5	30
B	4	4	6	5	15
C	5	3	3	2	10
D	4	−1	4	2	20
Requirement	20	25	15	15	

(*Ans.* $x_{11}=5$, $x_{12}=5$, $x_{13}=15$, $x_{14}=5$, $x_{21}=15$, $x_{34}=10$, $x_{42}=20$.)

11. Solve the transportation problem for which the cost, origin availabilities and destination requirements are given below.

Table 3·120

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	175 (Total)

[*Delhi M. Sc. 1968*]

(*Ans.* $x_{11}=20$, $x_{13}=10$, $x_{23}=20$, $x_{24}=10$, $x_{25}=20$, $x_{32}=40$, $x'_{35}=10$, $x_{36}=25$, $x_{45}=20$.)

12. Give a mathematical formulation of the transportation and simplex methods. What are the differences in the nature of problems that can be solved by these methods ?

[*Delhi M. Sc. 1965*]

13. Given below is the unit costs array with supplies $a_i=1, 2, 3$ and demands b_j, $j=1, 2, 3$ and 4.

Table 3·121

Sink

		1	2	3	4	a_i
	1	8	10	7	6	50
Source	2	12	9	4	7	40
	3	9	11	10	8	30
	b_j	25	32	40	23	120 (Total)

Find the optimal solution to the above Hitchcock problem.

[*Delhi M. Sc. 1975*]

14. The matrix below represents the weekly output of cattle feed in tons, which is obtained as a product of alcohol distillation in two plants P_1 and P_2. The cattle feed is to be transported to four warehouses for distribution. The transportation costs per ton are also shown in the matrix. Find out the transportation pattern so as to minimize the transportation cost.

Table 3·122

Warehouses

		W_1	W_2	W_3	W_4	Output
Plants	P_1	2	3	8	1	23
	P_2	5	4	7	7	27
	Demand	12	13	15	10	50

(*Ans.* $P_1W_1=12$, $P_1W_2=1$, $P_1W_4=10$, $P_2W_2=12$, $P_2W_3=15$.)

15. A manufacturing company has three factories F_1, F_2 and F_3 with monthly manufacturing capacities of 7000, 4,000 and 10,000 units of a product. The product is to be supplied to seven stores. The manufacturing cost of these factories are slightly different but the important factor is the shipping cost from each factory to a particular store. Table 3·123 represents the factory capacities, store requirements and unit cost in rupees of shipping from each factory to each store and slack. Here, slack is the difference between the total factory capacity and the total store requirement.

Table 3·123

Store

		S_1	S_2	S_3	S_4	S_5	S_6	S_7	slack	Factory capacity
	F_1	5	6	4	3	7	5	4	0	7,000
Factory	F_2	9	4	3	4	3	2	1	0	4,000
	F_3	8	4	2	5	4	8	3	0	10,000
Store demand		1,000	2,000,	4,500,	4,000	2,000	3,500	3,000	1,000	

Work out a transportation plan so as to minimize the transportation cost.

(*Ans.* $F_1S_1=1{,}000$, $F_1S_4=4{,}000$, $F_1S_7=1{,}000$, F_1 slack$=1{,}000$, $F_2S_6=3{,}500$, $F_2S_4=500$, $F_3S_2=2{,}000$, $F_3S_3=4{,}500$ $F_3S_5=2{,}000$ and $F_3S_7=4{,}500$.)

16. Consider four bases of operations B_i and three targets T_j. The tons of bombs per air-craft from any base that can be delivered to any target are given in the following table :

Table 3·124

Target (T_j)

		1	2	3
Base (B_i)	1	8	6	5
	2	6	6	6
	3	10	8	4
	4	8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all the three targets explaining each step.

[*Roorkee M.Sc. (Math.) 1977*]

(*Ans.* $x_{11}=50$, $x_{12}=100$, $x_{21}=150$, $x_{33}=150$, $x_{42}=100$, $x_{43}=50$; $Z_{opt}=3{,}300$.)

17. There are three sources or origins which store a given product. These sources supply these products to four dealers. The capacity of the sources and the demands of the dealers are given below :

Sources	Demands
$S_1=150$	$D_1=90$
$S_2=40$	$D_2=70$
$S_3=80$	$D_3=50$
	$D_4=60$

The cost of transporting the products from various sources to various dealers is shown below.

Table 3·125

	D_1	D_3	D_3	D_4
S_1	27	23	31	69
S_2	10	45	40	32
S_3	30	54	35	57

Find out the optimal . solution for transporting the products at a minimum cost.

[*Roorkee M. E. (Mech.) 1977*]

(*Ans.* $x_{11}=30$, $x_{12}=70$, $x_{13}=50$, $x_{24}=40$, $x_{31}=60$, $x_{34}=20$; $Z_{opt}=8{,}190$.)

18. General Electrodes is a big electrode manufacturing company. It has two factories and three main distribution centres in three cities. The supply and demand conditions for units of electrodes (truck loads) are given below along with unit cost of transportation. How should the trips be scheduled so that the cost of transportation is minimized ?

The present cost of transportation is around Rs. 3,100 per month. What can be the maximum savings by proper scheduling?

Table 3·126

Centres	A	B	C
Requirement	50	50	150
Cost per trip from X plant	25	35	10
,, ,, Y ,,	20	5	80
Capacity of plant X	150 units of electrodes		
,, ,, ,, Y	100 ,, ,, ,,		

[*Bombay Dip. Opr. Manag. 1972*]

(*Ans.* XC=150, YA=50, YB=50 ; Z_{min}=Rs. 2,750 ; max. savings=Rs. 350.)

19. A company has three factories and five warehouses. The transportation costs, factory capacities and warehouse requirements are as given below :

Table 3·127

Factories

		A	B	C	Warehouse requirement
	1	5	4	8	400
	2	8	7	4	400
Warehouses	3	6	7	6	500
	4	6	6	6	400
	5	3	5	4	·800
Factory capacity		800	800	1,100	

Determine the optimal transportation schedule.

[*Pb. Univ.* ***Mech.*** *Engg. April, 1978*]

20. A company has decided to manufacture some or all of five new products at three of their plants. The production capacity of each of these three plants is as follows :

Plant no.	Production capacity in total number of units
1	40
2	60
3	90

Sales potential of the five products is as follows :

Product no.	1	2	3	4	5
Market potential in units	30	40	70	40	60

Plant no. 3 cannot produce product no. 5. The variable cost per unit for the respective plant and product combination is given below.

Product no.	1	2	3	4	5
Plant no. 1	20	19	14	21	16
,, ,, 2	15	20	13	9	16
,, ,, 3	18	15	18	20	—

Based on above data, determine the optimum product to plant combination by using linear programming.

[*Bombay Dip. Opr. Manag. 1976*]

(*Ans.* $x_{13}=10$, $x_{15}=30$, $x_{23}=60$, $x_{31}=30$, $x_{32}=40$, $x_{33}=20$;

$Z_{min}=2{,}940$.)

21. A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated the required amounts of gravel which will be needed at these construction projects.

Project location	Weekly requirement (truck loads)
A	72
B	102
C	41

The company has three gravel pits located in towns W, X and Y. The gravel required by the construction projects can be supplied by these three plants. The amount of gravel which can be supplied by each plant is as follows :

Plant	W	X	Y
Amount available (truck loads)	76	82	77

The company has computed the delivery cost from each plant to each project site. These costs in rupees are shown in the following table :

Table 3·128

Cost per truck load

		A	B	C
	W	4	8	8
Plant	X	16	24	16
	Y	8	16	24

Schedule the shipment from each plant to each project in such a manner so as to minimize the total transportation cost within the constraints imposed by plant capacities and project requirements. Find the minimum cost. Is the solution unique ? If it is not, find alternative schedule with the same minimum cost.

[*Madras B.E. 1977*]

(*Ans.* $x_{12}=76$, $x_{21}=21$, $x_{23}=41$, $x_{31}=51$, $x_{32}=26$; $Z_{min}=$Rs. 2,424.)

22. A department store wishes to purchase the following quantities of ladies dresses :

Dress type :	A	B	C	D
Quantity :	150	100	75	250

Tenders are submitted by three different manufacturers who undertake to supply not more than the quantities below (all types of dresses combined)

Manufacturer :	W	X	Y
Total quantity :	350	250	150

The store estimates that its profit per dress will vary with the manufacturer as shown in the matrix below. How should orders be placed ?

Table 3.129

		Dress			
		A	B	C	D
	W	2.75	3.50	4.25	2.25
Manufacturer	X	3.00	3.25	4.50	1.75
	Y	2.50	3.50	4.75	2.00

[*Baroda Univ. B.E. 1975*]

23. Three warehouses are used to stock the same product. Orders are received at these warehouses and shipped on weekly basis. Due to variations in shipping rates from warehouses to different customers, the problem becomes which warehouse should supply which customer and what quantity during shipping period. The warehouses, location of customers and shipping rates are given in table 3.130. Find the solution to the problem.

Table 3.130

From/To	D_1	D_2	D_3	D_4	Supply
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Requirement	22	55	25	41	

[*Baroda Univ. B.E. 1973*]

24. A company has factories at A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 160, 150 and 190 respectively. Unit transportation costs in rupees are as follows :

Table 3.131

To

From	D	E	F	G
A	42	48	38	37
B	40	49	52	51
C	39	38	40	43

Determine the minimum distribution cost.

[*Gwalior B.E. Nov. 1975*]

25. (*a*) Describe the unit penalty method for solving a transportation problem.

(*b*) Solve the following transportation problem (cell entries represent unit costs) :

Table 3.132

							Required
	5	3	7	3	8	5	3
	5	6	12	5	7	11	4
	2	8	3	4	8	2	2
	9	6	10	5	10	9	8
Available	3	3	6	2	1	2	

[*Sambalpur May, 1977*]

26. A fertiliser company has three plants A, B and C which supply to six major distribution centres 1, 2, 3, 4, 5 and 6. The tableau below gives the transportation costs per case, the plant annual capacities and predicted annual demands at different centres in terms of thousands of cases. The variable production costs per case are Rs. 8.50, Rs. 9.40 and Rs. 7.20 respectively at plants A, B and C. Determine the minimum cost production-and-transportation allocation.

Table 3.133

Transportation cost, Rs. per case

Major distribution centres

		1	2	3	4	5	6	Annual production in thousands of cases
	A	2.50	3.50	5.50	4.50	1.50	4.00	2,200
Plant	B	4.60	3.60	2.60	5.10	3.10	4.10	3,400
	C	5.30	4.30	4.80	2.30	3.30	2.80	1,800
Annual demand in thousands of cases		850	750	420	580	1,020	920	

Prove that if the variable production costs are the same at every plant, one can obtain an optimal allocation by using transportation costs only.

[*Gujarat Univ. B.E. (Mech.) 1976*]

27. Describe the transportation problem. Give method of finding an initial feasible solution. Explain what is meant by an optimality test. Give the method of improving over the initial solution to reach the optimal feasible.

[*Bombay M. Com. 1975*]

28. Solve the transportation problem when cost coefficients, demands and supplies are given in the following table :

Table 3.134

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	—2	3	70
O_2	2	4	0	1	38
O_3	1	2	—2	5	52
Demand	40	28	30	42	

[*Kharagpur Dip. I.I.T. 1978*]

29. The unit cost of transportation from site i to site j are given below. At site $i=1, 2, 3$, stocks of 150, 200 and 170 units respectively are available. 300 units are to be sent to site 4 and the rest to site 5. Find the cheapest way of doing this.

Table 3.135

To

		1	2	3	4	5
	1	—	3	4	10	7
	2	1	—	2	16	6
From	3	7	4	—	12	13
	4	8	3	9	—	5
	5	2	1	7	5	—

[*Dibrugarh M. Sc. (Stat.) 1974*]

[**Hint** : In accordance with the restrictions of supply and

demand table 3.135 reduces to the following table :

Table 3.136

		To 4	5	
From	1	10	7	150
	2	16	6	200
	3	12	13	170
		310	220	

Table 3.136 can now be easily solved.]

30. Consider the following unbalanced problem :

Table 3.137

From \ To	1	2	3	Supply
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
Demand	75	20	50	

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose that there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations 1, 2 and 3 respectively. Find the optimal solution.

[*Roorkee M.E. 1977*]

[**Hint**. The balanced transportation table with dummy source and associated penalty costs is shown below.

Table 3·138

To / From	1	2	3	Supply
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
Dummy	5	3	2	40
Demand	75	20	50	

This table can now be solved by the usual Modi method.]

31. A production control superintendent finds the following information on his desk :

In departments A, B and C, the number of surplus pallets is 18, 27 and 21 respectively. In departments G, H, I and J, the number of pallets required is 14, 12, 23 and 17 respectively. The *time* in minutes to move a pallet from one department to another is given below.

Table 3·139

To / *From*	G	H	I	J
A	13	25	12	21
B	18	23	14	9
C	23	15	12	16

What is the optimal distribution plan to minimize the moving *time* ?

[*Pb. Univ. Mech. Engg. Nov., 1977*]

(*Ans.* $x_{AG}=14$, $x_{AI}=10$, $x_{BJ}=17$, $x_{CH}=12$ and $x_{CI}=9$.)

32. The following table gives the cost of transporting material from supply points A, B, C and D to demand points E, F, H and J.

Table 3·140

To / From	E	F	G	H	J
A	8	10	12	17	15
B	15	13	18	11	9
C	14	20	6	10	13
D	13	19	7	6	12

The present allocation is as follows :

A to E 90, A to F 10, B to F 150, C to F 10, C to G 50, C to J 120, D to H 210, D to J 70.

(*a*) Check if this allocation is optimum. If not, find an optimum schedule.

(*b*) If in the above problem the transportation cost from A to G is reduced to 10, what wlll be the new optimum schedule ?

[*Bombay Dip. Ind. Manag. 1975*]

33. The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.

Table 3·141

Market

		I	II	III	IV	Supply
Warehouse	A	5	2	4	3	22
	B	4	8	1	6	15
	C	4	6	7	5	8
Requirement		7	12	17	9	

The shipping clerk has worked out the following schedule from experience :

12 units from A to II, 1 unit from A to III, 9 units from A to IV, 15 units from B to III, 7 units from C to I and 1 unit from C to III.

(*a*) Check and see if the clerk has the optimal schedule.

(*b*) Find the optimum schedule and minimum total shipping cost.

(*c*) If the clerk is approached by a carrier of route C to II who offers to reduce his rate in the hope of getting some business, by how much must the rate be reduced before the clerk should consider giving him an order ?

[*Madras B.E. 1977*]

34. A firm is having two plants P_1 and P_2. The products are supplied to three markets M_1, M_2 and M_3.

For the four quarters of the coming year the production capacity of plants P_1, P_2 and the demand at M_1, M_2, M_3 and M_4 are given in the table below.

Table 3·142

Production Capacity

Quarter	*Plant P_1*	*Plant P_2*
1	4	8
2	5	9
3	6	10
4	7	11

Demand at Market

Quarter	M_1	M_2	M_3
1	3	4	2
2	3	2	6
3	10	6	4
4	10	6	4

The cost of transporting unit product from plant P_i to market M_j is C_{ij}. The firm is to decide a policy of shipping products to the market so as to minimize the total cost of transportation.

Formulate a linear programming model so as to optimize the objective. Discuss if the model can be solved by transportation technique of linear programming.

[*Pb. Univ. Prod. Engg. April, 1979*]

35. A company has factories at A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 250, 300 and 400 units respectively for regular production. If overtime production is utilised, factories A and B can produce 50 and 75 additional units respectively at overtime incremental costs of Rs. 4 and Rs. 5 respectively. The current warehouse requirements are 200, 225, 275 and 300 units respectively. Unit transportation

costs in rupees from factories to warehouses are as follows :

Table 3·143

From \ To	D	E	F	G
A	11	13	17	14
B	16	18	14	10
C	21	24	13	10

Determine the optimum distribution for this company to minimize costs.

[*Delhi M.B.A. 1977*]

[**Hint** : First, table 3·144 is made which takes into account the overtime production and the corresponding production costs.

Table 3·144

	D	E	F	G	Supply
A	10	13	17	14	250
	15	17	21	18	50
B	16	18	14	10	300
	21	23	19	15	75
C	21	24	13	10	400
Demand	200	225	275	300	

In the above table, total supply $=1,075$ units

total demand $=1,000$ units

Therefore, we add a dummy warehouse with demand of 75 units and cost coefficients zero in each of its cells. Table 3·145 results.

Table 3·145

	D	E	F	G	Dummy	Supply
A	11	13	17	14	0	250
	15	17	21	18	0	50
B	16	18	14	10	0	300
	21	23	19	15	0	75
C	21	24	13	10	0	400
Demand	200	225	275	300	75	

The initial feasible solution can now be obtained and can be optimized using Modi method.]

4

Assignment Model

In Chapters 2 and 3 we discussed the simplex and transportation techniques for solving linear programming problems. However, there are some special cases of linear programming problems whose solutions can be obtained by special techniques. They are easier to apply and greatly reduce the computational work required by simplex and transportation techniques. This chapter deals with one such special case—the assignment problem which finds many applications in allocation and scheduling, for example, in assigning planes or crews to commercial airline flights, trucks or drivers to different routes, men to offices and space to departments. If there are more jobs to do than can be done, we can decide either which job to leave undone or what resources to add. We present here a few examples which will make the reader conversant with the various situations where assignment models are most suitable.

4·1. Examples on Applications of Assignment model

EXAMPLE 4·1·1. (Assignment Problem)

A machine tool company decides to make four subassemblies

Table 4·1

Subassemblies \ *Contractors*	1	2	3	4
1	15	13	14	17
2	11	12	15	13
3	13	12	10	11
4	15	17	14	16

through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in table 4·1 in hundreds of rupees.

Assign the different subassemblies to contractors so as to minimize the total cost.

EXAMPLE 4·1-2

(Assignment Problem—Non-Square Matrix)

A company has one surplus truck in each of the cities A, B, C, D and E and one deficit truck in each of the cities 1, 2, 3, 4, 5 and 6. The distance between the cities in kilometres is shown in the matrix below (table 4·2). Find the assignment of trucks from cities in surplus to cities in deficit so that the total distance covered by vehicles is minimum.

Table 4·2

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

EXAMPLE 4·1-3

Assignment Problem—Maximization Problem

A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below,

Table 4.3

District

		1	2	3	4
Salesman	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit.

EXAMPLE 4.1-4

(Assignment Problem—Restrictions on Assignments)

Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. C_{ij}, the assignment cost of machine i to place j in rupees is shown below.

Table 4.4

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	—	5	4
M_3	—	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule.

EXAMPLE 4.1-5. (A Typical Assignment Problem)

A trip from Chandigarh to Delhi takes six hours by bus. A typical time table of the bus service in both directions is given below.

Table 4.5

TIME-TABLE

Departure from Chandigarh	*Chandigarh-Delhi Service-line or route number*	*Arrival at Delhi*
0·600	*a* →	12.00
07.30	*b* →	13.30
11 30	*c* →	17.30
19.00	*d* →	01.00
00.30	*e* →	06.30

Table 4·6

Arrival at Chandigarh	*Delhi Chandigarh Service line or route number*	*Departure from Delhi*
11.30	1 ←	05.30
15.00	2 ←	09.00
21.00	3 ←	15.00
00.30	4 ←	18.30
06.00	5 ←	00.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service times. There are five crews. There is a constraint that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip. The company has residential facilities for the crew at Chandigarh as well as at Delhi.

Key decision in this situation is which crew be assigned which line of service or which service line be connected with which other line so as to reduce the waiting time to the minimum.

4.2. Matrix Terminology

The matrix used in the assignment models consists of squares called 'cells', which when stacked form 'columns' vertically and 'rows' horizontally.

Tables 4·7

Tasl or Requirements

Agents or Means		1	2	3
	A	10	12	13
	B	8	9	11
	C	10	11	15

The cell located at the intersection of a row and a column is designated by its row and column headings. Thus the cell located at the intersection of row A and column 2 in table 4.7 is called cell (A, 2). Unit cost, profit, time, etc. are placed in each cell.

4.3. Definition of Assignment Model

The assignment problem is a special type of allocation problem. In both cases the objective is to fulfil the targets by means of available resources which are available in specified amounts. But the operating conditions are different. While in the allocation problem each target can be attained in one way only, in the assignment problem the individual targets can be attained in different ways.

During our discussion of degeneracy we found that a transportation problem is degenerate if, while deriving a feasible solution, an allocation to any cell satisfies the column as well as row requirements simultaneously. We also know that in the assignment problem, each resource can be assigned to only one job and each job requires only one resource. Hence the assignment problem is a completely degenerate form of the transportation problem.

The assignment problem may be defined as follows :

Given n facilities and n jobs, and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so that the given measure of effectiveness is optimized.

The assignment problem stated above can be translated into problems in many decision fields. As an example, consider the following situation :

The municipal committee of a city has a fleet of n tractors located at different places in a city. There are also n trailers lying at different places in the same city and it is desired to pick up and haul the trailers to the centralised depot. The problem is to assign each of the n tractors to corresponding trailers in such a way that a given measure of effectiveness (e.g., total cost involved, or the total distance travelled or the total time of travel for tractors) is optimized.

Such a problem can be represented by $n \times n$ or n^2 matrix which constitutes n ! possible ways of making assignments. One natural method of finding the optimal solution is to enumerate all the n ! possible ways, evaluate their total cost (measure of effectiveness) and select the assignment with minimum cost. It can be easily seen that this basic method becomes extremely laborious even for small or moderate values of n. For example, when $n=10$, a common situation, the number of possible arrangements is $n\ !=10\ !=$ 3, 623, 800. Evaluations of so large a number of arrangements will take a prohibitively large time. This proves the need of an easy computational technique for solving the assignment problem.

4·4. Comparison with Transportation Model

Assignment model may be regarded as a special case of

Table 4.8

		Jobs 1	2	...	n	*Supply* a_i
Facilities	1	C_{11}	C_{12}	...	C_{1n}	1
	2	C_{21}	C_{22}	...	C_{2n}	1
	⋮	⋮	⋮	⋮	⋮	⋮
	m	C_{m1}	C_{m2}	...	C_{mn}	1
Demand b_j		1	1	...	1	

transporation model. Here, the facilities represent the 'sources' while the jobs represent the 'destinations'. The supply available at each source is 1 i.e., $a_i = 1$ for all i. Similarly, the demand required at each destination is 1 i.e., $b_j = 1$, for all j. The cost of 'transporting' (assigning) facility i to job j is C_{ij}. The resulting transportation model can be represented as in table 4·8.

4·5. Mathematical Representation of Assignment Model

Mathematically, assignment model can be expressed as follows :

Let

$$x_{ij} = \begin{cases} 0, \text{ if the } i\text{th facility is not assigned to } j\text{th job.} \\ 1, \text{ if the } i\text{th facility is assigned to } j\text{th job.} \end{cases}$$

Then, the model is given by

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{n} C_{ij}\, x_{ij} \left\{ = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}\, x_{ij} \right\},$$

$$\text{subject to constraints } \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, 3 \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, 3, \ldots n,$$

and $x_{ij} = 0$ or 1.

This last condition may also be expressed as

$$x_{ij} = x^2_{ij}.$$

We see that if the last condition is replaced by $x_{ij} \geq 0$, we have a transportation model with all requirements and available resources equal to 1.

However, transportation technique cannot be used to solve this model because of degeneracy. Whenever we make an assignment, we automatically satisfy row and column requirements simultaneously (rim requirements being equal to 1), resulting in degeneracy. This special structure of assignment model allows a more convenient method of solution.

The technique used for solving assignment model makes use of two theorems.

Theorem I

It states, "In an assignment problem, if we add or subtract a constant to every element of a row (or column) in the cost matrix, then an assignment which minimizes the total cost on one matrix also minimizes the total cost on the other matrix".

Thus if constants u_i and v_j are subtracted from the ith row and jth column respectively, then the new cost elements will become

$$C'_{ij} = C_{ij} - u_i - v_j$$

and the new objective function will be

$$Z' = \sum_i \sum_j C'_{ij}\, x_{ij} = \sum_i \sum_j (C_{ij} - u_i - v_j) x_{ij}$$

$$= \sum_i \sum_j C_{ij}\, x_{ij} - \sum_i u_i \sum_j x_{ij} - \sum_j v_j \sum_i x_{ij}$$

Since, from the constraints of the model

$$\sum_i x_{ij} = \sum_j x_{ij} = 1,$$

$$\therefore \quad Z' = \sum_i \sum_j C_{ij}\, x_{ij} - \sum_i u_i - \sum_j v_j$$

or $Z' = Z - \text{Constant}$.

This shows that the minimization of the new objective function Z' yields the same solution as the minimization of original objective function Z.

Theorem II

It states "If all $C_{ij} \geqslant 0$ and we can find a set $X_{ij} = x_{ij}$ such that $\sum_i \sum_j C_{ij}\, x_{ij} = 0$,

then this solution is optimal".

The above two theorems indicate that if one can create a new C'_{ij} matrix with zero entries, and if these zero elements, or a subset thereof, constitute a feasible solution, then this feasible solution is the optimal solution.

Thus the method of solution consists of adding and subtracting constants from rows and columns until sufficient number of C_{ij}'s become zero to yield a solution with a value of zero. The actual procedure for solving assignment models will be described by taking up a few industrial situations.

4·6. Formulation and Solution of Assignment Models

In this section we shall consider a few examples which will make clear the techniques of formulation and solution of assignment models.

EXAMPLE 4·6·1

Solve example 4·1·1

FORMULATION OF MODEL

Step I

Key decision is what to whom i.e., which subassembly be assigned to which contractor or what are the 'n', optimum assignments on 1-1 basis.

Step II

Feasible alternatives are n ! possible arrangements for $n \times n$ assignment situation. In the given situation there are 4 ! different arrangements.

Step III

Objective is to minimize the total cost involved,

i.e., minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \; x_{ij}$.

In the given situation the objective is

$$\text{minimize } Z = \sum_{i=1}^{4} \sum_{j=1}^{4} C_{ij} \; x_{ij}.$$

Step IV

Constraints : (*a*) constraints on subassemblies are

$$x_{11} + x_{12} + x_{13} + x_{14} = 1,$$
$$x_{21} + x_{22} + x_{23} + {}_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1,$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1.$$

(*b*) Constraints on contractors are

$$x_{11} + x_{21} + x_{31} + x_{41} = 1,$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1,$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1,$$
$$x_{14} + x_{24} + x_{34} + x_{44} = 1.$$

Comparing this model to the transportation model, we find that $a_i = 1$ and $b_j = 1$. Thus assignmental model can be represented as in table 4·9.

Therefore, assignment model is a special case of transportation model in which

(*i*) all right-hand-side constants in the constraints are unity i.e., $a_i = 1, b_j = 1$.

Table 4·9

Contractors (facilities, agents or means)

	1	2	3	4	Supply a_i
Subassemblies 1	15	13	14	17	1
(jobs, tasks or requirements) 2	11	12	15	13	1
3	13	12	10	11	1
4	15	17	14	16	1
Demand b_j	1	1	1	1	

(*ii*) all coefficients of x_{ij} constraints are unity.

(*iii*) $m=n$.

SOLUTION OF THE MODEL

We shall apply *Flood's technique* for solving assignment problems since it results in substantial saving in time over the other techniques. Technique involves rapidly reducing the original matrix and finding a set of *n independent* zeros, one in each row and column, which gives an optimal solution. This technique also known as *Hungarian Method* or *Reduced Matrix Method* consists of the following steps :

Step I

Prepare a Square Matrix : Since the situation involves a square matrix, this step is not necessary.

Step II

Reduce the Matrix : This involves the following substeps :

Substep 1 : In the effectiveness matrix, subtract the minimum element of each row from all the elements of the row. See if there is at least one zero in each row and in each column. If it is so, stop here. If not, proceed to substep 2.

Substep 2 : Now subtract the minimum element of each column from all the elements of the column.

In the given situation, the minimum element in first row is 13. So, we subtract 13 from all the elements of the first row. Similarly we subtract 11, 10 and 14 from all the elements of row 2, 3 and 4 respectively. This gives at least one zero in each row as shown in table 4.10.

Table 4·10

Contractors

Subassemblies

2	0	1	4
0	1	4	2
3	2	0	1
1	3	0	2

Since column 4 contains no zero entry, we go to substep 2 giving the following matrix :

Table 4·11

2	0	1	3
0	1	4	1
3	2	0	0
1	3	0	1

Step III

Check if Optimal Assignment can be made in the Current Solution or not

Basis for making this check is that if the minimum number of lines crossing all zeros is less than n (in our example $n=4$), then an optimal assignment cannot be made in the current solution. If it is equal to n $(=4)$, then optimal assignment can be made in the current solution.

Approach for obtaining minimum number of lines crossing all zeros consists of the following substeps :

Substep 1 : Examine rows successively until a row with exactly one unmarked zero is found. Mark (□) this zero, indicating that an assignment will be made there. Mark (×) all other zeros in the *same column* showing that they cannot be used for making other assignments. Proceed in this manner until all rows have been examined.

In the given situation, row 1 has a single unmarked zero in column 2. Make an assignment as shown. Row 2 has a single unmarked zero in column 1, make an assignment. Row 4 has a single unmarked zero in column 3, make an assignment and delete the 2nd zero in column 3 Now, row 3 has a single zero in column 4, make an assignment here. This is shown in the matrix below.

Table 4·12

2	$\boxed{0}$	1	3
$\boxed{0}$	1	4	1
3	2	$\cancel{0}$	$\boxed{0}$
1	3	$\boxed{0}$	1

Substep 2 : Next examine columns for single unmarked zeros, marking them (□) and also marking (×) any other zeros in their rows.

In case there is no row or column containing single unmarked zero (there are more than one unmarked zeros), mark (□) one of the unmarked zeros arbitrarily and mark (×) all other zeros in its row *and* column. Repeat the process till no unmarked zero is left in the cost matrix.

Substep 3 : Repeat substeps 1 and 2 successively till one of the two things occurs :

(*a*) there may be no row and no column without assignment i.e., there is one assignment in each row and in each column. In such a case the optimal assignment can be made in the current solution i.e , the current feasible solution is an optimal solution. The minimum number of lines crossing all zeros will be equal to '*n*'.

(*b*) there may be some row and/or column without assignment. Hence optimal assignment cannot be made in the current solution.

The minimum number of lines crossing all zeros have to be obtained in this case.

In the present example, substeps 2 and 3 are not necessary since there is no column left unmarked. Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. Thus minimum total cost is

$$= \text{Rs.} \quad (13\times 1+11\times 1+11\times 1+14\times 1)\times 100$$
$$= \text{Rs.} \quad 4900.$$

And the optimal assignments policy is

Subassembly 1—contractor 2,
,, 2— ,, 1,
,, 3— ,, 4,
,, 4— ,, 3.

EXAMPLE 4·6·2.

Four different jobs can be done on four different machines. The set up and take down time costs are assumed to be prohibitively high for change overs. The matrix below gives the cost in rupees of producing job i on machine j.

Table 4.13

Machines

		M_1	M_2	M_3	M_4
	J_1	5	7	11	6
	J_2	8	5	9	6
Jobs	J_3	4	7	10	7
	J_4	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized ?

Solution : FORMULATION OF THE MODEL

Step I

Key decision is to find what job be assigned to which machine i.e., what are the 'n' optimum assignments on 1-1 basis.

Step II

Feasible alternatives are 4 ! possible arrangements for the given 4×4 assignment situation.

Step III

Objective is to minimize the total cost involved,

i.e., minimize $Z=\sum_{i=1}^{4} \sum_{j=1}^{4} C_{ij} \cdot x_{ij}$

Step IV Constraints are

(*a*) due to jobs : $x_{11}+x_{12}+x_{13}+x_{14}=1,$
$x_{21}+x_{22}+x_{23}+x_{24}=1,$
$x_{31}+x_{32}+x_{33}+x_{34}=1,$
$x_{41}+x_{42}+x_{43}+x_{44}=1.$

(*b*) due to machines : $x_{11}+x_{21}+x_{31}+x_{41}=1,$
$x_{12}+x_{22}+x_{32}+x_{42}=1,$
$x_{13}+x_{23}+x_{33}+x_{43}=1,$
$x_{14}+x_{24}+x_{34}+x_{44}=1.$

Also $x_{ij}=0$ or 1.

Thus the problem is to optimize (minimize) the above objective function Z subject to the above constraints.

SOLUTION OF THE MODEL

Hungarian method or Reduced matrix. method will be used to solve the above model. This method consists of the following steps :

Step I

Prepare a Square Matrix : Since the situation involves a square matrix, this step is not necessary.

Step II

Reduce the Matrix : This involves the following substeps :

Table 4.14

	M_1	M_2	M_3	M_4
J_1	0	2	6	1
J_2	3	0	4	1
J_3	0	3	6	3
J_4	7	1	5	0

Matrix after substep 1 (contains no zero in column 3)

→

Table 4.15

	M_1	M_2	M_3	M_4
J_1	0	2	2	1
J_2	3	0	0	1
J_3	0	3	2	3
J_4	7	1	1	0

First feasible solution

Matrix after substep 2

Substep 1 : In the effectiveness matrix, subtract the minimum element of each row from all the elements of the row. See if there is at least one zero in each row and in each column. If it is so, stop here. If not, proceed to substep 2.

Substep 2 : Now subtract the minimum element of each column from all the elements of the column.

Following these two substeps we get table 4·14 which further leads to table 4·15.

Step III

Check if Optimal Assignment can be made in the Current Solution or not.

Basis for making this check is that if the minimum number of lines crossing all zeros is less than $n(n=4$ here), then an optimal assignment cannot be made in the current solution. If it is equal to $n(=4)$, then optimal assignment can be made in the current solution.

Approach for obtaining minimum number of lines crossing all zeros consists of the following substeps :

Substep 1 : Examine rows sucessively until a row with exactly one unmarked zero is found. Mark (□) this zero, indicating that an assignment will be made there. Mark (×) all other zeros in the *same column* showing that they cannot be used for making other assignments. Proceed in this manner until all rows have been examined.

Table 4·16

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	2	2	1
J_2	3	$\boxed{0}$	$\cancel{0}$	1
J_3	$\cancel{0}$	3	2	3
J_4	7	1	1	$\boxed{0}$

Substep 2 : Next examine columns for single unmarked zeros, marking them (□) and also marking (×) any other zeros in their rows.

Substep 3 : Repeat substeps 1 and 2 successively till one of the two things occurs :

(*a*) there may be no row and no column without assignment. In such a case optimal assignment can be made in the current solution.

(*b*) there may be some row/or column without assignment. Optimal assignment cannot be made in the current solution in such case and minimum number of lines crossing all zeros have to be determined.

In the present example, after following substeps 1 and 2 we find that their repetition is unnecessary and also row 3 and column 3 are without any assignments (table 4·16). Hence we proceed as follows to find the minimum number of lines crossing all zeros :

Table 4·17

	M_1	M_2	M_3	M_4	
J_1	$\boxed{0}$	2	2	1	√
J_2	3	$\boxed{0}$	$\cancel{0}$	1	
J_3	$\cancel{0}$	3	2	3	√
J_4	7	1	1	$\boxed{0}$	
	√				

Substep 4 : Mark (√) the rows for which assignment has not been made. In our problem it is the third row.

Substep 5 : Mark (√) columns (not already marked) which have zeros in marked rows. Thus column 1 is marked (√).

Substep 6 : Mark (√) rows (not already marked) which have assignments in marked columns. Thus row 1 is marked (√).

Substep 7 : Repeat steps 5 and 6 until no more marking is possible. In the present case this repetition is not necessary.

Substep 8 : Draw lines through all unmarked rows and through all marked columns. This gives the minimum number of lines crossing all zeros. If the procedure is correct, there will be as many lines as the number of assignments. In this example, number of such lines is

3 which is less than $n (n=4$ here). Hence optimal assignment is not possible in the current solution.

Step IV.

Iterate Towards Optimality

Examine the elements that do not have a line through them. Select the smallest of these elements and subtract it from all the elements that do not have a line through them. Add this smallest element to every element that lies at the intersection of two lines. Leave the remaining elements of the matrix unchanged. Proceeding in this manner we get the following matrix :

Table 4·18

	M_1	M_2	M_3	M_4	
J_1	0	1	1	0	
J_2	4	0	0	1	*Second feasible solution*
J_3	0	2	1	2	
J_4	8	1	1	0	

Step V.

Check if Optimal Assignment can be made in the Current Feasible Solution or not

Repeating step III i.e., substeps 1 through 8 we get table 4·19.

Table 4·19

	M_1	M_2	M_3	M_4	
J_1	$\cancel{0}$	1	1	$\cancel{0}$	✓
J_2	4	$\boxed{0}$	$\cancel{0}$	1	
J_3	$\boxed{0}$	2	1	2	✓
J_4	8	1	1	$\boxed{0}$	✓
	✓			✓	

Since the minimum number of lines passing through all zeros is 3 (<4), optimal assignment cannot be made in the current solution.

Step VI.

Iterate Towards Optimality

Table 4·20

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$
J_2	5	$\boxed{0}$	$\cancel{0}$	2
J_3	$\cancel{0}$	1	$\boxed{0}$	2
J_4	8	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

Third feasible solution

Step VII.

Check if Optimal Assignment can be made in the Current Feasible Solution or not.

Repeat step III i.e., sub-steps 1 through 8 therein. Since there is no row with exactly one unmarked zero, we start considering the columns directly.

Make assignment in cell (J_1, M_1) and delete remaining zeros in row 1 and column 1. Make assignment in cell (J_2,m_2) and delete the other zeros in row 2 and column 2. Make assignment in cell (J_3, M_3) and delete other zero in row 3 and column 3. Make assignment in cell (J_4, M_4).

As there is assignment in each row and in each column, optimal assignment can be made in the current solution. Hence optimal assignment policy is

Job J_1 should be assigned to Machine M_1,

J_2 ,, ,, ,, ,, ,, M_2,

J_3 ,, ,, ,, ,, ,, M_3,

J_4 ,, ,, ,, ,, ,, M_4,

and optimum cost = Rs. $(5+5+10+3)$ = Rs. 23.

4·7. Variations of the Assignment Problem

We shall now consider three variations of the assignment problem.

4·7·1. Non-Square ($m \times n$) Matrix

Sometimes the assignment problem is presented in the form of a matrix which is not square. This non-square matrix should be modified to a square matrix by adding suitable dummies. Fictitious facilities or jobs (dummies) may be added to the matrix depending upon whether $m<n$ or $m>n$ respectively. Let us make it clear by considering an example.

EXAMPLE 4.7-1.

Solve example 4.1-2.

Solution. Hungarian method or reduced-matrix method will be used to obtain optimal assignment. This method consists of the following steps :

Step I.

Prepare a Square Matrix : As the situation involves a non-square matrix, it has to be modified to a square matrix by adding dummies. Add a dummy city with surplus vehicle. Since there is no distance associated with it, the corresponding cell values are made all zeros.

Table 4.21

Cities in deficit

		1	2	3	4	5	6
	A	12	10	15	22	18	8
	B	10	18	25	15	16	12
Cities with surplus	C	11	10	3	8	5	9
	D	6	14	10	13	13	12
	E	8	12	11	7	13	10
	Dummy	0	0	0	0	0	0

Step II.

Reduce the Matrix : Proceeding as in example 4.6-2, we get table 4.22.

Table 4.22

	1	2	3	4	5	6	
A	4	2	7	14	10	0	*Matrix after sub-step 1.*
B	0	8	15	5	6	2	*(contains zero in each row and in each column)*
C	8	7	0	5	2	6	
D	0	8	4	7	7	6	*(Initial) feasible solution)*
E	1	5	4	0	6	3	
Dummy	0	0	0	0	0	0	

Step III

Check if Optimal Assignment can be made in the Current Solution or not

Proceeding as in Example 4.6-2, the minimum number of lines crossing all zeros are given by table 4.23.

Table 4.23

	1	2	3	4	5	6	
A	4	2	7	14	10	$\boxed{0}$	
B	$\boxed{0}$	8	15	5	6	2	✓
C	8	7	$\boxed{0}$	5	2	6	
D	$\cancel{0}$	8	4	7	7	6	✓
E	1	5	4	$\boxed{0}$	6	3	
Dummy	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	
	✓						

As the minimum number of lines crossing all zeros is 4 (<6), optimal assignment cannot be made in the current feasible solution.

Step IV. Iterate Towards Optimal Solution

Proceeding as in example 4.6-2, we get table 4.24

Table 4·24

	1	2	3	4	5	6
A	6	2	7	14	10	0
B	0	6	13	3	4	0
C	10	7	0	5	2	6
D	0	6	2	5	5	4
E	3	5	4	0	6	3
Dummy	2	0	0	0	0	0

Second feasible solution

Step V. Check if Optimal Assignment can be made in the Current Feasible Solution or not

Table 4·25

	1	2	3	4	5	6	
A	6	2	7	14	10	$\boxed{0}$	✓
B	$\boxed{0}$	6	13	3	4	$\cancel{0}$	✓
C	10	7	$\boxed{0}$	5	2	6	
D	$\cancel{0}$	6	2	5	5	4	✓
E	3	5	4	$\boxed{0}$	6	3	
Dummy	2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	
	✓					✓	

As the minimum number of lines crossing all zeros is 5 (<6), optimal assignment cannot be made in the current solution.

Step VI. Iterate Towards Optical Solution

Table 4.26

	1	2	3	4	5	6
A	6	0	5	12	8	0
B	0	4	11	1	2	0
C	12	7	0	5	2	8
D	0	4	0	3	3	4
E	5	5	4	0	6	5
Dummy	4	0	0	0	0	2

Third feasible solution

Step VII. Ceeck if Optimal Assignment can be made in the Current Feasible Solution or not

Table 4.27

	1	2	3	4	5	6
A	6	$\boxed{0}$	5	12	8	$\cancel{0}$
B	$\cancel{0}$	4	11	1	2	$\boxed{0}$
C	12	7	$\boxed{0}$	5	2	8
D	$\boxed{0}$	4	0	3	3	4
E	5	5	4	$\boxed{0}$	6	5
Dummy	4	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2

Since there is no row and no column without assignment, the third feasible solution is the optimal solution. The optimal assignment pattern is

City A should supply the vehicle to city 2,
,, B ,, ,, ,, ,, ,, ,, 6,
,, C ,, ,, ,, ,, ,, ,, 3,
,, D ,, ,, ,, ,, ,, ,, 1,
,, E ,, ,, ,, ,, ,, ,, 4, and

minimum distance travelled $= (10+12+3+6+7)$ km $= 38$ km.

4.7-2. Maximization Problem

Sometimes the assignment problem may deal with maximization of an objective function. For example, the problem may be to assign persons to jobs in such a way that the expected profit is maximized. The maximization problem has to be reduced to minimization problem before Hungarian method may be applied. It is done by subtracting from the highest element, all the elements of the matrix. Let us consider a maximization problem and see how it can be solved.

EXAMPLE 4.7-2.

Solve Example 4.1-3.

Solution. As the given problem is of maximization type, it has to be reduced to minimization type before solving it by Hungarian method. This is achieved by subtracting all the elements of the matrix from the highest element in it.

The given problem is to maximize the profit which is given by the matrix

Table 4.28

District

Salesman		1	2	3	4
	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

The highest element is 16. Subtracting all the elements from 16, the problem reduces to

minimize the profit given by the matrix 4.29.

Table 4.29

	1	2	3	4
A	0	6	2	5
B	2	5	1	1
C	1	1	3	4
D	3	4	2	1

Hungarian method can now be applied which consists of the following steps :

Step I

Prepare a Square Matrix : This step is not required here.

Step II

Reduce the Matrix : Proceeding as in example 4.6-2, we get table 4 30.

Table 4.30

	1	2	3	4
A	0	6	2	5
B	1	4	0	0
C	0	0	2	3
D	2	3	1	0

Matrix after substep 1

Initial feasible solution

Step III. Check if Optimal Assignment can be made in the Current Feasible Solution or not

Proceeding as in example 4.6-2 we get

Table 4·31

	1	2	3	4
A	$\boxed{0}$	6	2	5
B	1	4	$\boxed{0}$	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	2	3
D	2	3	1	$\boxed{0}$

As there is one assignment in each row and in each column, optimal assignment can be made in the current feasible solution. Assignment policy shall be

salesman A should be assigned district 1,
,, B ,, ,, ,, ,, 3,
,, C ,, ,, ,, ,, 2,
,, D ,, ,, ,, ,, 4.

Maximum profit per day = Rs. (16+15+15+15) = Rs. 61.

4.7-3. Restrictions on Assignments

So far, we have considered examples in which all C_{ij}'s were finite elements. Sometimes, technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. This difficulty can be overcome by assigning a very high cost (infinite cost) to the corresponding cell. The activity will be automatically excluded from the optimal solution.

EXAMPLE 4.7-3

Solve example 4.1-4.

Solution : Step I. Prepare a Square Matrix

As the given matrix is non-square, we add a dummy machine and associate zero cost with the corresponding cells. As machine M_2 cannot be placed at C and M_3 cannot be placed at A, we assign infinite cost (∞) in cells (M_2, C) and (M_3, A), resulting in the following matrix.

Table 4·32

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	∞	5	4
M_3	∞	6	9	6	2
M_4	9	3	7	2	3
Dummy machine	0	0	0	0	0

Step II. Reduce the Matrix

Proceeding as in example 4·6-2 the following matrix results :

Table 4·33

	A	B	C	D	E	
M_1	0	2	6	1	2	*Matrix after substep 1*
M_2	3	0	∞	1	0	
M_3	∞	4	7	4	0	*Initial feasible solution*
M_4	7	1	5	0	1	
Dummy	0	0	0	0	0	

Step III. Check if Optimal Assignment can be made in the Current Feasible Solution or not

Proceeding as in example 4·6-2 we get table 4·34.

Table 4·34

	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	$\cancel{0}$
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{C}$	1
Dummy	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

As there is no row and no column without assignment, optimal assignment can be made in the initial feasible solution. The optimal assignment of various machines is as follows :

Machine M_1 to place A,

,, M_2 ,, ,, B,

,, M_3 ,, ,, C,

,, M_4 ,, ,, D,

and place C will remain vacant.

Total assignment cost = Rs. $(4+4+2+7)$ = Rs. 17.

EXAMPLE 4·7·4.

Solve example 4·1·5.

Solution. As service time is constant, it does not affect the key decision. If all the crew is asked to reside at Chandigarh (so that they start from and come back to Chandigarh with minimum halt time at Delhi), then waiting time at Delhi for different service line connections will be given by table 4·35.

Table 4·35

	1	2	3	4	5
a	17.5	21	3	6·5	12
b	16	19·5	1·5	5	10·5
c	12	15·5	21·5	1	6·5
d	4.5	8	14	17·5	23
e	23	2·5	8·5	12	17·5

Similarly, if crew is assumed to reside at Delhi (so that they start from and come back to Delhi with halt for minimum time at Chandigarh) then waiting time for different service line connections is

Table 4·36

	1	2	3	4	5
a	18·5	15	9	5·5	0
b	20	16·5	10·5	7	1·5
c	0	20·5	14·5	11	5·5
d	7·5	4	22	18·5	13
e	13	9·5	3·5	0	18·5

As the crew can be asked to reside at Chandigarh or Delhi, minimum waiting time from the above matrices can be obtained for the different route connections by choosing minimum value out of the two

waiting times, provided the value is more than 4 hours (minimum desired rest). These values of waiting times are shown below in table 4·37·

Tabe 4·37

	1	2	3	4	5
a	17·5	15	9	5·5	12
b	16	16·5	10·5	5	10·5
c	12	15·5	14·5	11	5·5
d	4·5	8	14	17·5	13
e	13	9·5	8·5	12	17.5

Hungarian method can now be employed for finding the optimal route connections which give minimum overall waiting time and hence the minimum cost of bus service operations. It consists of the following steps :

Step I. Prepare a Square Matrix

This step is not necessary here.

Step II.

Reduce the Matrix : Proceeding as in example 4.6-2, we get table 4·38.

Table 4·38

	1	2	3	4	5
a	12	9·5	3·5	0	6·5
b	11	11·5	5·5	0	5·5
c	6·5	10	9	5·5	0
d	0	3·5	9·5	13	8·5
e	4·5	1	0	3·5	9

Matrix after substep 1

After substep 2 we get the following matrix :

Table 4·39

	1	2	3	4	5	
a	11	8·5	3·5	0	6·5	*Initial feasible solution*
b	12	10·5	5·5	0	5·5	
c	6·5	9	9	5·5	0	
d	0	2·5	9·5	13	8·5	
e	4·5	0	0	3·5	0	

Step III. Check if Optimal Assignment can be made in the Current Feasible Solution or not

Proceeding as in example 4.6-2, we get

Table 4·40

	1	2	3	4	5	
a	12	8.5	3.5	$\boxed{0}$	6.5	✓
b	11	10.5	5.5	$\cancel{0}$	5.5	✓
c	6.5	9	9	5.5	$\boxed{0}$	
d	0	2.5	9.5	13	8.5	
e	4.5	$\boxed{0}$	$\cancel{0}$	3.5	9	
				✓		

As the minimum number of lines crossing all zeros is 4 *i.e.*, less than 5, optimal assignment cannot be made in the current feasible solution.

Step IV. Iterate towards Optimal Solution

Proceding as in example 4.6-2, we get table 4·41.

Table 4·41

	1	2	3	4	5
a	8·5	5	0	0	3
b	7·5	7	2	0	2
c	6·5	9	9	9	0
d	0	2·5	9·5	16·5	8·5
e	4·5	0	0	7	9

Second feasible solution

Step V. Check if Optimal Assignment can be made in the Current Feasible Solution or not

Table 4·42

	1	2	3	4	5
a	8.5	5	$\boxed{0}$	$\cancel{0}$	3
b	7.5	7	2	$\boxed{0}$	2
c	6.5	9	9	9	$\boxed{0}$
d	$\boxed{0}$	2.5	9.5	16.5	8.5
e	4.5	$\boxed{0}$	$\cancel{0}$	7	9

As there is no row or column without assignment, optimal assignment is possible in the current solution.

∴ We get the following information :

Table 4·43

Crew	*Residence at*	*Service number*	*Waiting time (hours)*
1	Chandigarh	$d1$	4·5
2	Delhi	$2e$	9 5
3	Delhi	$3a$	9·0
4	Chandigarh	$b4$	5·0
5	Delhi	$5c$	5·5

Total minimum waiting time $=(4{\cdot}5+9\ 5+9+5+5{\cdot}5)$ hours
$=33{\cdot}5$ hours $=33$ hrs., 30 minutes.

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Exercises

Section 4·6

1. (*a*) Show that assignment model is a special case of transportation model. [*Pb. Univ. Mech. Engg. April, 1977*]

(*b*) Consider the problem of assiging five operators to five machines. The assignment costs are given below.

Table 4·44

		Operators				
		I	II	III	IV	V
	A	10	5	13	15	16
	B	3	9	18	3	6
Machines	C	10	7	2	2	2
	D	5	11	9	7	12
	E	7	9	10	4	12

Assign the jobs to different machines so that total cost is minimized. [*Pb. Univ. Mech. Engg. April, 1976*]

2. (*a*) If in an assignment problem we add a constant to every element of a row (or column) in the effectiveness matrix, prove that an assignment that minimizes the total effectiveness in one matrix also minimizes the total effectiveness in the other matrix.

[*Sambalpur Univ. May, 1977*]

(*b*) A national car-rental service has a surplus of one car in each of the cities 1, 2, 3, 4, 5, 6 and a deficit of one car in each of the cities 7, 8, 9, 10, 11, 12. The distances in miles between cities with a surplus and cities with a deficit are displayed in matrix 4·45. How should the cars be despatched so as to minimize the total milage travelled ?

Table 4·45

To

		7	8	9	10	11	12
	1	41	72	39	52	25	51
	2	22	29	49	65	81	50
From	3	27	39	60	51	32	32
	4	45	50	48	52	37	43
	5	29	40	39	26	30	33
	6	82	40	40	60	51	30

[*Sambalpur Univ. May, 1977*]

(*Ans.* 1—11, 2—8, 3—7, 4—9, 5—10, 6—12; $Z_{min}=185$)

3. (*a*) Distinguish between transportation model and assignment model.

(*b*) Four different jobs are to be done on four different machines. The setup and production times are prohibitively high for change over. The matrix below indicates the cost of producing job *i* on machine *j* in rupees.

Table 4·46

Machines

		1	2	3	4
Jobs	1	5	7	11	6
	2	8	5	9	6
	3	4	7	10	7
	4	10	4	8	3

Assign jobs to different machines so that the total cost is minimized. [*Pb. Univ. Mech. Engg. Nov., 1977*]

4. Six machines M_1, M_2, M_3, M_4, M_5 and M_6 are to be located in six places P_1, P_2, P_3, P_4, P_5 and P_6. C_{ij}, the cost of locating machine M_i at place P_j is given in the matrix below.

Table 4.47

	P_1	P_2	P_3	P_4	P_5	P_6
M_1	20	23	18	10	16	20
M_2	50	20	17	16	15	11
M_3	60	30	40	55	8	7
M_4	6	7	10	20	25	9
M_5	18	19	28	17	60	70
M_6	9	10	20	30	40	55

Formulate an L.P. model to determine an optimal assignment. Write the objective function and the constraints in detail. Define any symbol used. Find an optimal layout by assignment technique of linear programming. [*Pb. Univ. Prod. Engg. April, 1977*]

5. (*a*) Discuss assignment model. Indicate a method of solving an assignment problem. [*Pb. Univ. Mech. Engg. April, 1978*]

(*b*) A national bus service has a surplus of one bus in each of the cities 1, 2, 3, 4, 5, and 6 and a deficit of one bus in each of the cities 7, 8, 9, 10, 11 and 12. The distances in miles between cities with a surplus and cities with a deficit bus are displayed in table 4.48. How should the buses be despatched to minimize the total milage travelled ?

Table 4.48

To

		7	8	9	10	11	12
	1	30	61	28	41	14	40
	2	11	18	38	54	70	39
From	3	16	28	49	40	21	21
	4	34	39	37	41	26	32
	5	18	29	28	15	19	22
	6	71	29	29	49	40	19

[*Ans. 1—11, 2—8, 3—7, 4—9, 5—10, 6—12* ; $Z_{min}=119$]

6. Five new machines are to be located in a machine shop. There are five possible locations in which the machines can be located. C_{ij}, the cost of placing machine i in place j is given in the table below.

Table 4.49

Place

Machine	1	2	3	4	5
1	15	10	25	25	10
2	1	8	10	20	2
3	8	9	17	20	10
4	14	10	25	27	15
5	10	8	25	27	12

It is required to place the machines at suitable places so as to minimize the total cost.

(*i*) Formulate an L.P. model to find an optimal assignment.

(*ii*) Solve the problem by assignment technique of L.P.

[*Pb. Univ. Prod. Engg. April, 1979*]

7. Find the optimal assignment for the assignment problem with the following cost matrix :

Table 4.50

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

(**Ans.** *A—III, B—IV, C—II, D—I* ; $Z_{min} = 16$

8. A project consists of four major jobs for which four contractors have submitted tenders. The tender amounts quoted in lakhs of rupees are given in the matrix below. Find the assignment which minimizes the total cost of the project. Each contractor has to be assigned at least one job.

Table 4.51

		Job			
		a	*b*	*c*	*d*
Contractor	1	10	24	30	15
	2	16	22	28	12
	3	12	20	32	10
	4	9	26	34	16

[*I.S.I. Dip. 1976*]

(*Ans.* 1—*b*, 2—*c*, 3—*d*, 4—*a* ; Z_{min}=71)

9. Solve the following assignment problem :

Table 4.52

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

[*Delhi B. Sc.* (*Math.*) *1977*]

(*Ans.* 1–I, 2–IV, 3–V, 4–III, 5–II ; Z_{min}=60)

10. A team of 5 horses and 5 riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any horse is shown below.

Table 4.53

Rider

		R_1	R_2	R_3	R_4	R_5
	H_1	5	3	4	7	1
	H_2	2	3	7	6	5
Horse	H_3	4	1	5	2	4
	H_4	6	8	1	2	3
	H_5	4	2	5	7	1

How should the horses be allotted to the riders so as to minimize the expected loss of the team ?

[*Delhi M. Sc. (Stat.) 1973*]

(*Ans.* H_1-R_5, H_2-R_1, H_3-R_4, H_4-R_3, H_5-R_2 ; $Z_{min}=8$)

11. Find the minimum cost solution for the 5×5 assignment problem whose cost coefficients are as given below.

Table 4.54

	1	2	3	4	5
1	−2	−4	−8	−6	−1
2	0	−9	−5	−5	−4
3	−3	−8	−9	−2	−6
4	−4	−3	−1	0	−3
5	−9	−5	−8	−9	−5

[*Roorkee M.E. (Mech.) 1977*]

(*Ans.* 1−3, 2−2, 3−5, 4−1, 5−4 ; $Z_{min}=36$)

Section 4.7-1

12. A company has four machines on which to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table :

Table 4.55

		Machine W	X	Y	Z
	A	18	24	28	32
Job	B	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost ?

[*Madras B.E. 1977*]

(*Ans.* A—W, B—X, C—Y : $Z_{min}=50$.)

13. Assign four trucks 1, 2, 3 and 4 to vacant spaces, 7, 8, 9, 10, 11 and 12 so that the distance travelled is minimized. The matrix below shows the distance.

Table 4.56

	1	2	3	4
7	4	7	3	7
8	8	2	5	5
9	4	9	6	9
10	7	5	4	8
11	6	3	5	4
12	6	8	7	3

(*Ans.* 7—3, 8—2, 9—1, 12—4 ; $Z_{min}=12$)

Section 4·7·2.

14. A department has four subordinates and four tasks to be performed. The subordinates differ in efficiency and tasks differ in their intrinsic difficulty. The estimates of the profit in rupees each man would earn is given in the effectiveness matrix. How should the tasks be allocated, one to each man, so as to maximize the total earnings ?

Table 4·57

		Task 5	6	7	8
Subordinate	1	5	40	20	5
	2	25	35	30	25
	3	15	25	20	10
	4	15	5	30	15

15. A company has five jobs to be done. The following matrix shows the return in rupees of assigning ith machine ($i=1, 2, \ldots, 5$) to the jth job ($j=1, 2, \ldots, 5$). Assign the five jobs to the five machines so as to maximize the total expected profit.

Table 4·58

		Job 1	2	3	4	5
Machine	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

[*Bombay B.Sc. (Stat.) 1974*]

(*Ans.* 1—3, 2—5, 3—4, 4—2, 5—1, $Z_{max}=50$)

16. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with the expected profit in rupees for each machinist on each job being as follows. Find the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined ?

Table 4·59

Job

Machinist	A	B	C	D	E
1	6·20	7·80	5·00	10·10	8·20
2	7·10	8·40	6·10	7·30	5·90
3	8·70	9·20	11·10	7·10	8·10
4	4·80	6·40	8·70	7·70	8·00

(*Delhi M.B.A. 1976*)

17. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows :

Table 4·60

Job

Machinist	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Find by using the assignment method, the assignment of machinists to jobs that will result in maximum profit. Which job should be declined ? [*Madras B.E. Mech. 1977*]

Section 4·7-3.

18. Consider the problem of assigning five operators to five machines. The assignment costs are given in table 4·61.

Table 4·61

Machine

Operator	M_1	M_2	M_3	M_4	M_5
A	7	7	—	4	8
B	9	6	4	5	6
C	11	5	7	—	5
D	9	4	8	9	4
E	8	7	9	11	3

Operator A cannot be assigned to machine M_3 and operator C cannot be assigned to machine M_4. Find the optimum assignment schedule.

19. Consider the problem of assigning 5 operators to 5 machines. The assignment costs are given below.

Table 4·62

Machine

Operator	1	2	3	4	5
1	5	5	—	2	6
2	7	4	2	3	4
3	9	3	5	—	3
4	7	2	6	7	2
5	6	5	7	9	1

Operator 1 cannot be assigned to machine 3 and operator 3 cannot be assigned to machine 4. Find the optimal assignment schedule. [*Pb. Univ. Mech. Engg. April, 1977*]

20. An airline that operates seven days a week has time-table as shown below. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing, the crew will be based at the city that results in smaller layover.

Table 4·63

Delhi – Jaipur			*Jaipur – Delhi*		
Flight No.	*Depart*	*Arrive*	*Flight No.*	*Depart*	*Arrive*
1	7·00 A.M.	8·00 A.M.	101	8·00 A.M.	9·15 A.M.
2	8·00 A.M.	9·00 A.M.	102	8·30 A.M.	9·45 A.M.
3	1·30 P.M.	2·30 P.M.	103	12·00 Noon	1·15 P.M.
4	6·30 P.M.	7·30 P.M.	104	5·30 P.M.	6·45 P.M.

For each pair also mention the town where the crew should be based. [*Agra M. Stat. 1974*]

(*Ans.* 3—101, 4—102, 1—103, 2—104; $Z_{min}=52{\cdot}5$ hrs.)

21. An air line that operates between Delhi (A) and Chandigarh (B) has the time-table as shown in table 4·64. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. Note that crews flying from A to B can be based either at A or at B. For any given pairing they will be based at the city that results in the smaller layover.

Table 4·64

Delhi—Chandigarh

Flight number	*Departure*	*Arrival*
1	6·00 A.M.	7·00 A.M.
2	7·30 A.M.	8·30 A.M.
3	10·30 A.M.	11·30 A.M.
4	2·00 P.M	3·00 P.M.
5	6·00 P.M.	7·00 P.M.
6	11·30 P.M.	0·30 A.M.

Chandigarh — Delhi

Flight number	*Departure*	*Arrival*
101	8·00 A.M.	9 15 A.M.
102	9 00 A.M.	10·15 A.M.
103	11·30 A.M.	0·45 P.M.
104	3·00 P.M.	4·15 P.M.
105	7·30 P.M.	8·45 P.M.
106	10·00 P. M.	11·15 P.M.

22. A company has four territories open and four salesmen available for assignment. The territories are not equally rich in their sales potential ; it is estimated that a typical salesman operating in each territory would bring in the following annual sales :

Table 4.65

Territory	*Annual sales (Rs.)*
I	60,000
II	50,000
III	40,000
IV	30,000

The four salesmen are also considered to differ in ability ; it is estimated that working under the same conditions, their yearly sales will be proportionately as follows :

Table 4.66

Salesman	*Proportion*
A	7
B	5
C	5
D	4

If the criterion is maximum expected total sales, the intutive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by the assignment technique. [*Meerut M. Sc.* (*Stat.*) *1971*]

23. (*a*) State mathematical model of assignment problem.

(*b*) The personnel manager of a medium size company decides to recruit two employees *D* and *E* in a particular section of the origanisation. The section has five fairly defined tasks 1, 2, 3, 4 and 5 ; and three employees *A*, *B* and *C* are already employed in the section. Looking to the rather specialised nature of task 3 and the special qualifications of the recruit *D* for task 3, the manager decides to assign task 3 to employee *D* and then assign the remaining tasks to remaining employees so as to maximize the total effectiveness. The index of effectiveness of each employee for different tasks is as under.

Table 4.67

Employee	Task 1	2	3	4	5
A	25	55	60	45	30
B	45	65	55	35	40
C	10	35	45	55	65
D	40	30	70	40	60
E	55	45	40	55	10

Assign the tasks for maximizing total effectiveness. Critically examine whether the decision of the manager to assign task 3 to employee *D* was correct. [*Gujarat Univ. B.E. April, 1976*]

5

Sequencing models and Related Problems

This chapter deals with the situations in which the effectiveness measure (time, cost, distance, etc.) is a function of the order or sequence of performing a series of jobs (tasks). The selection of the appropriate order in which waiting customers may be served is called *sequencing*. Sequencing problems can be classified in two groups :

1. In the first group, there are n jobs to be performed, where each job requires processing on some or all of m different machines. The order in which these machines are to be used for processing each job as well as the expected or actual processing time of each job on each of the machines is known. We can also measure the effectiveness for any given sequence of jobs at each of the machines and we wish to select from the $(n\,!)^m$ *theoretically feasible* alternatives, the one which is both *technologically feasible* and optimizes the effectiveness measure (*e.g.*, minimizes the total elapsed time from the start of the first job to the completion of the last job as well as idle time of machines). A technologically feasible sequence is one which satisfies the constraints (if any) on the order in which each job must be performed through the m machines. The technology of manufacturing processes renders many sequences technologically infeasible. For example, a part must be degreased before it is painted ; similarly, a hole must be drilled before it is threaded.

Although, theoretically, it is always possible to select the best sequence by testing each one ; in practice, it is impossible because of the large number of computations involved. For example, if there are 4 jobs to be processed at each of the 5 machines (*i.e.*, $n=4$

and $m=5$), the total number of theoretically possible different sequences will be $(4\,!)^5=7,962,624$. Of course, as already said, some of them may not be feasible because the required operations must be performed in a specified order. Obviously, any technique which helps us arrive at an optimal (or at least approximately so) sequence without trying all or most of the possibilities will be quite valuable.

2. The second group of problems deals with job shops having a number of machines and a list of tasks to be performed. Each time a task is completed by a machine, the next task to be started on it has got to be decided. Thus the list of tasks will change as fresh orders are received.

Unfortunately, both types of problems are intrinsically difficult. While solutions are possible for some simple cases of the first type, only some empirical rules have been developed for the second type till now.

A few situations where sequencing models have been successfully applied are given in the next section.

5·1. Examples on the Applications of Sequencing Models

EXAMPLE 5.1-1. (*Processing n jobs through two machines*) :

A machine operator has to perform two operations, turning and threading, on a number of different jobs. The time required to perform these operations (in minutes) for each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs.

Table 5·1.

Job	*Time for turning (minutes)*	*Time for threading (minutes)*
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3
6	11	1

Example 5.1·2 : (*Processing n jobs through three machines*) :

If an additional operation of knurling is also included, what should be the order of processing the jobs so as to minimize the total time required to turn out all the jobs ? Table 5·2 represents the given data.

Table 5 2.

Job	*Time for turning (minutes)*	*Time for threading (minutes)*	*Time for knurling (minutes)*
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

EXAMPLE 5·1·3 (*Processing two jobs through m machines*) :

Using graphical method, calculate the minimum time needed to process job 1 and 2 on five machines, A, B, C, D and E i.e., for each machine find the job which should be done first. Also calculate the total time needed to complete both jobs.

Table 5·3

Job 1	*Sequence*	A	B	C	D	E
	Time (hrs.)	1	2	3	5	1
Job 2	*Sequence*	C	A	D	E	B
	Time (hrs.)	3	4	2	1	5

5·2. Sequencing Problems

In sequencing problems, there are two or more customers to be served (or jobs to be done) and one or more facilities (machines) available for this purpose. We want to know when each job is to begin and what its due date is. We also want to know which facilities are required to do each job, in which order these facilities are required and how long each operation is to take.

Sequencing problems have been most commonly encountered in production shops where different products are to be processed over various combinations of machines.

However, sequencing problems can arise even where only one service facility is involved, for example, a number of programs waiting to get on a computer or a number of patients waiting for a doctor.

The following simplifying assumptions are usually made while dealing with sequencing problems :

(*i*) only one operation is carried out on a machine at a particular time.

(*ii*) each operation, once started, must be completed.

(*iii*) an operation must be completed before its succeeding operation can start.

(*iv*) only one machine of each type is available.

(*v*) a job is processed as soon as possible, but only in the order specified.

(*vi*) processing times are independent of order of performing the operations.

(*vii*) the transportation time *i.e.*, the time required to transport jobs from one machine to another is negligible.

(*viii*) jobs are completely known and are ready for processing when the period under consideration starts.

5·3. Processing each of n Jobs through m Machines

Let there be n jobs $(1, 2, 3, \ldots, n)$, each of which has to be processed, one at a time, on each of m machines (A, B, C,....) The order of processing each job through the machines is given (for example, job 1 is processed on machines A, C, B, in this order). **Also,** the time required for processing each job on each machine is given. The problem is to find among $(n\,!)^m$ possible sequences, that *technologically feasible* sequence for processing the jobs which gives the *minimum total elapsed time* for all the jobs.

Symbolically,

Let A_i = time required for job i on machine A,

B_i = time required for job i on machine B, etc., and

T = total elapsed time for jobs $1, 2, \ldots, n$ i.e., time from start of the first job to completion of the last job.

The problem is to determine a sequence $(i_1, i_2, \ldots, i_n)$ where $(i_1, i_2, \ldots, i_n)$ is a permutation of integers $(1, 2, \ldots, n)$. which will minimize T.

Analytic methods have been developed for solving only four simple cases :

(1) n jobs and two machines A and B ; all jobs processed in the order AB ; other limitations described in section 5·4.

(2) n jobs and three machines A, B and C ; all jobs processed in the order ABC, other limitations described in section 5.5.

(3) two jobs and m machines ; each job to be processed through the machines in a prescribed order, not necessarily the same for both jobs.

(4) n jobs and m machines A, B, C,.... K ; all jobs processed in the order ABC...K, other limitations described in section 5·7.

5·4. Processing n Jobs through two Machines

This sequencing problem is completely described as follows :

(*i*) only two machines are involved, A and B,

(*ii*) each job is processed in the order AB, and

(*iii*) the actual or expected processing times $A_1, A_2, \ldots, A_n$; $B_1, B_2, \ldots, B_n$ are known and represented by a table of the type shown below.

Table 5.4

Machine times for n jobs and two machines

Jobs i	*A*	*B*
1	A_1	B_1
2	A_2	B_2
3	A_3	B_3
.	.	.
.	.	.
i	.	.
i	A_i	B_i
⋮	⋮	⋮
n	A_n	B_n

The problem is to determine the sequence (order) of jobs which minimizes T, the total elapsed time from the start of first job to the completion of last job.

It can be shown that the shortest elapsed time occurs when all jobs are processed on the two machines in the same order. The solution procedure given below (without proof) is due to S.M. Johnson and R. Bellman. It consists of the following steps :

Step 1 : Examine the columns for processing times on machines A and B and find the smallest value [Min (A_i, B_i)].

Step 2 : If this value falls in column A, schedule this job first on machine A. If this value falls in column B, schedule this job last on machine A (because of the given order AB). If there are equal minimal values (there is tie) one in each column, schedule the one in the first column first on machine A ; and the one in the second column, last on machine A. If both equal values are in the first column (A), select the one with lowest entry in column B first. If the equal values are in the second column (B), select the one with the lowest entry in column A first.

Step 3 : Cross out the job assigned and continue the process (repeat steps 1 and 2), placing the jobs next to first or next to last till all the jobs are ordered. The resulting sequence will minimize T.

Some important assumptions made while following the above solution procedure are

(*i*) it is assumed that the order of completion of jobs has no significance i.e., no product is required more urgently than the other.

(*ii*) it is assumed that in-process storage space is available and that the cost of in-process inventory is either same for each job or is too small to be considered. This assumption, however, is correct only for processes involving short duration. For longer processes, inventory cost must be considered.

EXAMPLE 5·4·1 : Solve example 5·1·1.

Solution : Table 5·1 is again presented below as table 5·5.

Table 5·5

Job	*Turning time (minutes)*	*Threading time (minutes)*
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3
6	11	1

The solution procedure is described below.

By examining the columns, we find the smallest value. It is threading time of 1 minute for job 6 in second column. Thus we schedule job 6 last as shown below

					6

The reduced set of processing times becomes

Job	*Turning time (minutes)*	*Threading time (minutes)*
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3

The smallest value is turning time of 2 minutes for job 4 in first column. Thus we schedule job 4 first as shown below.

4					6

The reduced set of processing times becomes

Job	*Turning time (minutes)*	*Threading time (minutes)*
1	3	8
2	12	10
3	5	9
5	9	3

There are two equal minimal values ; turning time of 3 minutes for job 1 in first column and threading time of 3 minutes for job 5 in second column. According to the rules, job 1 is scheduled next to job 4 and job 5 next to job 6 as shown below.

4	1			5	6

The reduced set of processing times becomes

Job	*Turning time (minutes)*	*Threading time (minutes)*
2	12	10
3	5	9

The smallest value is turning time of 5 minutes for job 3 in first column. Therefore, we schedule job 3, next to job 1 and we get the optimal sequence as

4	1	3	2	5	6

Now we can calculate the elapsed time corresponding to the optimal sequence, using the individual processing times given in the problem. The details are shown in table 5·6.

Table 5·6

Job	*Turning operation*		*Threading operation*	
	Time in	*Time out*	*Time in*	*Time out*
4	0	2	2	8
1	2	5	8	16
3	5	10	16	25
2	10	22	25	35
5	22	31	35	38
6	31	42	42	43

Thus the minimum elapsed time is 43 minutes. Idle time for turning operation (m/c) is 1 minute (from 42nd minute to 43rd minute) and for threading operation (m/c) is $2+4=6$ minutes (from 0–2 and 38–42 minutes).

EXAMPLE 5.4-2

There are seven jobs, each of which has to go through the machines A and B in the order AB. Processing times in hours are given as

Job :	1	2	3	4	5	6	7
Machine A :	3	12	15	6	10	11	9
Machine B :	8	10	10	6	12	1	3

Determine a sequence of these jobs that will minimize the total elapsed time T. [*Meerut M.Sc.* (*Math.*) *1978*]

Solution.

By examining the processing times we find the smallest value. It is 1 hour for job 6 on machine B. Thus we schedule job 6 last as shown below.

						6

The reduced set of processing times becomes

Job :	1	2	3	4	5	7
Machine A :	3	12	15	6	10	9
Machine B :	8	10	10	6	12	3

There are two equal minimal values ; processing time of 3 hours for job 1 on machine A and processing time of 3 hours for job 7 on machine B. According to rules, job 1 is scheduled first and job 7 next to job 6 as shown below.

1					7	6

The reduced set of processing times becomes

Job :	2	3	4	5
Machine A :	12	15	6	10
Machine B :	10	10	6	12

Again there are two equal minimal values ; processing time of 6 hours for job 4 on machine A *as well as* on machine B. We may choose *arbitrarily* to process (schedule) job 4 *next to job 1 or next to job 7*. Let us schedule this job next to job 1 as shown below.

1	4				7	6

The reduced set of processing times becomes

Job :	2	3	5
Machine A :	12	15	10
Machine B :	10	10	12

There are three equal minimal values : processing time of 10 hours for job 5 on machine A and for jobs 2 and 3 on machine B. According to rules, job 5 is scheduled next to job 4 and job 2 next to job 7. The remaining job 3 is entered in the remaining cell *i.e.*, next to job 2. The optimal sequence is shown below.

1	4	5	3	2	7	6

Now we can calculate the elapsed time corresponding to the optimal sequence, using the individual processing times given in the problem. The details are shown below in table 5.7.

Table 5·7

Job	*Machine A*		*Machine B*		*Idle time for machine B*
	Time in	*Time out*	*Time in*	*Time out*	
1	0	3	3	11	3
4	3	9	11	17	0
5	9	19	19	31	2
3	19	34	34	44	3
2	34	46	46	56	2
7	46	55	56	59	0
6	55	66	66	67	7

Thus the minimum elapsed time is 67 hours. Idle time for machine A is 1 hour (66th-67th hour) and for machine B is 17 hours.

5·5. Processing n Jobs through three Machines

This sequencing problem is completely described as follows :

(*i*) only three machines A, B and C are involved,

(*ii*) each job is processed in the prescribed order ABC,

(*iii*) no passing of jobs is permitted (*i.e.*, the same order over each machine is maintained), and

(*iv*) the actual or expected processing times $A_1, A_2, \ldots, A_n$; $B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$ are known and represented by a table of the type shown below.

Table 5·8

Machine times for n jobs and three machines

Job	*A*	*B*	*C*
1	A_1	B_1	C_1
2	A_2	B_2	C_2
3	A_3	B_3	C_3
.	.	.	.
.	.	.	.
.	.	.	.
i	A_i	B_i	C_i
.	.	.	.
.	.	.	.
.	.	.	.
n	A_n	B_n	C_n

The problem, again, is to find the optimum sequence of jobs which minimizes T.

No general solution is available at present for such a case. However, the method of section 5·4 can be extended to cover the special cases where either one or both of the following conditions hold good (if neither of the conditions holds good, the method fails) :

(1) the minimum time on machine A is $\geqslant$ maximum time on machine B, and

(2) the minimum time on machine C is $\geqslant$ maximum time on machine B.

The method, described here without proof, is to replace the problem by an equivalent problem involving n jobs and two machines. These two (fictitious) machines are denoted by G and H and their corresponding processing times are given by

$$G_i = A_i + B_i,$$
$$H_i = B_i + C_i.$$

If this new problem with the prescribed order GH is solved by the method of section 5·4, the resulting optimal sequence will also be optimal for the original problem.

EXAMPLE 5·5-1.

Solve example 5.1-2.

Solution.

Table 5·2 is again presented below as table 5·9.

Table 5 9

Job	*Time for turning (minutes)*	*Time for threading (minutes)*	*Time for knurling (minutes)*
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Here, min $A_i=2$, max $B_i=8$, and min $C_i=8$. Since min $C_i=$max B_i, we can solve this example by the procedure described in section 5·5.

The equivalent problem involving 6 jobs and two fictitious operations G and H becomes

Processing times for 6 jobs and two fictitious operations

Job	G_i=*Turning*+*Threading* (*minutes*)	H_i=*Threading*+*Knurling* (*minutes*)
1	11	21
2	18	20
3	9	13
4	8	18
5	12	11
6	12	14

Examining the columns G_i and H_i, we find that the smallest value is 8 under operation G_i in row 4. Thus we schedule job 4 first as shown below.

4					

The reduced set of processing times becomes

Job	G_i	H_i
1	11	21
2	18	20
3	9	13
5	12	11
6	12	14

The next smallest value is 9 under column G_i for job 3. Hence we schedule job 3 as shown below.

4	3				

The reduced set of processing times becomes

Job	G_i	H_i
1	11	21
2	18	20
5	12	11
6	12	14

There are two equal minimal values, processing time of 11 minutes under column G_i for job 1 and processing time of 11 minutes under column H_i for job 5. According to the rules, job 1 is scheduled next to job 3 and job 5 is scheduled last as shown below.

4	3	1			5

The reduced set of processing times becomes

Job	G_i	H_i
2	18	20
6	12	14

The smallest value is 12 under column G_i for job 6. Hence we schedule job 6 next to job 1 and the optimal sequence becomes

4	3	1	6	2	5

Now we may calculate the elapsed time corresponding to the optimal sequence, using the individual processing times given in the problem. The details are shown in table 5·10.

Table 5·10

Job	*Turning operation*		*Threading operation*		*Knurling operation*	
	Time in	*Time out*	*Time in*	*Time out*	*Time in*	*Time out*
4	0	2	2	2	8	20
3	2	7	8	12	20	29
1	7	10	12	20	29	42
6	10	21	21	22	42	55
2	21	33	33	39	55	69
5	33	42	42	45	69	77

Thus the minimum elapsed time is 77 minutes. Idle time for turning operation is $77-42=35$ minutes, for threading operation is $2+1+11+13+(77-45)=17+32=49$ minutes and for knurling operation is 8 minutes.

EXAMPLE 5·5·2

There are five jobs, each of which is to be processed through three machines A, B, and C in the order ABC. Processing times in hours are

Table 5·11

Job	A	B	C
1	3	4	7
2	8	5	9
3	7	1	5
4	5	2	6
5	4	3	10

Determine the optimum sequence for the five jobs and the minimum elapsed time.

Solution.

Here, min $A_i=3$, max $B_i=5$, and min $C_i=5$. Since min $C_i=$max B_i, we can solve this example by the procedure described in section 5·5.

The equivalent problem involving 5 jobs and two fictitious machines G and H becomes

Machine times for five jobs and two machines

Job	G $(G_i=A_i+B_i)$	H $(H_i=B_i+C_i)$
1	7	11
2	13	14
3	8	6
4	7	8
5	7	13

Examining the columns G_i and H_i, we find that the smallest value of machining time is 7 (hrs.) under column G_i for jobs 1, 4 and 5. Since all the values are in the same column G_i, we schedule job 4 first since it has lowest entry 8 in column H_i, job 1 next as it has next higher entry 11 in column H_i and job 5 next to job 1 since it has still higher entry 13 in column H_i. This is shown below.

4	1	5		

The reduced set of machining times becomes

Job	G_i	H_i
2	**13**	**14**
3	**8**	**6**

The smallest value is 6 (hrs) in column H_i for job 3. Thus we schedule job 3 last and the optimum sequence becomes

4	1	5	2	3

Obviously, the other optimal solutions, because of ties will be

4	5	1	2	3

1	4	5	2	3

1	5	4	2	3

5	1	4	2	3

and

5	4	1	2	3

The minimum elapsed time can now be calculated corresponding to the first optimal sequence using the individual machining times given in the problem. The details are shown in table 5.12.

Table 5·12

Job	*Machine A*		*Machine B*		*Machine C*	
	Time in	*Time out*	*Time in*	*Time out*	*Time in*	*Time out*
4	0	5	5	7	7	13
1	5	8	8	12	13	20
5	8	12	12	15	20	30
2	12	20	20	25	30	39
3	20	27	27	28	39	44

Thus minimum elapsed time is 44 hours, idle time for machine A is $44-27=17$ hours, for machine B is $5+1+5+2+(44-28)=29$ hours and for machine C is 7 hours.

5·6. Processing two Jobs through m Machines

Let us consider the following situation :

(*a*) there are m machines, denoted by $A, B, C, \ldots, K$,

(*b*) only two jobs are to be performed : job 1 and job 2,

(c) the technological ordering of each of the two jobs through m machines is known. This ordering may not be the same for both jobs. Alternative ordering is not permissible for either job.

(d) the actual or expected processing times A_1, B_1, $C_1, \ldots, K_1$, A_2, B_2, C_2, ..., K_2 are known, and

(e) each machine can work only one job at a time and storage space for in-process inventory is available.

The problem is to minimize the total elapsed time T *i.e.*, to minimize the time from the start of first job to the completion of last job.

Such a problem can be solved by graphic method which is simple and provides good (though not necessarily optimal) results. This method will be explained with the help of an example.

EXAMPLE 5·6·1.

Solve example 5·1-3.

Solution.

Table 5-3 is reproduced as table 5·13 below.

Table 5·13

Job 1	*Sequence*	A	B	C	D	E
	Time (hrs.)	1	2	3	5	1
Job 2	*Sequence*	C	A	D	E	B
	Time (hrs.)	3	4	2	1	5

The graphic procedure is described with the help of following steps :

Step 1

Draw two axes at right angles to each other. Represent processing time on job 1 along horizontal axis and processing time on job 2 along vertical axis.

Step 2

Layout the machine times for the two jobs on corresponding axes in the given technological order. This is shown in figure 5·1.

Step 3

Machine A requires 1 hour for job 1 and 4 hours for job 2. A rectangle LMNP is, thus, constructed for machine A. Similar rectangles are constructed for machines B, C, D and E as shown.

Step 4

Make a program by starting from origin (O) and moving through the various stages of completion (points) till the point marked 'finish' is reached. Choose path consisting only of horizontal, vertical and 45° lines. A horizontal line represents work on job 1 while job 2 remains idle; a vertical line represents work on job 2 while job 1 remains idle and a 45° line to the base represents simultaneous work on both jobs.

Step 5

Find the optimal path (program). An optimal path is one that minimizes idle time for job 1 (horizontal movement). Likewise, an

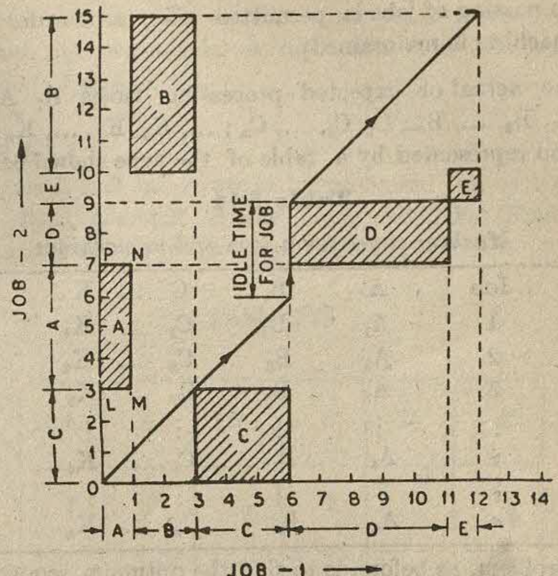

Fig. 5·1. Graphic solution of 2 job and 5 machine problem.

optimal path is one that minimizes idle time for job 2 (vertical movement). Obviously, the optimal path is one which coincides with 45° line to the maximum extent.

Further, both jobs cannot be processed simultaneously on one machine. Graphically, this means that diagonal movement through the blocked out areas is not allowed.

A good path, accordingly, is chosen by eye and drawn on the graph.

Step 6

Find the elapsed time. It is obtained by adding the idle time for either job to the processing time for that job. The idle time for the chosen path is found to be 3 hours for job 1.

$\therefore$ Total elapsed time $=12+3(=15+0)=15$ hours.

5·7. Processing n Jobs through m Machines

This sequencing problem is described as follows :

(*i*) there are n jobs to be performed, denoted by 1, 2, 3, ..., i, ..., n.

(*ii*) there are m machines, denoted by A, B, C, ..., K.

(*iii*) each job is to be processed in the prescribed order ABC ... K.

(*iv*) no passing of jobs is permitted (*i.e.*, same order over each machine is maintained).

(*v*) the actual or expected processing times A_1 A_2, ..., A_n, B_1, B_2, ..., B_n ; C_1, C_2, ..., C_n ; ..., K_1, K_2, ..., K_n are known and represented by a table of the type shown below.

Table 5·14

Machine times for n jobs and m machines

Job	A	B	C	...	K
1	A_1	B_1	C_1	...	K_1
2	A_2	B_2	C_2	...	K_2
3	A_3	B_3	C_3	...	K_3
⋮	⋮	⋮	⋮		⋮
i	A_i	B_i	C_i	...	K_i
⋮	⋮	⋮	⋮		⋮
n	A_n	B_n	C_n	...	K_n

The problem, as before, is to find the optimum sequence of jobs which minimizes T.

No general solution is available at present for such a case. However, the method of section 5.5 can be applied (extended) to cover the special cases where either *one* or *both* of the following conditions hold gold (if neither of the conditions hold good, the method fails) :

(*i*) the minimum time on machine A is $\geqslant$ maximum time on machines B, C, ..., K—1,

(*ii*) the minimum time on machine K is $\geqslant$ maximum time on machines B, C, ..., K—1.

The method is to replace the m machine problem by an equivalent two machine problem. These two (fictitious) machines are denoted by a and b and their corresponding processing times are given by

$$a_i = A_i + B_i + \ldots\ldots + (K-1)_i,$$

$$b_i = B_i + C_i + \ldots\ldots + (K-1)_i + K_i.$$

If this new problem with the prescribed order ab is solved by the method of section 5.5, the resulting optimal sequence will also be optimal for the original problem.

Further, if

$$B_i + C_i + \ldots + (K-1)_i = k,$$

where k is a fixed positive constant for all jobs ($i=1, 2, 3, \ldots, n$), then the given problem can be solved simply as n job two machine problem (where the two machines are A and K in the order AK) as per the method of section 5.4.

EXAMPLE 5·7-1

Four jobs 1, 2, 3 and 4 are to be processed on each of the five machines A, B, C, D and E in the order ABCDE. Find the total minimum elapsed time if no passing of jobs is permitted.

Table 5·15

Job	*A*	*B*	*C*	*D*	*E*
1	7	5	2	3	9
2	6	6	4	5	10
3	5	4	5	6	8
4	8	3	3	2	6

Solution.

Here, min $A_i = 5$, min $E_i = 6$ and max $(B_i, C_i, D_i) = 6, 5, 6$ respectively. Since min $E_i = \max(B_i, D_i)$, we can solve this problem by the procedure described in section 5·7.

The equivalent problem involving 4 jobs and 2 fictitious machines a and b becomes

Job	*Machine a*	*Machine b*
1	17	19
2	21	25
3	20	23
4	16	14

Examining the columns we find that the optimal sequence is

1	3	2	4

Now we may calculate the total elapsed time corresponding to the optimal sequence, using the individual processing times given in the original problem. The details are shown in table 5.16.

Table 5·16

Job	*A*	*B*	*C*	*D*	*E*
1	0—7	7—12	12—14	14—17	17—26
3	7—12	12—16	16—21	21—27	27—35
2	12—18	18—24	24—28	28—33	35—45
4	18—26	26—29	29—32	33—35	45—51

Thus the minimum elapsed time is 51 time units. Machines A, B, C, D and E remain idle for 25, 33, 37, 35 and 18 time units respectively.

EXAMPLE 5·7·2

Four jobs 1, 2, 3 and 4 are to be processed on each of the four machines A, B, C and D in the order ABCD. The processing times in minutes are given in table 5·16. Find, for no passing, the minimum elasped time.

Table 5·17

Job	*A*	*B*	*C*	*D*
1	58	14	14	48
2	30	10	18	32
3	28	12	16	44
4	64	16	12	42

Solution.

Here, min $A_i = 28$, min $D_i = 32$ and max $(B_i, C_i) = 16$, 18 respectively. Since min $(A_i, D_i) >$ max (B_i, C_i), we can solve this problem by the procedure described in section 5·7. The problem can be converted into a four job and two machine problem.

Further, since $B_1 + C_1 = B_2 + C_2 = B_3 + C_3 = B_4 + C_4 = 28$, a fixed positive constant, the given problem reduces to that of finding the optimal sequence for 4 jobs and 2 machines A and D in the order AD. Machines B and C do not have any effect on the optimality of the sequence.

Examining columns A and D only, the optimal sequence is given by

3	2	1	4

Now we may calculate the total elapsed time corresponding to the optimal sequence, using the individual processing times given in the original problem. The details are shown in table 5.17 *a*.

Table 5·17 a

Job	*A*	*B*	*C*	*D*
3	0–28	28–40	40–56	56–100
2	28–58	58–68	68–86	100–132
1	58–116	116–130	130–144	144–192
4	116–180	180–196	196–208	208–250

Thus the minimum elapsed time is 4 hrs. 10 min. Machines A, B, C and D remain idle for 70, 198, 190 and 84 minutes respectively.

5·8. Approaches to More Complex Sequencing Problems

So far, in this chapter we have considered only relatively simple problems. However, sequencing problems may be complicated by a number of conditions, such as

1. *Overlap* : This can happen when a job consists of making a number of similar items. A few first items coming out of one operation may go in for second operation before remaining items in the lot could go in for the first operation,

2. *Transportation time* : It may take a considerable amount of time to transport the items from one facility to another. The facilities may even be in different plants.

3. *Rework* : This may happen if one of the operations in sequence is an inspection operation. The defective items may have to be sent back to a previous operation for reworking, causing either a delay or splitting the jobs into two lots.

4. *Expediting* : Because of urgent demand, a particular job may have to be moved out of sequence and speeded up.

5. *Machine breakdown.* A machine may breakdown or operator may be absent or injured at work.

6. *Material shortages* : The material required for performing a particular operation may run out.

7. *Variable processing time* : The time required to perform a certain operation may vary from shift to shift as it frequently happens in multi-shift operations.

In addition, alternative routes for the jobs on different machines may be permitted where more than one machine of a given type exists. Moreover, machine times and/or costs may be of probabilistic nature. At present, each of these complex situations have to be treated individually and a tailor-made solution found.

There are two approaches for these complex situations. One approach is to split the problem into simpler sub-problems, each of which can, then, be handled by a particular technique. This approach, however, is open to all the shortcomings of treating a system in parts *i.e.* of suboptimization. This may lead to lower value of overall system effectiveness. As a rule, whenever suboptimization is unavoidable, because of inadequacy of available techniques, the operation researcher must keep his eye on the total system.

The second approach for complex sequencing problems is the use of *Monte Carlo Technique.* This technique makes use of the probability distribution of the characteristics of the system such as processing time on each machine, availability of the machines, alternative routes for the jobs, more than one machine of the same type, variations in working pace of operators in different shifts, etc.

At present, solutions to such complex sequencing problems are sought through simulation. The amount of computation required by simulation may be quite large. Hence, efforts have been made to

reduce the number of sequencings to be tested and also the number of trials required to test each sequence. High speed electronic computers have been used to facilitate simulation for complex sequencing problems. The solution of these complex problems by simulation has already begun. Further developments are expected in near future.

5·9. Routing Problems in Networks (Problems Related to Sequencing)

Routing problems in networks are the problems which are related to sequencing problems and have been receiving increased attention. Network problems occur quite often in transportation and communication processes. A typical network problem consists of finding a route from city 1 (origin) to city 2 (destination) having alternative paths at various stages of the journey. The cost of the journey—which may be measured by the time, money or distance—depends upon the route chosen and the problem is to find the route involving minimum cost. Theoretically, the solution lies in finding the cost associated with all the possible routes and selecting the best one. In practice, however, the number of these possibilities is too large to be tried one by one. For example, with 6 alternative paths the number of possibilities is 120 ; with 9 paths it increases to 40,320 while for 21 paths it becomes $2{\cdot}4329 \times 10^{23}$. Thus it is necessary to find a more efficient way of determining the 'best' value.

There are many problems (other than routing) associated with networks. But we shall, here, deal with only two types of routing problems :

1. The travelling salesman problem.
2. Minimal path problem.

5·10. The Travelling salesman Problem (Shortest Cyclic route Models)

There are a number of cities a salesman must visit. The distance (or time or cost) between every pair of cities is known. He starts from his home city, passes through each city once and only once and returns to his home city. The problem is to find the routes shortest in distance (or time or cost).

If the distance (or time or cost) between every pair of cities is independent of the direction of travel, the problem is said to be *symmetrical.* If for one or more pairs of cities the distance (or time

or cost) varies with the direction, the problem is called *asymmetrical.* For example, it takes more time to go up a hill between two stations than come down the hill between them ; similarly, a flight may take more time against the wind direction compared to that in the direction of wind.

If the salesman is to visit only two cities there is, of course, no choice. If the number of cities is three (A, B and C), of which the starting base is A, there are two possible routes A→B→C and A→C→B. For four cities, there are 6 possible routes :

A→B→C→D, A→B→D→C, A→C→B→D, A→C→D→B, A→D→B→C and A→D→C→B.

For eleven cities there are more than 3½ million possible routes ; in general, for n cities there are $(n-1)!$ possible routes. It may be noted that since the salesman has to visit all the n cities. the shortest route will be independent of the selection of the starting city.

Obviously, the problem is to find the best route without trying each one. Unfortunately, there is no analytical method which can be used satisfactorily. However, a few computational techniques for solving the problem have been suggested.

Such types of problems arise in the following areas o' management :

1. Postal deliveries
2. Inspection
3. School Bus routing
4. Television relays
5. Assembly lines, etc.

Consider, for example, the problem faced by a multiple product manufacturing firm. The firm manufactures a number of different products. The cost of setting up a line for a particular product depends upon the previous product made. The problem is to determine the sequence of products that will minimize the total setup cost.

The travelling salesman problem (such as described above) appears to be related to sequencing problem but actually it is more similar to assignment problem with the difference that there is an additional constraint. Mathematically the problem may be stated as follows :

Given C_{ij} as the cost of going from city i to city j and $x_{ij}=1$, for going directly frcm i to j and zero otherwise, minimize $\Sigma_i \Sigma_j C_{ij} x_{ij}$ subject to the additional constraint that x_{ij} *is to be so chosen that no city is visited twice before the tour of all the cities is completed.*

In particular, going from i directly to i is not permitted, which means $C_{ii}=\infty$. It should be noted that there must be only one $x_{ij}=1$ for each value of i and for each value of j.

The equivalent production problem is : n products are to be processed on a machine in a scheduled time. The setup cost if product i is followed by j is C_{ij}. Also $x_{ij}=1$ if product i is followed by j directly and o otherwise. Each product must be produced only once *i.e.*, changing from a product to itself is not allowed, or the corresponding setup cost $C_{ii}\equiv\infty$.

Thus the travelling salesman problem can be put in the form of an assignment problem as shown in table 5·18.

Table 5·18

To City

From city	1	2	3	...	n
1	∞	C_{12}	C_{13}	...	C_{1n}
2	C_{21}	∞	C_{23}	...	C_{2n}
3	C_{31}	C_{32}	∞	...	C_{3n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	C_{n1}	C_{n2}	C_{n3}	...	∞

The above assignment problem can be solved and it can be hoped that the solution will satisfy the additional constraint. If it does not, it can be adjusted by inspection. The method, however, works well for small problems only. For large problems, a more systematic approach, developed by J.D.C. Little and his colleagues is used.

EXAMPLE 5·10·1.

A salesman wants to visit cities A, B, C, D and E. He does not want to visit any city twice before completing his tour of all the cities and wishes to return to the point of starting journey. Cost of going from one city to another (in rupees) is shown in table 5·19. Find the least cost route.

Table 5·19

	A	B	C	D	E
A	0	2	5	7	1
B	6	0	3	8	2
C	8	7	0	4	7
D	12	4	6	0	5
E	1	3	2	8	0

Solution.

First Method. The given travelling salesman problem is first solved as assignment problem. If this optimal solution also satisfies the additional constraint, it is also the optimal solution to the given problem ; if not, it can be adjusted by inspection. Reduce the cost matrix by subtracting the lowest element in each row from all the elements of the row. Then subtract the lowest element in each column (if required) from all the elements of the column till there is zero in every row and every column. Tables 5.20 and 5·21 result. As going from $A \to A$, $B \to B$, etc. is not allowed, assign a large penalty (cost of journey), for these cells in the cost matrix.

Table 5·20

	A	B	C	D	E
A	∞	1	4	6	0
B	4	∞	1	6	0
C	4	3	∞	0	3
D	8	0	2	∞	1
E	0	2	1	7	∞

Reduced cost matrix with zero in every row

→

Table 5·21

	A	B	C	D	E
A	∞	1	3	6	0
B	4	∞	0	6	0
C	4	3	∞	0	3
D	8	0	1	∞	1
E	0	2	0	7	∞

Reduced cost matrix with zero in every row and every column

We now check if optimal assignment can be made in table 5.21 on not. Proceeding as in example 4.6-2 we get table 5.22.

Table 5·22

	A	B	C	D	E
A	∞	1	3	6	$\boxed{0}$
B	4	∞	$\boxed{0}$	6	$\cancel{0}$
C	4	3	∞	$\boxed{0}$	3
D	8	$\boxed{0}$	1	∞	1
E	$\boxed{0}$	2	$\cancel{0}$	7	∞

This table provides an optimum solution to the assignment problem but not to the travelling salesman problem as it gives $A \to E$, $E \to A$, $B \to C$, $C \to D$, $D \to B$ as the solution which means that the salesman should go from A to E and then come back to A without visting cities B, C and D. This violates the additional constraint that the salesman is not to visit any city twice before completing his tour of all the cities. Therefore, we now try to find the next best solution that also satisfies this additional constraint. The next minimum (non-zero) element in the matrix is 1. So, we shall try to bring element 1 into the solution. However, this element 1 occurs in three different cells. We shall consider all the three cases until an acceptable solution is obtained.

Case 1. We make 'unity assignment' in cell (A, B) instead of zero assignment in cell (A, E). Accordingly, zero assignment in cell (D, B) is changed to 'unity assignment' in cell (D, E). The resulting table is shown below.

Table 5·23

	A	B	C	D	E
A	∞	$\boxed{1}$	3	6	$\cancel{0}$
B	4	∞	$\boxed{0}$	6	$\cancel{0}$
C	4	3	∞	$\boxed{0}$	3
D	8	$\cancel{0}$	1	∞	$\boxed{1}$
E	$\boxed{0}$	2	$\cancel{0}$	7	∞

The resulting feasible solution is $A \to B$, $B \to C$, $C \to D$, $D \to E$, $E \to A$ and it involves a cost of Rs. $(2+3+4+5+1)=$Rs. 15.

Case 2. In table 5·22 we make 'unity assignment' in cell (D, C) instead of zero assignment in cell (D, B). As a result, we have to make assignments in cell (B, A) instead of cell (B, C) and in cell (E, B) instead of cell (E, A). Table 5·24 results.

Table 5·24

	A	B	C	D	E
A	∞	1	3	6	$\boxed{0}$
B	$\boxed{4}$	∞	$\cancel{0}$	6	$\cancel{0}$
C	4	3	∞	$\boxed{0}$	3
D	8	$\cancel{0}$	$\boxed{1}$	∞	1
E	$\cancel{0}$	$\boxed{2}$	$\cancel{0}$	7	∞

The resulting solution is $A \to E$, $E \to B$, $B \to A$, $C \to D$, $D \to C$, which is not feasible as it does not satisfy the additional constraint.

Case 3. In table 5·22, we make 'unity assignment' in cell (D, E) instead of zero assignment in cell (D, B). Accordingly, zero assignment in cell (A, E) is changed to 'unity assignment' in cell (A, B). The resulting table is same as table 5·23 which gives the feasible solution $A \to B$, $B \to C$, $C \to D$, $D \to E$, $E \to A$ with a cost Rs. 15. Hence the least cost route is $A \to B \to C \to D \to E \to A$ with cost of Rs. 15.

Second Method. The iterative procedure developed by Little, Musty, Sweeney and Karel (1963) will be used to solve this problem. This is called *Little's method* or *Branch and bound method.* This method consists of the following steps :

Step 1

(*i*) As going from A → A, B → B, etc. is not allowed, assign a large penalty (cost of journey), ∞ for these cells in the cost matrix, which we will call set S. We get table 5·24 shown below representing the cost matrix (C_{ij}).

Table 5·25

	A	B	C	D	E
A	∞	2	5	7	1
B	6	∞	3	8	2
C	8	7	∞	4	7
D	12	0	6	∞	5
E	1	3	2	8	∞

Cost matrix (C_{ij})

(*ii*) *Reduce the matrix* : Subtract the lowest element in each row from all the elements of the row. Then subtract the lowest element in each column (if required) from all the elements of the column, till there is a zero in every row and every column. The total reduction r is the sum of the elements subtracted. We call the resulting matrix as (C'_{ij}).

Table 5·26

	A	B	C	D	E
A	∞	1	4	6	0
B	4	∞	1	6	0
C	4	3	∞	0	3
D	8	0	2	∞	1
E	0	2	1	7	∞

Reduced cost matrix with zero in every row

Table 5·27

	A	B	C	D	E
A	∞	1	3	6	0
B	4	∞	0	6	0
C	4	3	∞	0	3
D	8	0	1	∞	1
E	0	2	0	7	∞

Reduced cost matrix (C'_{ij}) *with zero in every row and every column*

$\therefore$ Total reduction $r=(1+2+4+4+1)+1=12+1=13$.

Step 2

Find the penalty of *not using* each zero cell in (C'_{ij}). We argue that if we do not use the link (h, k), we must use other element in row h and some element in column k. Thus the cost of not using (h, k) is

at least equal to the sum of the smallest element in row h and the smallest element in column k. These penalties have been recorded in the top left corners of the zero cells in table 5·28·

Table 5·28

	A	B	C	D	E
A	∞	1	3	6	$^{1}0$
B	4	∞	$^{0}0$	6	$^{0}0$
C	4	3	∞	$^{9}\surd 0$	3
D	8	$^{2}0$	1	∞	1
E	$^{4}2$	2	$^{0}0$	7	∞

For example, consider zero in cell (A, E). The sum of smallest elements in row A and column E [excluding zero in cell (A, E)] is $1+0=1$. For cell (B, C), the sum is $=0+0=0$ and for cell (C, D), the sum is $=3+6=9$ and so on.

Step 3

Let (h, k) be the zero entry with the highest penalty. In case of tie, select arbitrarily. We now divide the given set S into two subsets : S (h, k) which contain the link (h, k) and S $(\overline{h, k})$, which do not contain the link (h, k). We next calculate *lower bounds* on the costs of all routes in each subset.

We know that if (h, k) is not used, in addition to the reduction r, there will be cost of at least P_{hk}. Therefore, the lower bound θ $(\overline{h, k})$ is given by

$$\theta(\overline{h, k})=r+p_{hk}.$$

In the example $r=13$ and $P_{cd}=9$ $\quad\therefore\quad \theta\,(\overline{C, D})=13+9=22.$

$\therefore$ Total cost of the *route not containing the highest penalty link* $=\theta\,(\overline{C, D})=22.$

Step 4

Find the total cost of route containing the highest penalty lin k *i.e.*, lower bound for S(h, k). This is done as follows :

We observe that if we use the link (h, k), we cannot use the link (k, h); for, if we use (h, k) and (k, h) we would go from h to k and back to h without visiting other cities. We avoid the use of (k, h) by assigning a very heavy cost *i.e.* $C'_{kh}=\infty$. Moreover, if we use (h, k) we will not use any other link in row h and column k. This is ensured by deleting row h and column k. In the remaining matrix we select our element from each row and column so that the cost will be at least the amount by which the remaining matrix can be reduced. Let this be r_{hk}. Then the lower bound $\theta\,(h, k)$ for $S(h, k)$ is given by $\theta\,(h, k)=r+r_{hk}$.

In the example, $C'_{DC}=\infty$ and we delete row C and column D. The resulting matrix can be reduced by only zero, giving θ (C, D)$=13+0=13$.

The results obtained in steps 3 and 4 are recorded in figure 5·2.

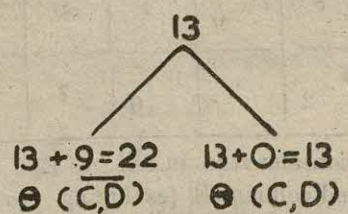

Fig. 5·2

Table 5·29

	A	B	C	D	E
A					
B					
C					
D			∞		
E					

Table 5·30

	A	B	C	E
A	∞	1	3	0
B	4	∞	0	0
D	8	0	∞	1
E	0	2	0	∞

Step 5

Find out of θ $(\overline{C, D})$ and θ (C, D), the one which is lower for further partitioning the subset S (h, k). If θ (C, D) is lower, return to step 2 and repeat it on the cost matrix obtained in step 4 (table 5·30). If θ $(\overline{C, D})$ is chosen, return to the original reduced matrix (of step 1), put cost in cell (C, D)$=\infty$ and repeat the steps from 2 onwards.

In the example, θ (C, D) is chosen as it is lower. Applying step 2 we get table 5·31 in which penalties have been recorded in the top left corners of cells with zero entries.

Table 5·31

	A	B	C	E
A	∞	1	3	$^{1}\,0$
B	4	∞	$^{0}\,0$	$^{0}\,0$
D	8	$^{2}\,0$	∞	1
E	$^{4}\checkmark 0$	2	$^{0}\,0$	∞

Step 6

Cell (E, A) has the highest penalty of 4. We, now, divide the subset S(h, k) into two parts, one containing the cell (E, A) and the other not containing the cell (E, A) and calculate lower bounds on the costs of all routes in each part.

Total cost of route not containing the highest penalty link $=$ θ $(\overline{E, A})=13+4=17$.

Step 7

Find the total cost of route containing the highest penalty link. This is done as follows : If we can use the link (E, A), we cannot use the link (A, E). Thus we put $C'_{AE}=\infty$ and delete row E and column A. In the remaining matrix we select one element from each row and column so that the cost will be at least the amount by which the remaining matrix can be reduced.

In the example, lower bound for the matrix of table 5·30 i.e., the total cost of route containing the highest penalty link (E, A) is = θ (E, A) = 13 + 0 = 13, as the resulting matrix can be reduced by only zero.

The results obtained in steps 6 and 7 are recorded in figure 5·3 and tables 5·32 and 5·33.

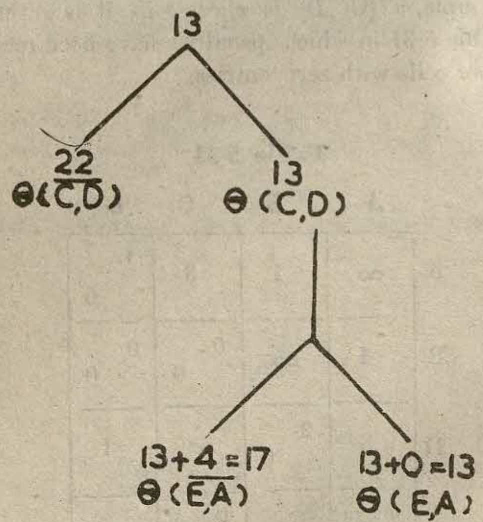

Fig. 5·3

Table 5·32

	A	B	C	E
A				∞
B				
D				
E				

→

Table 5·33

	B	C	E
A	1	3	∞
B	∞	0	0
D	0	∞	1

Step 8

Chose out of θ ($\bar{E}$, $\bar{A}$), and θ (E, A), the one which is lower.

Since θ (E, A) is lower, we return to step 2 and repeat it on the cost matrix obtained in step 7 (tables 5·33), after reducing it so that each row and column contains zero entry (table 5·34) cell. Penalties have been recorded in the top left corners of the cell with zero entries in table 5·35.

Table 5·34

	B	C	E
A	0	2	∞
B	∞	0	0
D	0	∞	1

$\rightarrow$

Table 5·35

	B	C	E
A	$^{2}\sqrt{0}$	2	∞
B	∞	$^{2}\,0$	$^{1}\,0$
D	$^{1}\,0$	∞	1

Step 9

Cell (A, B) has the highest penalty of 2 [cell (B, C) also has the same penalty of 2 ; out of these two cells, we choose cell (A, B) arbitrarily].

$\therefore$ Cost of route not containing the cell (A, B) ; $\theta\,(\overline{A, B}) = 13+2 = 15$.

Step 10

If we use the link (A, B), we have the chain (E, A), (A, B). Hence we must exclude the link (B, E). Thus we put $C'_{BE} = \infty$. We delete row A and column B.

Cost of route containing the cell (A, B); θ (A, B) $= 13+1 = 14$.

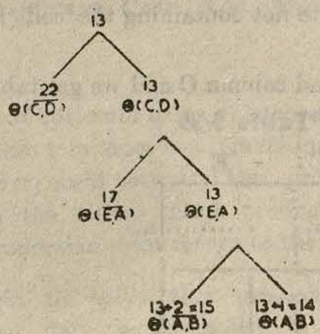

Fig. 5·4

The results obtained in steps 9 and 10 are shown in figure 5·4 and tables 5·36 and 5·37.

Table 5·36

	B	C	E
A			
B			
D			

Table 5·37

	C	E
B	0	∞
D	∞	1

Step 11

Reduce the matrix of step 10 so that each row and each column has at least one zero entry cell. This is shown in table 5·38.

Table 5·38

	C	E
B	∞ 0 √	∞
D	∞	∞ 0

Penalties are recorded on the top left corners of the cells containing zero entries in table 5·38.

Step 12

Let us choose cell (B, C) containing a penalty of ∞.

$\therefore$ Cost of route not containing the cell (B, C) $=14+\infty=\infty$.

Step 13

Delete row B and column C and we get table 5·39.

Table 5·39

	C	E
B		
D		0

Cost of route containing cell (B, C) $=14+0=14$.

The results of steps 12 and 13 are shown in figure 5·5.

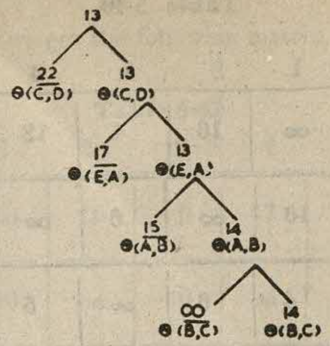

Fig. 5·4

$\therefore$ The least cost route is $C \to D \to E \to A \to B \to C$.

$$\begin{aligned}\text{Total cost of journey} &= \text{cell }(C, D) + \text{cell }(D, E) + \text{cell }(E, A) \\ &\quad + \text{cell }(A, B) + \text{cell }(B, C) \\ &= \text{Rs. }(4+5+1+2+3) = \text{Rs. } 15.\end{aligned}$$

EXAMPLE 5·10·2

Products 1, 2, 3, 4 and 5 are to be processed on a machine. The setup costs in rupees per change depend upon the product presently on the machine and the setup to be made and are given by the following data :

$C_{12}=16$, $C_{13}=4$, $C_{14}=12$, $C_{23}=6$, $C_{34}=5$, $C_{25}=8$, $C_{35}=6$, $C_{45}=20$; $C_{ij}=C_{ji}$. $C_{ij}=\infty$ for all values of i and j not given in the data. Find the optimum sequence of products in order to minimize the total setup cost.

[*Punjab Univ. B.Sc. (Mech.) Engg. Nov., 1981*]

Solution.

Determination of optimum order of processing the products so that the setup costs are minimum is a travelling salesman problem. The setup (change over) costs between the products are analogous to distances between the cities. Each product must be produced only once and the production must return to the first product.

We first begin to solve this problem as an assignment problem. The given data is expressed in the form of a table (table 5·40).

Table 5·40

	1	2	3	4	5
1	∞	16	4	12	∞
2	16	∞	6	∞	8
3	4	6	∞	5	6
4	12	∞	5	∞	20
5	∞	8	6	20	∞

Hungarian method or reduced matrix method will be used to obtain optimal assignment. This method consists of the following steps :

Step I

Prepare a Square Matrix. This step is not necessary in this example.

Step II

Reduce the Matrix. Proceeding as in example 4.6-2, we get table 5.41.

Table 5·41

	1	2	3	4	5
1	∞	12	0	8	∞
2	10	∞	0	∞	2
3	0	2	∞	1	2
4	7	∞	0	∞	15
5	∞	2	0	14	∞

Matrix after substep I (contains zero in each row)

After substep 2 we get the following matrix :

Table 5·42

	1	2	3	4	5
1	∞	10	0	7	∞
2	10	∞	0	∞	0
3	0	0	∞	0	0
4	7	∞	0	∞	13
5	∞	0	0	13	∞

Matrix after substep 2 (Contains zero in each row and in each column) Initial feasible solution

Step III

Check if Optimal Assignment can be made in the Current Feasible Solution or not. Proceeding as in Example 4·6·2 we get

Table 5·43

	1	2	3	4	5	
1	∞	10	$\boxed{0}$	7	∞	✓
2	10	∞	$\cancel{0}$	∞	$\boxed{0}$	
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$	
4	7	∞	$\cancel{0}$	∞	13	✓
5	∞	$\boxed{0}$	$\cancel{0}$	13	∞	

As the minimum number of lines crossing all zeros is 4 *i.e.*, less than 5, optimal assignment cannot be made in the current feasible solution.

Step IV

Iterate towards Optimal Solution

Proceeding as in example 4·6-2 we get

Table 5·44

	1	2	3	4	5
1	∞	3	0	0	∞
2	10	∞	7	∞	0
3	0	0	∞	0	0
4	0	∞	0	∞	6
5	∞	0	7	13	∞

Second feasible solution

Step V

Check if Optimal Assignment can be made in the Current Feasible Solution or not

Table 5·45

	1	2	3	4	5
1	∞	3	~~0~~	$\boxed{0}$	∞
2	10	∞	7	∞	$\boxed{0}$
3	$\boxed{0}$	~~0~~	∞	~~0~~	~~0~~
4	~~0~~	∞	$\boxed{0}$	∞	6
5	∞	$\boxed{0}$	7	13	∞

As there is no row or column without assignment, optimal assignment is possible in the current solution. Table 5·45, however, provides an optimal solution to the assignment problem but not to the given travelling salesman problem (sequencing problem) as it

gives 1→4, 4→3, 3→1, 2→5 and 5→2 as the solution which means that the products should be processed in the order 1—4—3—1, without processing the products 2 and 5. This violates the additional constraint that each product must be processed only once and only after having processed all the products, the production should return to product 1. So we try to find the 'next best' solution that also satisfies this additional constraint.

The next minimum (non-zero) element is 3 in cell (1, 2). We make assignment to entry 3 in this cell instead of zero assignment in cell (1, 4). Accordingly, zero assignment in cell (5, 2) is changed to assignment in cell (5, 4) with entry 13. This gives table 5·46.

Table 5·46

	1	2	3	4	5
1	∞	$\boxed{3}$	$\cancel{0}$	$\cancel{0}$	∞
2	10	∞	7	∞	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$
4	$\cancel{0}$	∞	$\boxed{0}$	∞	6
5	∞	$\cancel{0}$	7	$\boxed{13}$	∞

The resulting feasible solution 1—2, 2—5, 5—4, 4—3, 3—1 is also the optimal solution. Thus the optimal sequence for the processing of products is 1—2—5—4—3—1 and it involves a cost of Rs. (16+8+4+5+20)=Rs. 53.

5·11. Minimal Path Problem (Shortest Acyclic Route Models)

The travelling salesman problem is a routing problem involving rather severe constraints. Another routing problem arises when we wish to go from one place to another or to several other places and we are to select the shortest route (involving least distance or time or cost) out of many alternatives, to reach the desired station. Such acyclic route network problems can be easily solved by graphic methods.

A *network* is defined as a set of points or *nodes* which are connected by lines or links. A way of going from one node (*the origin*) to another (*the destination*) is called a *route or path*. The links in a network may be one way (in either direction) or two-way (in both directions). The numbers on the links in the network represent the time, cost or distance involved in traversing them. It is assumed that the way in which we enter a node has no effect on the way of leaving it—an assumption which does not hold good in travelling salesman problem.

EXAMPLE 5·11·1 :

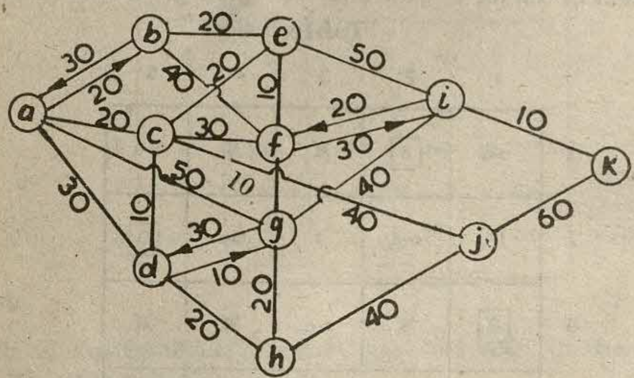

Fig. 5·6.

A person wishes to reach the destination '*k*' while starting from the station '*a*' in the network shown in figure 5·6. Links work in both directions unless marked otherwise.

The numbers on the links represent cost (in rupees) of going from one node to another. Find the route that involves the least cost.

Solution. It is the shortest route problem but does not have the restrictions imposed in a travelling salesman problem. For instance, it is not necessary to include all the nodes in the optimal shortest route ; moreover, it is not a cyclic route problem as it is not required to come back to the starting point '*a*'.

Obviously, in order to find the shortest route between '*a*' and '*k*' we must find the shortest route from '*a*' to every other point in the network. The graphic method used for this purpose consists of the following steps :

Step 1 : Starting with the origin '*a*', draw all links by which one can go from '*a*' to other nodes and represent the cost from '*a*' on each of these nodes. This is shown in figure 5·7.

Step 2 : In case there are links between any of the nodes obtained in step 1, determine for each of these links if the indirect route from '*a*' is shorter than the direct route. Draw the shorter route as a solid line and the longer route as a dotted line. Insert the shortest cost found on each such node.

For example, the cost of going from '*a*' to '*g*' through '*d*' is lower than the cost of going from '*a*' to '*g*' directly. Hence link *ag* is drawn dotted. In case of a tie both links are drawn solid. Thus one can go from '*a*' to '*d*' directly or through '*c*' at the same cost. Hence the links connecting them are drawn solid, as shown in figure 5·8.

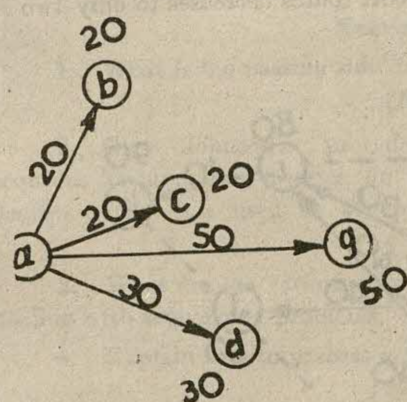

Fig. 5·7.

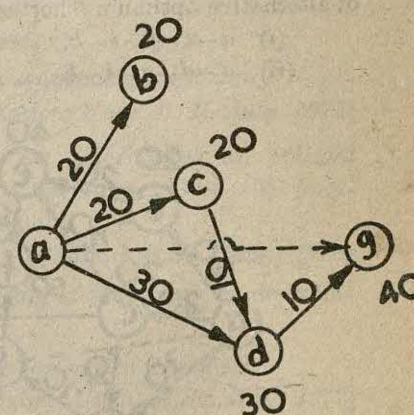

Fig. 5·8.

Step 3 : Add nodes to which one can go from the nodes represented in step 2 and repeat step 2. This is shown in figure 5·9.

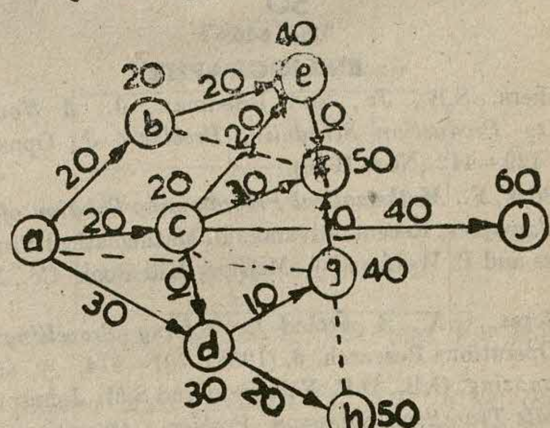

Fig. 5·9.

Step 4 : Continue till completed. This is shown in figure 5·10. Solid lines show the routes that can be taken from 'a' to every other node. Evidently there are a number of alternative paths giving least cost. They are

(*i*)	$a—b—e—f—i—k$	(involving 6 nodes),
(*ii*)	$a—c—f—i—k$	(,, 5 ,,),
(*iii*)	$a—c—d—g—f—i—k$	(,, 7 ,,),
(*iv*)	$a—d—g—f—i—k$	(,, 6 ,,),
(*v*)	$a—d—g—i—k$	(,, 5 ,,).

All the routes have the same cost (Rs. 90) of travelling from 'a' to 'k'.

If, however, an additional constraint is imposed e.g., person is to visit minimum number of stations before reaching 'k', the number of alternative optimum (shortest) cost routes decreases to only two :

(*i*) $a—c—f—i—k$,

(*ii*) $a—d—g—i—k$.

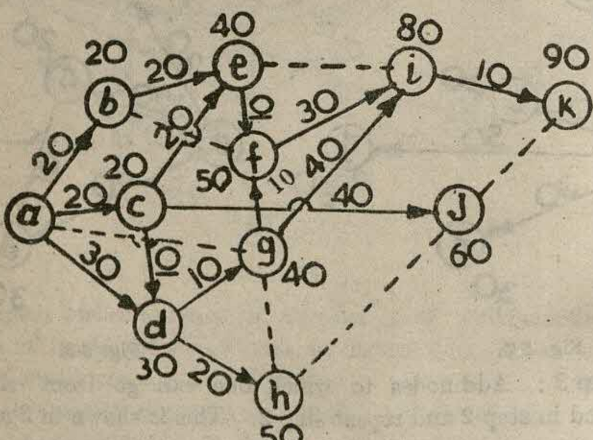

Fig. 5·10.

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Exercises

1. What is 'no passing rule' in a sequencing algorithm ?

[*Bangalore Univ. B. E. July, 1978*]

2. Give Johnson's procedure for determining an optimal sequence for processing n items on two machines. Give the justification of the rule used in the procedure.

[*Aligarh B.Sc. (Stat.) 1976*]

3. What are the various simplifying assumptions made while dealing with sequencing problems.

4. Explain how to process n jobs through m machines.

[*Sambalpur Univ. M.Sc. May, 1977*]

Section 5·4

5. We have five jobs, each of which must go through the two machines A and B in the order AB. Processing times are given in the table below.

Table 5·47

Processing time in hours

Job	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the elapsed time T.

[*Meerut M.Sc. (Math.) 1975, Delhi M.Sc. (Stat.) 1969*]

(Ans. 2-4-3-5-1; T_{min}=30 hours.)

6. A book-binder has one printing press, one binding machine and manuscripts of a number of different books. The times required to perform the printing and binding operations for each book are known. We wish to determine the order in which books should be processed in order to minimize the total time required to turn out all the books.

Table 5·48

Book :	1	2	3	4	5	6
Printing time :	30	120	50	20	90	110
Binding time :	80	100	90	60	30	10

[*Sambalpur M·Sc. May, 1977*]

(**Ans.** 4-1-3-2-5-6; T_{min}=430 hours.)

7. A ready made garment manufacturer has to process 7 items through two stages of production, viz., cutting and sewing. The time taken for each of these items at the different stages are given below in appropriate units.

Table 5·49

Item :	1	2	3	4	5	6	7
Cutting time :	5	7	3	4	6	7	12
Sewing time :	2	6	7	5	9	5	8

Find an order in which these items are to be processed through these stages so as to minimize the total processing time.

[*I·S·I. (Dip.) 1976*]

(**Ans.** 3-4-5-7-2-5-1)

8. A company has eight large machines which receive preventive maintenance. The maintenance team is divided into two crews *A* and *B*. Crew *A* takes the machine 'power' and replaces parts according to a given maintenance schedule. The second crew resets the machine and puts it back into operation. At all times the no passing rule is considered to be in effect. The servicing times for each machine are given below.

Table 5·50

Machine	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
Crew A	5	4	22	16	15	11	9	4
Crew B	6	10	12	8	20	7	2	21

Determine the optimal sequence of scheduling the factory maintenance crews to minimize their idle time and represent it on a chart. [*Bangalore Univ. M.E. July, 1978*]

(**Ans**. *b-a-e-c-d-f-h-g*)

Section. 5·5

9. A foreman wants to process four different jobs on three machines : a shaping machine, a drilling machine and a tapping machine, the sequence of operations being shaping-drilling-tapping. Decide the optimal sequence for the four jobs to minimize the time elapsed from the start of first job to the end of last job if the process times are

Table 5·51

Job	*Shaping (minutes)*	*Drilling (minutes)*	*Tapping (minutes)*
1	13	3	18
2	18	8	4
3	8	6	13
4	23	10	8

(**Ans.** 3-1-4-2; $T_{min}=74$ minutes ; Idle times : 12, 47,31 minutes respectively)

10. If a third stage of production is added, viz., pressing and packing with the processing times of exercise no. 7, find an order in which these seven items are to be processed so as to minimize the time taken to process all the items through all the three stages.

Item :	1	2	3	4	5	6	7
Processing time							
Pressing and packing :	10	12	11	13	12	10	11

[*I.S.I.* (*Dip.*) *1976*]

(**Ans.** 1-4-3-6-2-5-7; $T_{min}=86$ time units; Idle times : 42, 44, 7 time units respectively)

11. Find the sequence, for the following eight jobs, that will minimize the total elapsed time for the completion of all the jobs. Each job is processed in the same order *CAB*. Entries give the time in hours on the machines.

Table 5·52

Jobs		1	2	3	4	5	6	7	8
Times	*A*	4	6	7	4	5	3	6	2
on	*B*	8	10	7	8	11	8	9	13
machines	*C*	5	6	2	3	4	9	15	11

[*Bombay B.Sc. (Stat.) 1975*]

(**Ans.** 4-1-3-5-2-7-8-6; $T_{min}=81$ hours.)

Section 5·6

12. Use graphical method to minimize the time needed to process the following jobs on machines A, B, C, D and E. Find the total time elapsed to complete both jobs. Also find for each job, the machine on which it should be processed first.

Table 5·53

Job 1	*Sequence*	*A*	*B*	*C*	*D*	*E*
	Time (hrs.)	2	3	5	2	1
Job 2	*Sequence*	*D*	*C*	*A*	*B*	*E*
	Time (hrs.)	6	2	3	1	3

[**Ans.** $T_{min}=19$ hrs. ; A (1), D (2)]

13. There are two jobs to be processed through four machines A, B, C and D. The prescribed technological orders are

Job 1 : ABCD

Job 2 : DBAC

Processing times in hours are given in the following table :

Table 5·54

Job	*Machine*			
	A	*B*	*C*	*D*
1	2	4	5	1
2	6	4	2	3

Calculate the total minimum elapsed time to complete the two jobs. [*Delhi M.Sc. (Math.) 1978*]

(**Ans.** $T_{min}=15$ hrs.)

14. There are two jobs to be processed through five machines *A, B, C, D* and *E*. The prescribed technological order is

Job 1 : A B C D E

Job 2 : B C A D E.

The process times in hours are given in table 5·55.

Table 5·55

Job 1		*Job 2*	
Sequence of machines	*Time*	*Sequence of machines*	*Time*
A	3	*B*	5
B	4	*C*	4
C	2	*A*	3
D	6	*D*	2
E	2	*E*	6

Find out the optimal sequencing of jobs on machines and the minimum time required to process these jobs. [*Agra M. Stat. 1974*]

[**Ans.** $T_{min}=22$ hrs.; *A* (1), *B*, (2)]

Section 5·7

15. There are four jobs each of which has to be processed on machines *A, B, C, D, E* and *F* in the order *A B C D E F*. Processing time in hours is given below. Find out the optimal sequencing of jobs, minimum time required to process these jobs and the idle time for each of the machines.

Table 5·56

Job	*A*	*B*	*C*	*D*	*E*	*F*
1	15	8	6	14	6	26
2	17	7	9	10	15	22
3	21	7	12	9	11	19
4	18	6	11	12	14	17

(**Ans.** 1-2-4-3; $T_{min}=133$ hrs.; Idle times : 66, 105, 95, 88, 87 and 49 hrs. respectively

16. Solve the following sequencing problem, giving an optimal solution when no passing is allowed.

Table 5·57

		Job 1	2	3	4	5
Machine	A	14	7	12	8	10
	B	5	6	4	7	3
	C	3	2	4	1	5
	D	10	12	8	15	16

(**Ans.** 2-4-5-1-3; $T_{min}=76$ time units; Idle times : 25, 51, 61, 15 time units respectively)

Section 5·10

17. A machine operator processes five types of items on his machine each week, and must choose a sequence for them. The setup cost per change depends on the items presently on the machine and the setup to be made according to the following table :

Table 5·58

		To item A	B	C	D
From item	A	∞	4	7	3
	B	4	∞	6	3
	C	7	6	∞	7
	D	3	3	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on his machine ?

[*Agra M. Stat. 1974*]

(**Ans.** $A\to D\to B\to C\to A$ and $A\to C\to B\to D\to A$; min. cost=29)

18. Given the matrix of setup costs below, show how to sequence production so as to minimize setup cost per cycle.

Table 5·59

To

From	*A*	*B*	*C*	*D*	*E*
A	∞	3	6	2	3
B	3	∞	5	2	3
C	6	5	∞	6	4
D	2	2	6	∞	6
E	3	3	4	6	∞

(**Ans.** $A \to D \to B \to C \to E \to A$; min. cost $=$ **16**)

19. Find the least cost route for the travelling salesman problem shown below.

Table 5·60

To city

From city	1	2	3	4	5	6	7
1	∞	8	14	8	6	10	3
2	8	∞	12	7	6	5	5
3	10	9	∞	13	5	13	10
4	7	6	13	∞	7	10	8
5	7	4	9	10	∞	6	9
6	8	5	13	7	6	∞	4
7	4	5	11	9	6	5	∞

(**Ans.** $1 \to 7 \to 2 \to 6 \to 4 \to 3 \to 5 \to 1$; least cost $=$ **45**)

20. Find the shortest route from 'a' to 'i' by using graphic method. The numbers on the links represent the distance in kilometres.

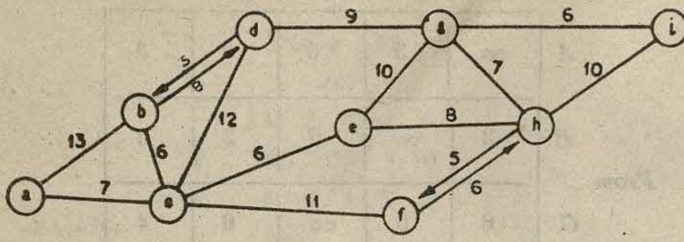

Fig. 5 11

(**Ans**. $a \to c \to e \to g \to i$; 29 km.)

6

Advanced Topics in Linear Programming

6·1. Duality in Linear Programming

For every L.P. problem (linear programme) there is a related unique L.P. problem (another linear programme) involving the same data which also describes the original problem (programme).

The given original programme is called the *primal programme* (P). This programme can be solved by transposing (reversing) the rows and columns of the algebraic statement of the problem. Inverting the programme in this way results in *dual programme* (D). A solution to the dual programme may be found in a manner similar to that used for the primal. The two programmes have very closely related properties so that *optimal* solution of the dual gives complete information about the *optimal* solution of the primal and vice-versa.

Duality is an extremely important and interesting feature of linear programming. The various useful aspects of this property are

(*i*) if the primal problem contains a large number of rows (constraints) and a smaller number of columns (variables.), the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence it offers an advantage in many applications.

(*ii*) it gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of the problem. This is termed as post optimality or sensitivity analysis.

(*iii*) duality in linear programming has certain far reaching consequences of economic nature.

(*iv*) Calculation of the dual checks the accuracy of the primal solution.

(*v*) duality in linear programming shows that each linear programme is equivalent to a two-person zero-sum game. This indicates that fairly close relationships exist between linear programming and the theory of games.

6·1-1. Dual Problem when Primal is in Canonical form

The general linear programming problem in canonical form as discussed in sections 2·6 and 2·7-1 is

$$\left.\begin{array}{l} \text{maximize} \quad Z=c_1x_1+c_2x_2+c_3x_3+\ldots+c_nx_n, \\ \text{subject to} \quad a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n \leqslant b_1, \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots+a_{2n}x_n \leqslant b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_2+a_{m2}x_2+a_{m3}x_3+\ldots+a_{mn}x_n \leqslant b_n, \end{array}\right\} \quad \ldots(6\cdot1)$$

where $x_1, x_2, x_3, \ldots, x_n$, all $\geqslant 0$.

If the above problem is referred to as primal, then, its associated dual will be

$$\left.\begin{array}{l} \text{minimize} \quad W=b_1y_1+b_2y_2+b_3y_3+\ldots+b_my_m, \\ \text{subject to} \quad a_{11}y_1+a_{21}y_2+a_{31}y_3+\ldots+a_{m1}y_m \geqslant c_1, \\ \qquad a_{12}y_1+a_{22}y_2+a_{32}y_3+\ldots+a_{m2}y_m \geqslant c_2, \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad a_{1n}y_1+a_{2n}y_2+a_{3n}y_3+\ldots+a_{mn}y_m \geqslant c_n, \end{array}\right\} \quad \ldots(6\cdot2)$$

where the dual variables $y_1, y_2, y_3, \ldots, y_m$, all $\geqslant 0$.

Equations (6·1) and (6·2) are called *symmetric primal-dual pairs.*

The above pair of programmes can be written as

Primal	*Dual*
maximize $Z=\sum_{j=1}^{n} c_j x_j$,	minimize $W=\sum_{i=1}^{m} b_i y_i$,
subject to $\sum_{j=1}^{n} a_{ij}x_j \leqslant b_i$, $i=1, 2, 3, \ldots m$,	subject to $\sum_{i=1}^{m} a_{ij} y_i \geqslant c_j$, $j=1, 2, 3, \ldots, n$,
where $x_j \geqslant 0$, $j=1, 2, 3, \ldots, n$.	where $y_i \geqslant 0$, $i=1, 2, 3, \ldots, m$.

From the above two programmes, the following points are clear :

(1) if the primal contains n variables and m constraints, the dual will contain m variables and n constraints.

(2) the maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(3) the maximization problem has ($\leqslant$) constraints while the minimization problem has ($\geqslant$) constraints.

(4) the constants $c_1, c_2, c_3, \ldots, c_n$ in the objective function of the primal appear in the constraints of the dual.

(5) the constants $b_1, b_2, b_3, \ldots, b_m$ in the constraints of the primal appear in the objective function of the dual.

(6) the variables in both problems are non-negative.

The constraint relationships of the primal and dual can be represented in a single table shown below.

Table 6·1

	x_1	x_2	x_3	$\cdots$	x_n	
y_1	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leqslant b_1$
y_2	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leqslant b_2$
y_3	a_{31}	a_{32}	a_{33}	$\cdots$	a_{3n}	$\leqslant b_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leqslant b_m$
	$\geqslant c_1$	$\geqslant c_2$	$\geqslant c_3$	$\cdots$	$\geqslant c_n$	

EXAMPLE 6·1·1·1

Construct the dual to the primal problem

maximize $Z=3x_1+5x_2$,

subject to $2x_1+6x_2\leqslant 50$,

$3x_1+2x_2\leqslant 35$,

$5x_1-3x_2\leqslant 10$,

$x_2\leqslant 20$,

where $x_1\geqslant 0$, $x_2\geqslant 0$.

Solution :

Let y_1, y_2, y_3 and y_4 be the corresponding dual variables, then the dual problem is given by

minimize $W=50y_1+35y_2+10y_3+20y_4$,

subject to $2y_1+3y_2+5y_3\geqslant 3$,

$6y_1+2y_2-3y_3+y_4\geqslant 5$,

where y_1, y_2, y_3, y_4, all $\geqslant 0$.

As the dual problem has lesser number of constraints than the primal (2 instead of 4), it requires lesser work and effort to solve it. This follows from the fact that the computational difficulty in the linear programming problem is mainly associated with the number of constraints rather than the number of variables.

EXAMPLE 6·1·1·2

Construct the dual of the problem

minimize $Z=3x_1-2x_2+4x_3$,

subject to the constraints

$$\begin{aligned}
&3x_1+5x_2+4x_3\geqslant 7,\\
&6x_1+x_2+3x_3\geqslant 4,\\
&7x_1-2x_2-x_3\leqslant 10,\\
&x_1-2x_2+5x_3\geqslant 3,\\
&4x_1+7x_2-2x_3\geqslant 2,\\
&x_1,\ x_2,\ x_3\geqslant 0.
\end{aligned}$$

[*Meerut B Sc. (Math.) 1970*]

Solution :

As the given problem is of minimization, all constraints should be of $\geqslant$ type. Multiplying the third constraint by -1 on both sides we get

$$-7x_1+2x_2+x_3\geqslant -10.$$

The dual of the given problem will be

maximize $W=7y_1+4y_2-10y_3+3y_4+2y_5$,

subject to

$$\begin{aligned}
&3y_1+6y_2-7y_3+y_4+4y_5\leqslant 3,\\
&5y_1+y_2+2y_3-2y_4+7y_5\leqslant -2,\\
&4y_1+3y_2+y_3+5y_4-2y_5\leqslant 4,\\
&y_1,\ y_2,\ y_3,\ y_4,\ y_5 \text{ all}\geqslant 0,
\end{aligned}$$

where y_1, y_2, y_3, y_4 and y_5 are the dual variables associated with the first, second, third, fourth and fifth constraint respectively.

6.1.2 Dual Problem when Primal is in the Standard Form

As discussed in section 2·7·2, all constraints are equations $(=)$ in standard form. In this section we shall find that an equality constraint in the primal corresponds to an unconstrained variable in the dual and vice versa. Consider the problem

maximize $Z=c_1x_1+c_2x_2$,

subject to

$$\begin{aligned}
&a_{11}x_1+a_{12}x_2=b_1,\\
&a_{21}x_1+a_{22}x_2\leqslant b_2,\\
&x_1\geqslant 0,\ x_2\geqslant 0.
\end{aligned}$$

The first constraint of equality form, when expressed in canonical form is equivalent to

$$a_{11}x_1+a_{12}x_2\leqslant b_1 \text{ and } a_{11}x_1+a_{12}x_2\geqslant b_1.$$

Or it can be expressed as

$$a_{11}x_1+a_{12}x_2\leqslant b_1,$$
$$-a_{11}x_1-a_{12}x_2\leqslant -b_1.$$

Let y'_1, y''_1 and y_2 be the dual variables associated with the first, second and third constraints. Then the dual problem is

maximize $W=b_1(y'_1-y''_1)+b_2y_2$,

subject to $a_{11}(y'_1-y''_1)+a_{21}y_2\geqslant c_1$,
$a_{12}(y'-y''_1)+a_{22}y_2\geqslant c_2$,
$y'_1\geqslant 0,\ y''_1\geqslant 0,\ y_2\geqslant 0.$

Note that the term $(y'_1-y''_1)$ occurs in the objective function as well as all the constraints of the dual. This will always be true whenever there is an equality constraint in the primal. If we put $y_1'-y''_1=y_1$, then the new variable y_1, which is the difference between two non-negative variables, becomes unrestricted in sign and the dual problem becomes

minimize $W=b_1y_1+b_2y_2$,

subject to $a_{11}y_1+a_{21}y_2\geqslant c_1$,
$a_{12}y_1+a_{22}y_2\geqslant c_2$,

y_1 unrestricted in sign, $y_2\geqslant 0$.

This shows that the dual variable which corresponds to an equality constraint must be unrestricted in sign. Conversely, when a primal variable is unrestricted in sign, its dual constraint must be in equation form.

In general, if the primal problem in standard form is

$$\text{maximize } Z=\sum_{j=1}^{n} c_jx_j,$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j=b_i,\quad i=1, 2, 3,\ldots, m,$$

$$x_j\geqslant 0,\qquad j=1, 2, 3,\ldots, n;$$

then the dual problem is

$$\text{minimize } W=\sum_{i=1}^{m} b_iy_i,$$

$$\text{subject to } \sum_{i=1}^{m} a_{ij}y_i\geqslant c_j,\ j=1, 2, 3,\ldots, n,$$

y_i are unrestricted in sign for all i

On the other hand, if the primal problem is

$$\text{maximize } Z=\sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \quad i=1, 2, 3, \ldots, m,$$

x_j are unrestricted in sign for all j,

then the dual problem is given by

$$\text{minimize } W=\sum_{i=1}^{m} b_i y_i,$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i = c_j, \quad j=1, 2, 3, \ldots, n,$$

$$y_i \geqslant 0, \qquad i=1, 2, 3, \ldots, m.$$

Note that in this case the dual is in the standard form.

From the above discussion it is concluded that we can work with any of the forms of primal while dealing with duality theory.

EXAMPLE 6.1-2·1

Construct the dual of the problem

maximize $Z=3x_1+10x_2+2x_3$,

subject to $2x_1+3x_2+2x_3 \leqslant 7$,

$3x_1-2x_2+4x_3=3$,

$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$

Solution :

The standard form of this problem is

maximize $Z=3x_1+10x_2+2x_3+0s_1$,

subject to $2x_1+3x_2+2x_3+s_1=7$,

$3x_1-2x_2+4x_3=3$,

$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ s_1 \geqslant 0.$

The dual is given by

minimize $W=7y_1+3y_2$,

subject to $2y_1+3y_2 \geqslant 3$,

$3y_1-2y_2 \geqslant 10$,

$2y_1+4y_2 \geqslant 2$,

$y_1 \geqslant 0$, y_2 unrestricted in sign.

Note that the last constraint, $y_1 \geqslant 0$ corresponds to s_1 in the primal.

It, thus, follows that if the ith primal constraint is an equation, the ith dual variable will be unrestricted in sign. Further, dual variables which correspond to primary equality constraints only are unrestricted; those corresponding to the primary inequations are non-negative.

Though the canonical and standard forms yield the same dual, it is easier to handle the standard form, particularly when the primal contains all the three types of constraints.

EXAMPLE 6.1-2·2

Construct the dual of the problem

maximize $Z=3x_1+17x_2+9x_3$,

subject to $x_1-x_2+x_3 \geqslant 3$,

$-3x_1+2x_3 \leqslant 1$,

$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$

Solution.

Firstly the $\geqslant$ constraint is converted into $\leqslant$constraint by multiplying both sides by -1.

i.e., $$-x_1+x_2-x_3 \leqslant -3.$$

Now the dual of the given problem is

minimize $W=-3y_1+y_2$,

subject to $-y_1-3y_2 \geqslant 3$,

$y_1 \geqslant 17$,

$-y_1+2y_2 \geqslant 9$,

$y_1 \geqslant 0,\ y_2 \geqslant 0.$

EXAMPLE 6·1.2·3

Construct the dual of the problem

minimize $Z=x_2+3x_3$,

subject to $2x_1+x_2 \leqslant 3$,

$x_1+2x_2+6x_3 \geqslant 5$,

$-x_1+x_2+2x_3=2$,

$x_1,\ x_2,\ x_3 \geqslant 0.$

Solution.

As the given problem is of minimization, all constraints should be of $\geqslant$ type. Multiplying the first constraint by -1 on both sides we get

$$-2x_1-x_2 \geqslant -3.$$

The equation $-x_1+x_2+2x_3=2$ can be expressed as a pair of inequalities

$$-x_1+x_2+2x_3 \geqslant 2 \text{ and } -x_1+x_2+2x_3 \leqslant 2,$$

or $$-x_1+x_2+2x_3 \geqslant 2 \text{ and } x_1-x_2-2x_3 \geqslant -2.$$

Thus the given problem becomes

minimize $Z=0x_1+x_2-3x_3$,

subject to $-2x_1-x_2 \geqslant -3$,

$x_1+2x_2+6x_3 \geqslant 5$,

$-x_1+x_2+2x_3 \geqslant 2$,

$$x_1-x_2-2x_3 \geqslant -2,$$
$$x_1,\ x_2,\ x_3 \geqslant 0.$$

Let y_1, y_2, y_3' and y_3'' be the associated non-negative dual variables. Then the dual of the problem is

maximize $W=-3y_1+5y_2+2y'_3-2y_3''$,

subject to $\quad -2y_1+y_2-y_3'+y_3'' \leqslant 0,$

$-y_1+2y_2+y_3'-y_3'' \leqslant 1,$

$6y_2+2y_3'-2y_3'' \leqslant 3,$

$y_1,\ y_2,\ y'_3\ y_3''$, all $\geqslant 0$

Substituting $y_3'-y_3''=y_3$, where y_3 is unrestricted in sign, the dual problem becomes

maximize $W=-3y_1+5y_2+2y_3$,

subject to $\quad -2y_1+y_2-y_3 \leqslant 0,$

$-y_1+2y_2+y_3 \leqslant 1,$

$6y_2+2y_3 \leqslant 3,$

$y_1,\ y_2 \geqslant 0$, y_3 unrestricted in sign.

EXAMPLE 6·1·2·4

Construct the dual of the primal problem

maximize $Z=2x_1+x_2+x_3$,

subject to $\quad x_1+x_2+x_3 \geqslant 6,$

$3x_1-2x_2+3x_3=3,$

$-4x_1+3x_2-6x_3=1,$

$x_1,\ x_2,\ x_3 \geqslant 0.$

Solution :

As the given problem is of maximization, all constraints should be of $\leqslant$ type. Multiplying the first constraint by -1 on both sides we get

$$-x_1-x_2-x_3 \leqslant -6.$$

The equation $3x_1-2x_3+3x_3=3$ can be expressed as a pair of inequalities

$$3x_1-2x_2+3x_3 \geqslant 3 \quad \text{and} \quad 3x_1-2x_2+3x_3 \leqslant 3,$$

or
$$3x_1-2x_2+3x_3 \leqslant 3 \quad \text{and} \quad -3x_1+2x_2-3x_3 \leqslant -3.$$

Similarly, the equation $-4x_1+3x_2-6x_3=1$ can be expressed as

$$-4x_1+3x_2-6x_3 \leqslant 1 \quad \text{and} \quad 4x_1-3x_2+6x_3 \leqslant -1.$$

Thus the given problem becomes

maximize $Z=2x_1+x_2+x_3$,

subject to $\quad -x_1-x_2-x_3 \leqslant -6,$

$3x_1-2x_2+3x_3 \leqslant 3,$

$-3x_1+2x_2-3x_3 \leqslant -3,$

$$-4x_1+3x_2-6x_3\leqslant 1,$$
$$4x_1-3x_2+6x_3\leqslant -1,$$
$$x_1,\ x_2,\ x_3\geqslant 0.$$

Let y_1, y_2', y_2'', y_3', y_3'' be the associated non-negative dual variables. Then the dual of this problem is

minimize $W=-6y_1+3(y_2'-y_2'')+(y_3'-y_3'')$,

subject to $-y_1+3(y_2'-y_2'')-4(y_3'-y_3'')\geqslant 2,$
$$-y_1-2(y_2'-y_2'')+3(y_3'-y_3'')\geqslant 1,$$
$$-y_1+3(y_2'-y_2'')-6(y_3'-y_3'')\geqslant 1,$$
$$y_1,\ y'_2,\ y_2'',\ y_3',\ y_3'',\ \text{all}\geqslant 0.$$

Substituting $y_2'-y_2''=y_2$ and $y_3'-y_3''=y_3$, where y_2 and y_3 are both unrestricted in sign, the dual problem becomes

minimize $W=-6y_1+3y_2+y_3$,

subject to $-y_1+3y_2-4y_3\geqslant 2,$
$$-y_1-2y_2+3y_3\geqslant 1,$$
$$-y_1+3y_2-6y_3\geqslant 1,$$
$y_1\geqslant 0$, y_2 and y_3 unrestricted in sign.

EXAMPLE 6·1·2·5

Obtain the dual of the following primal problem :

minimize $Z=3x_1-2x_2+x_3$,

subject to $2x_1-3x_2+x_3\leqslant 5,$
$$4x_1-2x_2\geqslant 9,$$
$$-8x_1+4x_2+3x_3=8,$$
$x_1,\ x_2\geqslant 0$, x_3 is unrestricted.

Solution :

Since x_3 is unrestricted, let it be replaced by x'_3-x_3'', where $x_3'\geqslant 0$, $x_3''\geqslant 0$. Then the given problem becomes

minimize $Z=3x_1-2x_2+x_3'-x_3''$,

subject to $2x_1-3x_2+x_3'-x_3''\leqslant 5,$
$$4x_1-2x_2\geqslant 9,$$
$$-8x_1+4x_2+3x_3'-3x_3''=8,$$
$$x_1,\ x_2,\ x_3',\ x_3''\geqslant 0.$$

Since it is a minimization problem, the first constraint is multiplied by -1 on both sides to give
$$-2x_1+3x_2-x_3'+x_3''\geqslant -5.$$

The equation $-8x_1+4x_2+3x_3'-3x_3''=8$ can be expressed as a pair of inequalities
$$-8x_1+4x_2+3x_3'-3x_3''\geqslant 8 \quad\text{and}\quad -8x_1+4x_2+3x_3'-3x_3''\leqslant 8,$$
or $\quad -8x_1+4x_2+3x_3'-3x_3''\geqslant 8 \quad$ and $\quad 8x_1-4x_2-3x_3'+3x_3''\geqslant 8.$

Thus the given problem becomes

$$\begin{aligned}
\text{minimize } Z &= 3x_1 - 2x_2 + x_3' - x_3'',\\
\text{subject to } & -2x_1 + 3x_2 - x_3' + x_3'' \geqslant -5,\\
& 4x_1 - 2x_2 \geqslant 9,\\
& -8x_1 + 4x_2 + 3x_3' - 3x_3'' \geqslant 8,\\
& 8x_1 - 4x_2 - 3x_3' + 3x_3'' \geqslant -8,\\
& x_1, x_2, x_3', x_3'' \geqslant 0.
\end{aligned}$$

Let y_1, y_2, y_3', y_3'' be the associated non-negative dual variables. Then the dual of this problem is

$$\begin{aligned}
\text{maximize } W &= -5y_1 + 9y_2 + 8(y_3' - y_3''),\\
\text{subject to } & -2y_1 + 4y_2 - 8(y_3' - y_3'') \leqslant 3,\\
& 3y_1 - 2y_2 + 4(y_3' - y_3'') \leqslant -2,\\
& -y_1 \qquad + 3(y_3' - y_3'') \leqslant 1,\\
& y_1 \qquad - 3(y_3' - y_3'') \leqslant -1.\\
& y_1, y_2, y_3', y_3'' \geqslant 0.
\end{aligned}$$

Replacing $y_3' - y_3''$ by y_3, where y_3 is unrestricted in sign and combining the last two inequations so as to form an equation, this dual may be written as

$$\begin{aligned}
\text{maximize } W &= -5y_1 + 9y_2 + 8y_3,\\
\text{subject to } & -2y_1 + 4y_2 - 8y_3 \leqslant 3,\\
& 3y_1 - 2y_2 + 4y_3 \leqslant -2,\\
& -y_1 \qquad + 3y_3 = 1,\\
& y_1, y_2 \geqslant 0, \ y_3 \text{ unrestricted in sign.}
\end{aligned}$$

6·1-3. Duality Theorems

This section presents some of the fundamental theorems which describe the relationship between the primal problem and its dual.

Theorem 1. *The dual of the dual is the primal.*

Proof. Let the primal problem in canonical form be

$$\left.\begin{aligned}
\text{maximize } Z = \ & c_1x_1 + c_2x_2 + \dots + c_nx_n,\\
\text{subject to } \ & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leqslant b_1,\\
& a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leqslant b_2,\\
& \ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\ \ \vdots\\
& a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leqslant b_m,\\
& \text{where } x_1, x_2, \dots, x_n, \text{ all} \geqslant 0;
\end{aligned}\right\} \quad \dots(6\cdot3)$$

then the dual of this problem is

$$\left.\begin{aligned}
\text{minimize } W \ & = b_1y_1 + b_2y_2 + \dots + b_my_m,\\
\text{subject to } \ & a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geqslant c_1,\\
& a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geqslant c_2,\\
& \ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\ \ \vdots\\
& a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geqslant c_n,\\
& \text{where } y_1, y_2, \dots, y_m, \text{ all} \geqslant 0 \text{ are the}\\
& \qquad\qquad \text{associated dual variables.}
\end{aligned}\right\} \quad \dots(6\cdot4)$$

Multiplying each of the relations (6·4) throughout by -1 we obtain

$$\left.\begin{array}{ll}\text{minimize } (-W)=-b_1y_1-b_2y_2-\ldots-b_my_m, & \\ \text{subject to} & -a_{11}y_1-a_{21}y_2-\ldots-a_{m1}y_m\leqslant-c_1, \\ & -a_{12}y_1-a_{22}y_2-\ldots-a_{m2}y_m\leqslant-c_2, \\ & \vdots \\ & -a_{1n}y_1-a_{2n}y_2-\ldots-a_{mn}y_m\leqslant-c_n, \\ & \text{where } y_1, y_2, \ldots, y_m, \text{ all}\geqslant 0.\end{array}\right\} \quad \ldots(6\cdot5)$$

By constructing the dual of relations (6·5) and multiplying throughout by -1 we obtain a system identical to relations (6·3). This proves the theorem.

Theorem 2. *The value of the objective function Z for any feasible solution of the primal is $\leqslant$ the value of the objective function W for any feasible solution of the dual.*

Proof. The general linear programming problem (primal) involving n decision variables and m constraints in canonical form is given by relations (6·3) and its associated dual involving m decision variables and n constraints is given by relations (6·4).

Multiply the first inequation of (6·3) by y_1, the second inequation of (6·3) by y_2, etc., and add them all. We get

$$\left.\begin{array}{l}(a_{11}x_1y_1+a_{12}x_2y_1+\ldots+a_{1n}x_ny_1)+(a_{21}x_1y_2+a_{22}x_2y_2+\ldots \\ \quad +a_{2n}x_ny_2)+\ldots+(a_{m1}x_1y_m+a_{m2}x_2y_m+\ldots \\ \quad +a_{mn}x_ny_m)\leqslant b_1y_1+b_2y_2+\ldots+b_my_m.\end{array}\right\} \quad \ldots(6\cdot6)$$

Similarly, multiply the first inequation of (6·4) by x_1, the second inequation by x_2, etc., and add them all. We get

$$\left.\begin{array}{l}(a_{11}x_1y_1+a_{21}x_1y_2+\ldots+a_{m1}x_1y_m)+(a_{12}x_2y_1+a_{22}x_2y_2 \\ \quad +\ldots+a_{m2}x_2y_m)+\ldots+(a_{1n}x_ny_1+a_{2n}x_ny_2+\ldots \\ \quad +a_{mn}x_ny_n)\geqslant c_1x_1+c_2x_2+\ldots+c_nx_n.\end{array}\right\} \quad \ldots(6\cdot7)$$

Now the sums on the left hand side of inequations (6·6) and (6·7) are equal. Hence

$$c_1x_1+c_2x_2+\ldots+c_nx_n\leqslant b_1y_1+b_2y_2+\ldots+b_my_m.$$

i.e., $\qquad Z\leqslant W.$

In other words, any feasible solution to the primal is less than or equal to any feasible solution to the dual. Thus if the feasible solution to the dual approximates $-\infty$, the primal can have no feasible solution. Similarly, if the feasible solution to the primal approximates $+\infty$, the dual can have no feasible solution. In either case, there is no finite optimum for the primal or the dual. If, however, both problems have finite optimal solutions, it can be reasonably expected that these two solutions are the same. This important conclusion leads to the following theorems :

Theorem 3. If either the primal or the dual problem has an unbounded solution, then the solution to the other problem is infeasible.

Theorem 4. If both primal and dual problems have feasible solutions, then both have optimal solutions and Max. Z = Min. W.

Theorem 5. If the primal has a feasible solution but the dual does not have, then the primal will not have a finite optimum solution and vice versa.

6·1·4. Properties of Primal and Dual Optimal Solutions

1. Values for the non-basic variables of the primal are given by the base row of the dual solution, under the slack variables (if there are any), with changed sign and under the artificial variables (if there is no slack variable in a constraint) after deleting the constant M.
2. Values for the slack variables of the primal are given by the base row under the non-basic variables of the dual solution with changed sign.
3. The value of the objective function is same for primal and dual solutions, but of opposite sign.

The above properties will be clear from the following example.

EXAMPLE 6·1·4·1.

A feed mixing operation can be described in terms of the two activities. The required mixture must contain four kinds of ingredients w, x, y and z. Two basic feeds A and B, which contain the required ingredients are available in the market. One Kg. of A contains 0·1 kg. of w, 0·1 Kg. of y and 0·2 Kg. of z. Likewise, one Kg. of feed B contains 0·1 Kg. of x, 0·2 Kg. of y and 0·1 Kg. of z. The daily per head requirement is of at least 0·4 Kg. of w, 0·6 Kg. of x, 2 Kg. of y and 1·6 Kg. of z. Feed A can be bought for £0·07 per Kg. and B can be bought for £0·05 per Kg. The availabilities, requirements and costs are summarized in the table below.

Table 6·2

Ingredient	*Feed A*	*Feed B*	*Requirement*
	(Kg.)	*(Kg.)*	*Kg.)*
w	0·1	0·0	04
x	0·0	0·1	0·6
y	0·1	0·2	2·0
z	0·2	0·1	1·8
cost	£0·07	£0·05	

Determine the quantities of feeds A and B in the mixture so that the total cost is minimum.

Solution : Formulation of L.P. Model

Step 1 : *Key decision* is to find the extent of feeds A and B.

Step 2 : Let these extents be x_1 and x_2.

Step 3 : *Feasible alternatives* are sets of values of x_1 and x_2, where $x_1 \geqslant 0$, $x_2 \geqslant 0$. ...(6.8)

Step 4 : *Objective* is to minimize the cost

i.e., minimize $Z=0{\cdot}07x_1+0{\cdot}05x_2$ ...(6·9)

Step 5 : *Constraints* are imposed by the requirements.

Thus $0{\cdot}1x_1+0x_2 \geqslant 0{\cdot}4,$ ...(6 10 *a*)

$0x_1+0{\cdot}1x_2 \geqslant 0{\cdot}6,$ ...(6·10 *b*)

$0{\cdot}1x_1+0{\cdot}2x_2 \geqslant 2{\cdot}0,$ ...(6·10 *c*)

$0{\cdot}2x_1+0{\cdot}1x_2 \geqslant 1{\cdot}8.$ ...(6·10 *d*)

Thus we get a linear optimization model in which we are to minimize equation (6·9) subject to constraints (6·10 *a*), (6·10 *b*), (6·10 *c*), (6·10 *d*) and the non-negativity restriction (6·8).

SOLUTION OF THE MODEL

Method (1) : Using the Primal Problem

Minimizing $Z=0{\cdot}07x_1+0{\cdot}05x_2$ amounts to

maximizing $Z_1=-0{\cdot}07x_1-0{\cdot}05x_2$.

Step 1 : Make the problem as N + S co-ordinate problem

At first thought it appears that we should introduce slack variables s_1, s_2, s_3 and s_4 for converting inequalities into equalities as shown below:

$$0{\cdot}1x_1+0x_2-s_1=0{\cdot}4,$$
$$0x_1+0{\cdot}1x_2-s_2=0{\cdot}6,$$
$$0{\cdot}1x_1+0{\cdot}2x_2-s_3=2{\cdot}0,$$
$$0{\cdot}2x_1+0{\cdot}1x_2-s_4=1{\cdot}8,$$

where $x_1, x_2, s_1, s_2, s_3, s_4$, all $\geqslant 0$.

Step 2 : Make N co-ordinates assume zero value

Putting $x_1=0$ and $x_2=0$, we get $s_1=-0{\cdot}4$, $s_2=-0{\cdot}6$, $s_3=-2{\cdot}0$ and $s_4=-1{\cdot}8$ as the first basic solution.

But negative values of the variables are unacceptable. Therefore, we introduce artificial variables A_j (A_1, A_2 A_3, and A_4) and the above equations can be written as

$$0{\cdot}1x_1+0x_2-s_1+A_1=0{\cdot}4,$$
$$0x_1+0{\cdot}1x_2-s_2+A_2=0{\cdot}6,$$
$$0{\cdot}1x_1+0{\cdot}2x_2-s_3+A_3=2{\cdot}0,$$
$$0{\cdot}2x_1+0{\cdot}1x_2-s_4+A_4=1{\cdot}8,$$

where $x_1, x_2, s_1, s_2, s_3, s_4, A_1, A_2, A_3, A_4$ all $\geqslant 0$ which gives the first feasible solution.

Now artificial variables with values greater than zero destroy the equality required by the general linear programming model. Therefore A_j (A_1, A_2 A_3, A_4) must not appear in the final solution. To achieve this, these artificial variables are assigned a large penalty (a large negative value, $-M$) in the objective function which can be written as

$$\text{maximize } Z_1=-0{\cdot}07x_1-0{\cdot}05x_2-MA_1-MA_2-MA_3-MA_4$$
$$+0s_1+0s_2+0s_3+0s_4$$

Step 3 :

The above information can be represented in the form of a simple table (table 6·3).

Table 6.3

		body						identity					
Objective function c_j		-0.07	-0.05	0	0	0	0	$-M$	$-M$	$-M$	$-M$		
e_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	b	θ
$-M$	A_1	0.1	0	-1	0	0	0	1	0	0	0	0.4	∞
$-M$	A_2	0	(0.1)	0	-1	0	0	0	1	0	0	0.6	6 ← (Key row)
$-M$	A_3	0.1	0.2	0	0	-1	0	0	0	1	0	2.0	10
$-M$	A_4	0.2	0.1	0	0	0	-1	0	0	0	1	1.8	18
$E_j = \Sigma e_i a_{ij}$		$-0.4M$	$-0.4M$	M	M	M	M	$-M$	$-M$	$-M$	$-M$		
$c_j - E_j$		$-0.07+0.4M$	$-0.05+0.4M$ ↑ K	$-M$	$-M$	$-M$	$-M$	M	M	M	M		

Step 4

(*i*) Making key element unity we get

Table 6.4

c_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	b
$-M$	A_1	0.1	0	-1	0	0	0	1	0	0	0	0.4
$-M$	A_2	0	(1)	0	-10	0	0	0	10	0	0	6.0
$-M$	A_3	0.1	0.2	0	0	-1	0	0	0	1	0	2.0
$-M$	A_4	0.2	0.1	0	0	0	-1	0	0	0	1	1.8

(*ii*) Replacing A_2 by x_2 and making three more iterations, the final matrix is

Table 6.5

c_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	s_4	A_1	A_2	A_3	A_4	b
$-0{\cdot}07$	x_1	1	0	0	0	3.3	$-6{\cdot}7$	0	0	-3.3	6.7	5.33
$-0{\cdot}05$	x_2	0	1	0	0	-6.6	3.4	0	0	6.6	-3.4	7.34
0	s_2	0	0	0	1	-0.66	$\cdot34$	0	-1	0.66	$-.34$	.134
0	s_1	0	0	1	0	0.33	$-.67$	-1	0	$-.33$	67	.133
c_j-E_j		0	0	0	0	$-.099$	$-.299$	$-M$	$-M$	$.099-M$	$.299-M$	

$\therefore$ Quantities of feeds A and B in the mixture are 5.33 kg. and 7.34 kg.

Total cost of the mixture $=$ £ $(0.07\times5.33+0.05\times7.34)=$ £0.74 (neglecting $-$ve sign)

Method II : Using the Dual Problem

The primal problem is

$$\begin{aligned}\text{minimize}\quad & Z=0{\cdot}07x_1+0.05x_2,\\ \text{subject to}\quad & 0.1x_1+0x_2\geqslant0.4,\\ & 0x_1+0.1x_2\geqslant0.6,\\ & 0.1x_1+0.2x_2\geqslant2.0,\\ & 0.2x_1+0.1x_2\geqslant1.8,\\ & x_1\geqslant0,\ x_2\geqslant0.\end{aligned}$$

The dual of this problem will be

$$\begin{aligned}\text{maximize}\quad & W=0.4y_1+0.6y_2+2y_3+1.8y_4,\\ \text{subject to}\quad & 0.1y_1+0y_2+0.1y_3+0.2y_4\leqslant0.07,\\ & 0y_1+0.1y_2+0.2y_3+0.1y_4\leqslant0.05,\\ & y_1, y_2, y_3, y_4 \text{ all } \geqslant0.\end{aligned}$$

Step 1 :

Make the problem as N+S co-ordinate problem

Adding the slack variables s_1 and s_2, the constraints become

$$0.1y_1+0y_2+0.1y_3+0.2\ y_4+s_1=0.07,$$
$$0y_1+0.1y_2+0.2y_3+0.1y_4+s_2=0.05,$$

and the objective function becomes

maximize $W=0.4y_1+0.6y_2+2y_3+1.8y_4+os_1+os_2$,
where $y_1, y_2, y_3, y_4, s_1, s_2$, all $\geqslant 0$.

Step 2 :

Make N-coordinates assume zero value

i.e. if $y_1=y_2=y_3=y_4=0$, then $s_1=0.07$ and $s_2=0.05$, which is the first feasible solution.

Step 3 :

Perform optimality test

Find whether the above solution can be improved or not. Represent the above data in the form of a matrix.

Table 6.6

	Objective function c_j	0.4	0.6	2	1.8	0	0		
e_i	variables in current solution	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	0.1	0	0.1	0.2	1	0	0.07	0.7
0	s_2	0	0.1	(0.2)	0.1	0	1	0.05	0.25← (Key row)
$E_j=\Sigma e_i a_{ij}$		0	0	0	0	0	0		
c_j-E_j		0.4	0.6	2	1.8	0	0		
				↑ K					

Step 4 :

(*i*) Make key element unity.

Table 6.7

e_i	C.S.V.	y_1	y_2	y_3	y_4	s_1	s_2	b
0	s_1	0.1	0	0.1	0.2	1	0	0.07
0	s_2	0	0.5	(1)	0.5	0	5	0.25

(ii) Replace s_2 by y_3

Table 6.8

e_i	C.S.V.	y_1	y_2	y_3	y_4	s_1	s_2	b	θ	
0	s_1	0.1	−0.05	0	−0.15	1	−0.5	0.045	0.3	← (Key row)
2	y_3	0	0.5	1	0.5	0	5	0.25	0.5	
c_j - E_j		0.4	−0.4	0	0.8	0	−1			
					↑ K					

Step 5 :

Iterating again, the final matrix becomes

Table 6.9

e_i	C.S.V.	y_1	y_2	y_3	y_4	s_1	s_2	b
1.8	y_4	0.0667	−0.0334	0	1	6.6667	−3.3334	0.3
2.0	y_3	−0.3334	0.6667	1	0	−3.3334	6.6667	0.1
c_j—E_j		—0.1333	—0.1333	0	0	−5.333	−7.333	

Optimal values are $y_1=y_2=0$, $y_3=0{\cdot}1$ and $y_4=0{\cdot}3$,
and optimal cost $=£[0.4(0)+0.6(0)+1.8(0.3)]$

$$=£(0.2+0.54)=£0.74.$$

Optimal values of x_1 and x_2 are given under the slacks of the dual with changed sign.

∴ Optimal values of x_1 and x_2 are 5.333 and 7.333 respectively *i.e.*, optimal quantities of feeds A and B in the mixture are 5.333 Kg. and 7.333 Kg. respectively.

Similarly, optimal values of dual variables y_3 and y_4 ($y_3=0.1$ and $y_4=0.3$ from table 6.9) could be obtained from the optimal primal solution given by table 6.5 either under the slack variables y_3 and y_4 with changed sign or under the corresponding artificial variables A_3 and A_4 by deleting the constant M.

6·1·5. Economic Interpretation of the Dual Variables

The dual variables have an interesting interpretation from the cost or economic point of view. To illustrate this point let us reconsider example 6.1-3.1 on feed mixing operation. The dual of this problem is

$$\text{maximize } W=0.4y_1+0.6y_2+2y_3+1.8y_4, \qquad ...(6.11)$$

$$\text{subject to}\quad \left.\begin{array}{l} 0.1y_1+0y_2+0.1y_3+0.2y_4 \leqslant 0.07, \\ 0y_1+0.1y_2+0.2y_3+0.1y_4 \leqslant 0.05, \end{array}\right\} \quad \ldots(6.12)$$

$$y_1, y_2, y_3, y_4, \text{ all } \geqslant 0,$$

and the optimal values of the dual variables are

$$y_1=0,\ y_2=0,\ y_3=0.1 \text{ and } y_4=0.3.$$

As the R.H.S. of constraints (6·12) denotes monetary units (£), L.H.S. must also be expressed in monetary units. In the first term 0·1 y_1 of the first constraint, 0·1 is the amount in kg. of ingredient w in one kg. of feed A and hence y_1 must denote the cost per kg. of ingredient w. Similarly, y_2, stands for the cost per kg. of ingredient x and so on. y_1, y_2, y_3 and y_4 are called the *shadow prices* of ingredients w, x, y and z respectively. They represent not the *actual market prices* but the *true accounting values* or the *imputed values* of the respective ingredients. Thus the true accounting values per kg. of ingredients w, x, y, and z to the company preparing the mixture are £ 0, £ 0, £ 0·1 and £ 0·3 respectively. $y_1=0$ and $y_2=0$ means that the accounting values of ingredients w and x are zero each.

6·2 DUAL SIMPLEX METHOD

Dual simplex method is helpful in finding the solution of an L.P. problem for a number of different right hand side vectors b_i. It is also used when new constraints are added to an L.P·P. for which the optimal solution has already been obtained. In such situations we have an infeasible basic primal solution whose associated dual solution is feasible. Dual simplex method, developed by C.E. Lemke is used for these situations since it clears the feasibility in the problem. It is precisely the regular (standard) simplex method applied to the dual problem, but is so constructed that it can operate with the same tableau as the primal problem. In this method this solution starts optimum and infeasible and remains infeasible until the true optimum is reached at which the solution becomes feasible. Thus whereas the regular simplex method starts with a basic feasible but non-optimal solution and works towards optimality, the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility. This method consists of the following steps :

Step 1 :

Convert the problem into maximization problem if it is initially in the minimization form.

Step 2 :

Convert $\geqslant$ type constraints, if any, into $\leqslant$ type by multiplying both sides of such constraints by -1.

Step 3 :

Convert the inequality constraints into equalities by the addition of slack variables and obtain the initial basic solution. **Express the above information in the form of a table called the dual simplex table.**

Step 4 :

Compute $c_j - E_j$ for every column.

(a) If all $c_j - E_j$ are negative or zero and all b_i are non-negative, the solution found above is the optimum basic feasible solution.

(b) If all $c_j - E_j$ are negative or zero and at least one b_i is negative, then proceed to step 5.

(c) If any $c_j - E_j$ is non-negative, the method fails.

Step 5 :

Select the row that contains the most negative b_i. **This row is called the key row or the pivot row.** The corresponding **basic variable leaves the current solution (basis matrix).**

Step 6 :

Look at the elements of the key row.

(a) If all elements are non-negative, the problem does not have a feasible solution.

(b) If at least one element is negative, find the ratios of the corresponding elements of $c_j - E_j$ row to these elements. Ignore the ratios associated with positive or zero elements of the key row. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the entering variable. Mark the key element or the pivot element.

Step 7 :

Make this key elements unity. Carry out the row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is obtained in a finite number of steps or there is an indication of non-existence of a feasible solution.

EXAMPLE 6.2-1

Solve by dual simplex method the following problem :

minimize $Z = 2x_1 + 2x_2 + 4x_3$,

subject to $2x_1 + 3x_2 + 5x_3 \geqslant 2$,

$3x_1 + x_2 + 7x_3 \leqslant 3$,

$x_1 + 4x_2 + 6x_3 \leqslant 5$,

$x_1, x_2, x_3 \geqslant 0$. [*Sambalpur M. Sc. (Math.) 1977*]

Solution.

It consists of the following steps :

Step 1 :

The given minimization problem is converted into maximization problem by writing

maximize $G=-Z=-2x_1-2x_2-4x_3$.

Step 2 :

The first constraint is of $\geqslant$ type. It is converted into $\leqslant$ type by multiplying throughout by -1. Thus the constraint becomes

$$-2x_1-3x_2-5x_3\leqslant-2.$$

Step 3 :

The problem in canonical form is now converted into standard form by adding slack variables s_1, s_2 and s_3 in the constraints. Thus the problem is expressed as

$$\text{maximize } G=-2x_1-2x_2-4x_3+0s+0s_2+0s_3,$$

$$\text{subject to} \quad -2x_1-3x_2-5x_3+s_1=-2,$$

$$3x_1+x_2+7x_3+s_2=3,$$

$$x_1+4x_2+6x_3+s_3=5.$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3 \text{ all } \geqslant 0.$$

Putting $x_1=x_2=x_3=0$, the initial basic solution is $s_1=-2$, $s_2=3$, $s_3=5$. Since s_1 is negative, solution is infeasible. The above information is expressed in table 6·10 called starting dual simplex table.

Table 6·10

Objective function c_j		-2	-2	-4	0	0	0	
e_i	v.c.s.	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-2	(-3)	-5	1	0	0	-2←key row
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
$E_j=\Sigma e_i a_{ij}$		0	0	0	0	0	0	
c_j-E_j		-2	-2	-4	0	0	0	
			↑K					

Initial basic infeasible solution

Step 4 :

Compute c_j-E_j where $E_j=\Sigma\, e_i a_{ij}$. As all c_j-E_j are either negative or zero and b_1 is negative, the solution is optimal but infeasible. We proceed to step 5.

Step 5 :

As $b_1=-2$, the first row is the key row and s_1 is the outgoing variable.

Step 6 :

Find the ratios of elements of $c_j - E_j$ row to the elements of key row. Neglect the ratios corresponding to positive or zero elements of key row. The desired ratios are

$$\frac{-2}{-2}=1,\ \frac{-2}{-3}=\frac{2}{3}\ \text{and}\ \frac{-4}{-5}=\frac{4}{5}.$$

Since $\frac{2}{3}$ is the smallest ratio, 'x_2'-column is the key column; x_2 is the incoming variable and (-3) is the key element.

Step 7 :

Make the key element unity. This is shown in table 6·11.

Table 6·11

e_i	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	$\frac{2}{3}$	(1)	$\frac{5}{3}$	$-\frac{1}{3}$	0	0	$\frac{2}{3}$
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5

Key element unity

Step 8 :

Replace s_1 by x_2. This is shown in table 6·12.

Table 6·12

	c_j	-2	-2	-4	0	0	0	
e_i	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b
-2	x_2	$\frac{2}{3}$	1	$\frac{5}{3}$	$-\frac{1}{3}$	0	0	$\frac{2}{3}$
0	s_2	$\frac{7}{3}$	0	$\frac{16}{3}$	$\frac{1}{3}$	1	0	$\frac{7}{3}$
0	s_3	$\frac{-5}{3}$	0	$\frac{-2}{3}$	$\frac{4}{3}$	0	1	$\frac{7}{3}$
$E_j=\Sigma e_i a_{ij}$		$\frac{-4}{3}$	-2	$\frac{-10}{3}$	$\frac{2}{3}$	0	0	
c_j-E_j		$\frac{-2}{3}$	0	$\frac{-2}{3}$	$\frac{-2}{3}$	0	0	

Optimal basic feasible solution

As all $c_j - E_j$ are negative or zero and all b_j are positive, the solution given by table 6·12 is optimal. The optimal solution is

$$x_1=0,$$
$$x_2=\frac{2}{3},$$
$$x_3=0,$$

$$\text{Max } G = -2\times 0 - 2\times \frac{2}{3} - 4\times 0$$

$$= -\frac{4}{3},$$

or $\quad \text{Min } Z = \frac{4}{3}.$

EXAMPLE 6.2.2

Use dual simplex method to

maximize $Z = -3x_1 - 2x_2$,

subject to

$$x_1 + x_2 \geqslant 1,$$
$$x_1 + x_2 \leqslant 7,$$
$$x_1 + 2x_2 \geqslant 10,$$
$$x_2 \leqslant 3,$$
$$x_1, x_2 \geqslant 0.$$

[*Delhi M. Sc.* (*Math.*) *1972*]

Solution.

Proceeding as in example 6·2·1 we express the given problem as

maximize $Z = -3x_1 - 2x_2$.

subject to

$$-x_1 - x_2 \leqslant -1,$$
$$x_1 + x_2 \leqslant 7,$$
$$-x_1 - 2x_2 \leqslant -10,$$
$$x_2 \leqslant 3,$$
$$x_1, x_2 \geqslant 0.$$

Adding slack variables the problem can be expressed as

maximize $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$,

subject to

$$-x_1 - x_2 + s_1 = -1,$$
$$x_1 + x_2 + s_2 = 7,$$
$$-x_1 - 2x_2 + s_3 = -10,$$
$$x_2 + s_4 = 3,$$
$$x_1, x_2, x_3, x_4, \text{ all} \geqslant 0.$$

The initial basic infeasible solution is $x_1 = 0$, $x_2 = 0$, $s_1 = -1$, s_2 7, $s_3 = -10$, $s_4 = 3$. This is expressed in table 6·13.

Table 6·13

Objective function c_j		-3	-2	0	0	0	0		
e_i variables in current solution		x_1	x_2	s_1	s_2	s_3	s_4	b	
0	s_1	-1	-1	1	0	0	0	-1	
0	s_2	1	1	0	1	0	0	7	
0	s_3	-1	(-2)	0	0	1	0	-10	$\leftarrow$— key row
0	s_4	0	1	0	0	0	1	3	
$E_j=\Sigma\, e_i a_{ij}$		0	0	0	0	0	0		*First basic*
c_j-E_j		-3	-2	0	0	0	0		*infeasible solution*
			$\uparrow$ K						

$$\frac{c_j-E_j}{a_{ij}}=\frac{-3}{-1}=3, \text{ for '}x_1\text{'-column and}$$

$$=\frac{-2}{-2}=1, \text{ for '}x_2\text{'-column.}$$

$\therefore$ 'x'$_2$—column is the key column and (-2) is the key element. This element is made unity in table 6·14.

Table 6·14

e_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	s_4	b	
0	s_1	-1	-1	1	0	0	0	-1	
0	s_2	1	1	0	1	0	0	7	
0	s_3	$\frac{1}{2}$	(1)	0	0	$-\frac{1}{2}$	0	5	*key element unity*
0	s_4	0	1	0	0	0	1	3	

Replacing s_3 by x_2 we get table 6·15.

Table 6·15

	c_j	-3	-2	0	0	0	0		
e_i	C.S.V.	x_1	x_2	s_1	s_2	s_3	s_4	b	
0	s_1	$\frac{-1}{2}$	0	1	0	0	0	4	
0	s_2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2	
-2	x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5	
0	s_4	$\left(-\frac{1}{2}\right)$	0	0	0	$\frac{1}{2}$	1	-2	$\leftarrow$—key row
$E_j=\Sigma e_i a_{ij}$		-1	-2	0	0	1	0		
c_j-E_j		-2	0	0	0	-1	0		*second*
$\frac{c_j-E_j}{a_{ij}}$		1	—	—	—	—	—		*basic infeasible solution*
		$\uparrow$ K							

Make key element unity. This is shown in table 6·16.

Table 6·16

e_i	C·S·V.	x_1	x_2	s_1	s_2	s_3	s_4	b	
0	s_1	$\frac{-1}{2}$	0	1	0	$-\frac{1}{2}$	0	4	
0	s_2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2	
-2	x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5	
0	s_4	(1)	0	0	0	-1	-2	4	*Key element unity*

Replace s_4 by x_1. This is shown in table 6·17.

Table 6·17

		c_j	-3	-2	0	0	0	0	0	
e_i	C·S·V.		x_1	x_2	s_1	s_2	s_3	s_4	b	
0	s_1		0	0	1	0	-1	-1	6	
0	s_2		0	0	0	1	1	1	0	
-2	x_2		0	1	0	0	0	1	3	
-3	x_1		1	0	0	0	-1	-2	4	
$E_j=\Sigma e_i a_{ij}$			-3	-2	0	0	3	4		
c_j-E_j			0	0	0	0	-3	-4		*O ptimal feasible solution*

Table 6·17 gives the optimal feasible solution, which is

$$x_1=4,\ x_2=3 \text{ and } Z_{max}=-3\times4-2\times3=-18.$$

*6·3 THE REVISED SIMPLEX METHOD

The simplex method discussed in chapter 2 performs calculations on the entire tableau during each iteration. If a linear programming problem involving a large number of variables and constraints is to be solved by this method, it will require a large storage space and time on a computer. Some computational techniques have been developed which require much less computer storage and time than that required by the simplex method. The three important and efficient computational techniques are : *the revised simplex method* or *simplex method with multipliers*, *the decomposition method* and *the bounded variables method*. Only the first and third techniques will be discussed in this text.

While solving a problem with simplex method, successive iterations are obtained by using row operations. This requires storing the entire tableau in the memory of the computer, which may not be feasible for very large problems. Luckily, it is really not

*The reader should refer to appendix A-3 while going through sections 6-3 and 6-4.

necessary to calculate the entire tableau during each iteration. The only information needed in moving from one tableau to the next is

(1) The $c_j - E_j$ row to determine the non-basic variable that enters the basis.

(2) The pivot column.

(3) The current basic variables and their values (right-hand-side constants) to determine the minimum positive ratio and thereby to determine the basic variable that leaves the basis.

The above information is directly obtained from the original equations by making use of the inverse of the current basic matrix. This method will now be illustrated with the help of an example.

EXAMPLE 6·3·1

$$\text{Maximize} \quad Z = 6x_1 + 3x_2 + 4x_3 - 2x_4 - x_5,$$
$$\text{subject to} \quad 2x_1 + 3x_2 + 3x_3 + x_4 = 10,$$
$$x_1 + 2x_2 + x_3 \quad + x_5 = 8,$$
$$x_1, \ldots, x_5 \geqslant 0.$$

Solution :

Since variable x_4 appears only in the first constraint equation with a unit coefficient, it is a basic variable in that equation. Similarly, x_5 is a basic variable. The basic feasible solution is $x_1 = x_2 = x_3 = 0$, $x_4 = 10$ and $x_5 = 8$. For easy reference the tableaus of the regular simplex method are shown below.

Table 6·18

	c_j	6	3	4	−2	1		
e_i	C·S·V.	x_1	x_2	x_3	x_4	x_5	b	θ
−2	x_4	2	3	(3)	1	0	10	10/3 ←
1	x_5	1	2	1	0	1	8	4
$E_j = \Sigma e_i a_{ij}$		−3	−4	−5	−2	1		
$c_j - E_j$		9	7	9 ↑	0	0	*Initial basic feasible solution*	

Table 6·19

e_i	C·S·V.	x_1	x_2	x_3	x_4	x_5	b
−2	x_4	2/3	1	(1)	1/3	0	10/3
1	x_5	1	2	1	0	1	8 *Key element unity*

Table 6·20

	c_j	6	3	4	-2	1		
e_i	C.S.V.	x_1	x_2	x_3	x_4	x_5	b	θ
4	x_3	$\left(\frac{2}{3}\right)$	1	1	$\frac{1}{3}$	0	$\frac{10}{3}$	5←
1	x_5	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	1	$\frac{14}{3}$	14
$E_j=$	$\Sigma e_i a_{ij}$	3	5	4	1	1		
c_j-E_j		3 ↑	-2	0	-3	0	*Second feasible solution*	

Table 6.21

e_i	C.S.V.	x_1	x_2	x_3	x_4	x_5	b	
4	x_3	(1)	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	0	5	
1	x_5	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	1	$\frac{14}{3}$	*Key element unity*

Table 6.22

	c_j	6	3	4	-2	1		
e_i	C.S.V.	x_1	x_2	x_3	x_4	x_5	b	
6	x_1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	0	5	
1	x_5	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	
$E_j=\Sigma e_i a_{ij}$		6	$\frac{19}{2}$	$\frac{17}{2}$	$\frac{5}{2}$	1		
c_j-E_j		0	$-\frac{13}{2}$	$-\frac{9}{2}$	$-\frac{9}{2}$	0	*Optimal solution*	

$\therefore$ Optimal solution is $x_1=5$, $x_2=x_3=x_4=0$, $x_5=3$,

and $$z_{max}=6\times5+1\times3=33.$$

The revised simplex method works on the principle that any tableau corresponding to a basic feasible solution can be obtained directly from the original equations by matrix-vector operations. Let the column vectors $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ and $\mathbf{P}_5$ denote the original columns of x_1, x_2, x_3, x_4 and x_5 and let the column vector $\mathbf{b}$ represent the right-hand-side constants. Thus

$$\mathbf{P}_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \mathbf{P}_2=\begin{bmatrix} 3 \\ 2 \end{bmatrix},\ \mathbf{P}_3=\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ \mathbf{P}_4=\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{P}_5=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{b}=\begin{bmatrix} 10 \\ 8 \end{bmatrix}.$$

Table 6.20 in which x_3 and x_5 are the basic variables may be generated directly by matrix theory as follows:

Define a *basis matrix* **B** whose elements are the original columns of the basic variables x_3 and x_5. Thus

$$\mathbf{B}=[\mathbf{P}_3,\ \mathbf{P}_5]=\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}.$$

The inverse of the basis matrix, denoted by $\mathbf{B}^{-1}$ is obtained as follows :

Since $|\mathbf{B}|=3-0=3(\neq 0)$, **B** is nonsingular and hence $\mathbf{B}^{-1}$ exists.

$$(\mathbf{BI})=\left[\begin{array}{cc:cc} 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right].$$

Divide the first row by 3 : $\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{3} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

Subtract first row from the second : $\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 1 \end{array}\right]$

Then $\mathbf{B}^{-1}=\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}.$

Then according to matrix theory any column in table 6.20 can be obtained by premultiplying the corresponding original column in table 6.18 by the inverse of the basis matrix, $\mathbf{B}^{-1}$. For example,

$$\bar{\mathbf{P}}_1=\mathbf{B}^{-1}\mathbf{P}_1=\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix},$$

$$\bar{\mathbf{P}}_2=\mathbf{B}^{-1}\mathbf{P}_2=\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ etc.}$$

and $$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ \frac{14}{3} \end{bmatrix}.$$

where $\bar{\mathbf{P}}_1$, $\bar{\mathbf{P}}_2$, etc. respresent the column vectors and $\bar{\mathbf{b}}$ represents the right-hand-side constants of table 6.20.

We know that there are two key steps in the simplex method, namely, the determination of the nonbasic variable that enters the basis and the basic variable that leaves the basis. These two steps are carried out in the revised simplex method as shown below.

In the regular simplex method, the $(\bar{c}_j - \bar{E}_j)$ row for table 6.20 is calculated as follows :

$$(\bar{c}_j - \bar{E}_j) = c_j - \Sigma c_i a_{ij} = c_j - c_i \bar{\mathbf{P}}_j$$

Also $\bar{\mathbf{P}}_j = \mathbf{B}^{-1}\mathbf{P}_j$.

$$\therefore \quad (\bar{c}_j - \bar{E}_j) = c_j - c_i \mathbf{B}^{-1}\mathbf{P}_j$$

Let the vector Π denote $c_i\mathbf{B}^{-1}$. The elements of vector Π are called the **simplex multipliers**.

$$\therefore \quad (\bar{c}_j - \bar{E}_j) = c_j - \Pi\mathbf{P}_j, \text{ for all } j.$$

For example in table 6·20,

$$\Pi = (\pi_1, \pi_2) = c_i\mathbf{B}^{-1} = (4, 1)\begin{bmatrix} 1/3 & 0 \\ -1/3 & 0 \end{bmatrix} = (1, 1).$$

$$\therefore \quad (\bar{c}_1 - \bar{E}_1) = c_1 - \Pi\mathbf{P}_1 = 6 - (1, 1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 6 - (2+1) = 3,$$

$$(\bar{c}_2 - \bar{E}_2) = c_2 - \Pi\mathbf{P}_2 = 3 - (1, 1)\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
$$= 3 - (3+2) = -2,$$

$$(\bar{c}_4 - \bar{E}_4) = c_4 - \Pi\mathbf{P}_4 = -2 - (1, 1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2 - 1 = -3.$$

The incoming non-basic variable is the one corresponding to the maximum positive value of $(\bar{c}_j - \bar{E}_j)$. As $\bar{c}_1 - \bar{E}_1$ has the maximum positive value (3), x_1 is the non-basic variable that enters the basis of table 6·20.

Next we are to find the basic variable that is to leave the basis by the minimum positive ratio rule. For this we have to determine the elements in 'X_1' column and the right-hand-side constants for table 6·20. As already shown,

$$\overline{\mathbf{P}_1}=\mathbf{B}^{-1}\mathbf{P}_1=\begin{bmatrix}1/3 & 0\\ -1/3 & 1\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}2/3\\ 1/3\end{bmatrix},$$

and $$\overline{\mathbf{b}}=\mathbf{B}^{-1}\mathbf{b}=\begin{bmatrix}1/3 & 0\\ -1/3 & 1\end{bmatrix}\begin{bmatrix}10\\ 8\end{bmatrix}=\begin{bmatrix}10/3\\ 14/3\end{bmatrix}.$$

The first row gives minimum positive ratio and hence the basic variable x_3 will be replaced by x_1. Hence the new set of basic variables are x_1 and x_5.

The next new basis matrix $\mathbf{B}=(\mathbf{P}_1, \mathbf{P}_5)=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$.

$\mathbf{B}^{-1}$ can be computed as follows :

$\because$ $|\mathbf{B}|=2-0=2$ $(\neq 0)$. $\mathbf{B}$ is non-singular and hence $\mathbf{B}^{-1}$ exists.

$$(\mathbf{BI})=\left[\begin{array}{cc:cc}2 & 0 & 1 & 0\\ 1 & 1 & 0 & 1\end{array}\right].$$

Divide the first row by 2 : $\left[\begin{array}{cc:cc}1 & 0 & 1/2 & 0\\ 0 & 1 & 0 & 1\end{array}\right]$

Subtract first row from the second : $\left[\begin{array}{cc:cc}1 & 0 & 1/2 & 0\\ 0 & 1 & -1/2 & 1\end{array}\right]$

$$\therefore \quad \mathbf{B}^{-1}=\begin{bmatrix}1/2 & 0\\ -1/2 & 1\end{bmatrix}.$$

The new right-hand-side constants are

$$\overline{\mathbf{b}}=\mathbf{B}^{-1}\mathbf{b}=\begin{bmatrix}1/2 & 0\\ -1/2 & 1\end{bmatrix}\begin{bmatrix}10\\ 8\end{bmatrix}=\begin{bmatrix}5\\ 3\end{bmatrix}$$

Thus the new basic feasible solution is

$$x_1=5,\ x_2=x_3=x_4=0,\ x_5=3 \text{ and } Z_{max}=33.$$

To check the optimality of this solution. we need $(\bar{c}_j-\mathrm{E}_j)$ coefficients for x_2, x_3 and x_4.

$$(\bar{c}_j-\overline{\mathrm{E}_j})=c_j-\pi\mathbf{P}_j,$$

where $\Pi=(\pi_1,\ \pi_2)=e_i\ \mathbf{B}^{-1}=(6,\ 1)\begin{bmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 1\end{bmatrix}=\left(\frac{5}{2}, 1\right).$

$$\therefore \quad (\bar{c}_2-\overline{\mathrm{E}_2})=c_2-\Pi\mathbf{P}_2=3-\left(\frac{5}{2}, 1\right)\begin{bmatrix}3\\ 2\end{bmatrix}$$

$$=3-\frac{19}{2}=-\frac{13}{2},$$

$$(\bar{c}_3-\overline{\mathrm{E}_3})=c_3-\Pi\mathbf{P}_3=4-\left(\frac{5}{2}, 1\right)\begin{bmatrix}3\\ 1\end{bmatrix}$$

$$=4-\frac{17}{2}=-\frac{9}{2},$$

$$(\bar{c}_4-\overline{E_4})=c_4-\Pi\mathbf{P}_4=-2-\left(\frac{5}{2},1\right)\begin{bmatrix}1\\0\end{bmatrix}$$

$$=-2-\frac{5}{2}=-\frac{9}{2},$$

As all $(\bar{c}_j-\overline{E_j})$ coefficients are negative, the current solution is optimal.

From the above discussion it may be concluded that any information contained in a simplex tableau can be obtained directly from the original equations if the inverse of the basis matrix of that tableau is known. This, in turn, can be obtained from the original equations if the current basic variables in that table are known. Thus the revised simplex method can generate any information that is available in the regular simplex method. However, it generates only the relevant information that is required to perform the simplex steps.

Actually, while solving a problem by revised simplex method, the inverse of the basis is not obtained by inverting the matrix of basic columns because inverting a matrix is costly and needs more time on a digital computer. The inverse of the basis matrix, at each step, is obtained by a simple pivot operation on the previous inverse. To illustrate this we refer back to the tables 6·18 to 6·22 already computed.

In the initial basic feasible solution of table 6·18, x_4 and x_5 are the basic variables. The initial basic matrix is

$$[\mathbf{P}_4, \mathbf{P}_5]=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\mathbf{I} \text{ (identity matrix)}.$$

The new column coefficients corresponding to x_4 and x_5 in any subsequent table are obtained by premultiplying $\mathbf{P}_4$ and $\mathbf{P}_5$ by the inverse of current basis matrix. Thus

$$\bar{\mathbf{P}}_4=\mathbf{B}^{-1}\mathbf{P}_4 \text{ and } \bar{\mathbf{P}}_5=\mathbf{B}^{-1}\mathbf{P}_5.$$

$$\therefore \quad [\bar{\mathbf{P}}_4, \bar{\mathbf{P}}_5]=\mathbf{B}^{-1}[\mathbf{P}_4, \mathbf{P}_5]=\mathbf{B}^{-1}\mathbf{I}=\mathbf{B}^{-1}.$$

Thus the 'x_4' and 'x_5'-columns in any table give the inverse of the basis for that table. For instance, in table 6·20, the 'x_4' and 'x_5'-columns are given by $\begin{bmatrix}\frac{1}{3} & 0\\-\frac{1}{3} & 1\end{bmatrix}$ which was the computed inverse of the basis for the table. The same is true for table 6·22 as well. Thus the new basis inverse can be easily obtained by considering the columns of initial basic variables and updating them by pivot operation. Similarly, the right-hand-side constants of

any table can be obtained by updating their values by the pivot operation at each iteration. Thus the revised simplex method makes use of a reduced simplex table which contains the columns of the initial basic variables, the right-hand-side constants along with current basic variables. This reduced simplex table corresponding to table 6·23 is shown below.

Table 6·23

Basis	$\mathbf{B}^{-1}$		$\bar{\mathbf{b}}$
x_3	$\frac{1}{3}$	0	$\frac{10}{3}$
x_5	$-\frac{1}{3}$	1	$\frac{14}{3}$

From $\mathbf{B}^{-1}$, the simplex multipliers and the $(\bar{c}_j - \bar{E}_j)$ coefficients are calculated. This results in selecting x_1 as the entering variable. The 'x_1' column (pivot column) for table 6·20 is then computed as follows :

$$\bar{\mathbf{P}}_1 = \mathbf{B}^{-1}\mathbf{P}_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Knowing $\bar{\mathbf{P}}_1$ and $\bar{\mathbf{b}}_1$, the minimum positive ratio test can be performed, which identifies x_3 as the outgoing variable. This means that the pivot column $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ should be reduced to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This can be done by

(1) Multiplying the first row by $\frac{3}{2}$.

(2) Multiplying this modified row by $\frac{1}{3}$ and subtracting it from the second row.

The pivot operation, when carried on the reduced table 6·23 yields the following table :

Table 6·24

Basis	$\mathbf{B}^{-1}$		$\bar{\mathbf{b}}$
x_1	$\frac{1}{2}$	0	5
x_5	$-\frac{1}{2}$	1	3

The new basic feasible solution is $x_1=5$, $x_5=3$, $x_2=x_3=x_4=0$. The new inverse of the basis is $\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$. Using the new basis inverse, the simplex multipliers and $(\bar{c}_j\text{-}\bar{E}_j)$ coefficients can be calculated to check the optimality of the solution provided by table 6·24.

EXAMPLE 6.3-2

$$\begin{aligned} \text{Minimize } Z &= -4x_1+x_2+2x_3, \\ \text{subject to } \quad & 2x_1-3x_2+2x_3 \leqslant 12, \\ & -5x_1+2x_2+3x_3 \geqslant 4, \\ & 3x_1 \qquad -2x_3 = -1, \\ & x_1,\, x_2,\, x_3 \geqslant 0. \end{aligned}$$

Solution :

In standard form the problem reduces to

$$\begin{aligned} \text{minimize } Z &= -4x_1+x_2+2x_3, \\ \text{subject to } \quad & 2x_1-3x_2+2x_3+x_4=12, \\ & -5x_1+2x_2+3x_3-x_5=4, \\ & -3x_1 \qquad +2x_3 \qquad =1, \\ & x_1,\ldots,\, x_5 \geqslant 0. \end{aligned}$$

Since the second and third equations do not contain basic variables, artificial variables x_6 and x_7 are added and the problem reduces to

$$\begin{aligned} \text{minimize } Z &= -4x_1+x_2+2x_3+0x_4+0x_5+Mx_6+Mx_7, \\ \text{subject to } \quad & 2x_1-3x_2+2x_3+x_4=12, \\ & -5x_1+2x_2+3x_3 \qquad -x_5+x_6=4, \\ & -3x_1 \qquad +2x_3 \qquad\qquad x_7=1, \\ & x_1,\,\ldots,\, x_7 \geqslant 0. \end{aligned}$$

Let $\mathbf{P}_1,\ldots,\mathbf{P}_7$ and $\mathbf{b}$ denote the column vectors corresponding to $x_1,\ldots,x_7$ and the right-hand-side constants respectively. Thus

$$\mathbf{P}_1=\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \quad \mathbf{P}_2=\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{P}_3=\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix},$$

$$\mathbf{P}_4=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}_5=\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{P}_6=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{P}_7=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{b}=\begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix}.$$

Now (x_4, x_6, x_7) forms the initial basis.

$$\therefore \underset{(3\times3)}{\mathbf{B}}=[\mathbf{P}_4, \mathbf{P}_6, \mathbf{P}_7]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\mathbf{I},$$

$$\mathbf{B}^{-1}=\mathbf{I} \quad \text{and} \quad \bar{\mathbf{b}}=\mathbf{B}^{-1}\mathbf{b}=\mathbf{b}.$$

The initial table of the revised simplex method is given below. The last two columns of this table are added later.

Table 6·25

Basis	$\mathbf{B}^{-1}$			*Constants*	*Variable to enter*	*Pivot columns*
x_4	1	0	0	12		2
x_6	0	1	0	4	x_3	3
x_7	0	0	1	1		(2)

The simplex multipliers are

$$\pi=(\pi_1, \pi_2, \pi_3)=c_i\mathbf{B}^{-1}=(0, \text{M}, \text{M})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=(0, \text{M}, \text{M}).$$

$$\therefore \quad (\bar{c}_1-\bar{E}_1)=c_1-\pi\mathbf{P}_1=-4-(0, \text{M}, \text{M})\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}=8\text{M}-4,$$

$$(\bar{c}_2-\bar{E}_2)=c_2-\pi\mathbf{P}_2=1-(0, \text{M}, \text{M})\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}=1-2\text{M},$$

$$(\bar{c}_3-\bar{E}_3)=c_3-\pi\mathbf{P}_3=2-(0, \text{M}, \text{M})\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}=2-5\text{M},$$

$$(\bar{c}_5-\bar{E}_5)=c_5-\pi\mathbf{P}_5=0-(0, \text{M}, \text{M})\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}=\text{M}.$$

Since $(\bar{c}_3-\bar{E}_3)$ is most negative, x_3 is the variable that enters the basis. The pivot column is

$$\bar{\mathbf{P}}_3=\mathbf{B}^{-1}\mathbf{P}_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

The entering variable x_3 and the elements in the pivot column are now entered in table 6.25. Applying the minimum positive ratio rule, the ratios are $(6, \frac{4}{3}, \frac{1}{2})$. Hence x_7 is the outgoing variable. This is shown by putting the corresponding pivot-element in brackets *i.e.*, (2). Now the pivot column $\begin{bmatrix} 2 \\ 3 \\ (2) \end{bmatrix}$ must be reduced to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

This is obtained by

(*i*) subtracting row 3 from row 1,

(*ii*) multiplying row 3 by $\frac{3}{2}$ and subtracting from row 2,

(*iii*) dividing row 3 by 2.

The new $\mathbf{B}^{-1}$ and constants are given in table 6.26.

Table 6.26

Basis	$\mathbf{B}^{-1}$			*Constants*	*Variable to enter*	*Pivot column*
x_4	1	0	-1	11		-3
x_6	0	1	$-\frac{3}{2}$	$\frac{5}{2}$	x_2	(2)
x_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$		0

The simplex multipliers corresponding to table 6.26 are

$$\pi=(\pi_1, \pi_2, \pi_3)=c_i\mathbf{B}^{-1}=(0, M, 2)\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$=\left(0, M, 1-\frac{3M}{2}\right).$$

$$\therefore \quad (\bar{c}_1-\bar{E}_1)=c_1-\pi\mathbf{P}_1= -4-\left(0, M, 1-\frac{3M}{2}\right).\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$$

$$=-4-\left\{-5M-3+\frac{M}{2}\right\}=-1+\frac{9}{2}M,$$

$$(\bar{c}_2-\bar{E}_2)=c_2-\pi P_2=1-(0,\ M,\ 1-\frac{3}{2}\ M)\begin{bmatrix}-3\\2\\0\end{bmatrix}=1-\{2M\}$$

$$=1-2M,$$

$$(\bar{c}_5-\bar{E}_5)=c_5-\pi P_5=0-(0,\ M,\ 1-\frac{3}{2}\ M)\begin{bmatrix}0\\-1\\0\end{bmatrix}=0-\{-M\}$$

$$=M.$$

$(\bar{c}_7-\bar{E}_7)$ is not calculated since x_7 is an artificial variable. Since $(\bar{c}_2-\bar{E}_2)$ is the most negative, x_2 enters the basis and the pivot column becomes

$$\bar{P}_2=B^{-1}P_2=\begin{bmatrix}1&0&-1\\0&1&-\frac{3}{2}\\0&0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}-3\\2\\0\end{bmatrix}=\begin{bmatrix}-3\\2\\0\end{bmatrix}.$$

The ratios are $\left(-\frac{11}{3},\ 5,\ \infty\right)$. Hence x_6 is the variable that leaves the basis (x_6 will be discarded from further consideration). This is shown by putting the corresponding pivot element in brackets *i.e.*, (2). The pivot column $\begin{bmatrix}-3\\(2)\\0\end{bmatrix}$ is now reduced to $\begin{bmatrix}0\\1\\0\end{bmatrix}$. This is done by

(*i*) multiplying row 2 by $\frac{3}{2}$ and adding it to row 1,

(*ii*) dividing row 2 by 2.

The new B^{-1} and constants are given in table 6.27.

Table 6.27

Basis	B^{-1}			*Constants*	*Variable to enter*	*Pivot column*
x_4	1	$\frac{3}{2}$	$-\frac{13}{4}$	$\frac{59}{4}$		$\left(\frac{17}{4}\right)$
x_2	0	$\frac{1}{2}$	$-\frac{3}{4}$	$\frac{5}{4}$	x_1	$-\frac{1}{4}$
x_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$		$-\frac{3}{2}$

The simplex multipliers of table 6.27 are

$$\pi=(\pi_1, \pi_2, \pi_3)=c_i\mathbf{B}^{-1}=(0, 1, 2)\begin{bmatrix} 1 & \frac{3}{2} & -\frac{13}{4} \\ 0 & \frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$=\left(0, \frac{1}{2}, \frac{1}{4}\right).$$

$$\therefore \quad (\bar{c}_1-\bar{\mathbf{E}}_1)=c_1-\pi\mathbf{P}_1=-4-\left(0, \frac{1}{2}, \frac{1}{4}\right)\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$$

$$=-4-\left\{-\frac{5}{2}-\frac{3}{4}\right\}=-\frac{3}{4},$$

$$(\bar{c}_5-\bar{\mathbf{E}}_5)=c_5-\pi\mathbf{P}_5=0-\left(0, \frac{1}{2}, \frac{1}{4}\right)\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}=\frac{1}{2}.$$

Since $(\bar{c}_1-\bar{\mathbf{E}}_1)$ is the most negative, x_1 enters the basis and the pivot column becomes

$$\bar{\mathbf{P}}_1=\mathbf{B}^{-1}\mathbf{P}_1=\begin{bmatrix} 1 & \frac{3}{2} & -\frac{13}{4} \\ 0 & \frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}=\begin{bmatrix} \frac{17}{4} \\ -\frac{1}{4} \\ -\frac{3}{2} \end{bmatrix}$$

The ratios are $\left(\frac{59}{17}, -5, -3\right)$. Hence x_4 is the variable that leaves the basis.

The pivot column $\begin{bmatrix} \left(\frac{17}{4}\right) \\ -\frac{1}{4} \\ -\frac{3}{2} \end{bmatrix}$ must be reduced to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

This is done by

(*i*) multiplying row 1 by $\frac{1}{17}$ and adding it to row 2,

(*ii*) multiplying row 1 by $\frac{6}{17}$ and adding it to row 3,

(*iii*) dividing row 1 by $\frac{17}{4}$.

The new $\mathbf{B}^{-1}$ and constants are given in table 6.28.

Table 6.28

Basis	B^{-1}			Constants
x_1	$\frac{4}{17}$	$\frac{6}{17}$	$\frac{-13}{17}$	$\frac{59}{17}$
x_2	$\frac{1}{17}$	$\frac{10}{17}$	$\frac{-16}{17}$	$\frac{36}{17}$
x_3	$\frac{6}{17}$	$\frac{9}{17}$	$\frac{-11}{17}$	$\frac{97}{17}$

The simplex multipliers of table 6.28 are

$$\pi=(\pi_1, \pi_2, \pi_3)=c_i\,\mathbf{B}^{-1}=(-4, 1, 2). \begin{bmatrix} \frac{4}{17} & \frac{6}{17} & \frac{-13}{17} \\ \frac{1}{17} & \frac{10}{17} & \frac{-16}{17} \\ \frac{6}{17} & \frac{9}{17} & \frac{-11}{17} \end{bmatrix}$$

$$=\left(\frac{-3}{17}, \frac{4}{17}, \frac{14}{17}\right).$$

$$\therefore\ (\bar{c}_4-\bar{\mathbf{E}}_4)=c_4-\pi\mathbf{P}_4=0-\left(\frac{-3}{17}, \frac{4}{17}, \frac{14}{17}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}=\frac{3}{17},$$

$$(\bar{c}_5-\bar{\mathbf{E}}_5)=c_5-\pi\,\mathbf{P}_5=0-\left(\frac{-3}{17}, \frac{4}{17}, \frac{14}{17}\right)\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}=\frac{4}{17}.$$

Since $(\bar{c}_4-\bar{\mathbf{E}}_4)$ as well as $(c_5-\bar{\mathbf{E}}_5)$ are non-negative, talbe 6.28 gives optimal solution, which is $x_1=59/17$, $x_2=36/17$, $x_3=97/17$,

$x_4=0$, $x_5=0$ and optimal value of objective function is

$$Z_{min}=(-4, 1, 2)\begin{bmatrix} \frac{59}{17} \\ \frac{36}{17} \\ \frac{97}{17} \end{bmatrix}$$

$$=\left\{-\frac{236}{17}+\frac{36}{17}+\frac{194}{17}\right\}=-\frac{6}{17}.$$

6.3-1. Advantages of the Revised Simplex Method over the Standard Simplex Method

1. For L.P. problems involving much larger number of variables than the number of constraints, the amount of computations is very much reduced because the revised simplex method works with a reduced tableau whose size is determined by the number of constraints.

2. The revised simplex method works with a reduced tableau as it stores only the basic variables, the basis inverse and the constants. Hence less new information needs to be stored in the memory of the computer from one iteration to the other.

3. Data can be recorded in lesser space. The original data is usually given in fixed decimals of three or four digits. The data can be stored more accurately and compactly since the revised simplex method works only with the original data.

4. There is less accumulation of round-off errors since no calculations are done on a column unless it is ready to enter the basis.

5. The theory of the revised simplex method, especially the importance of the basis inverse and the simplex multipliers is quite helpful in understanding the advanced topics in linear programming such as duality theory and sensitivity analysis.

6.4 Bounded Variables

A linear programming problem may have, in addition to the regular constraints, some or all variables with lower and upper limits. Such a problem, in general, may be expressed as

$$\text{maximize } Z=\mathbf{CX},$$
$$\text{subject to } \mathbf{AX}=\mathbf{b},$$
$$\mathbf{L}\leqslant\mathbf{X}\leqslant\mathbf{U},$$

where vector $\mathbf{X}=(x_1, x_2, \ldots, x_n)^T$,

vector $\mathbf{C}=(c_1, c_2, \ldots, c_n)$,

vector $\mathbf{b}=(b_1, b_2, \ldots, b_m)^T$,

$$\text{matrix } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and the superscript T is used to indicate the transpose. Also,

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}, \ \mathbf{U} \geqslant \mathbf{L} \geqslant 0.$$

Here $\mathbf{A}$ is $m \times n$ real matrix and $\mathbf{U}$ and $\mathbf{L}$ are the upper and lower bounds for $\mathbf{X}$ respectively. These limits are ∞ and 0 respectively if a variable is unbounded.

The inequality constraints $\mathbf{L} \leqslant \mathbf{X} \leqslant \mathbf{U}$ can be converted into equality constraints by introducing slack and/or surplus variables $\mathbf{X}'$ and $\mathbf{X}''$ as shown below.

$$\mathbf{X} + \mathbf{X}' = \mathbf{U},$$
$$\mathbf{X} - \mathbf{X}'' = \mathbf{L},$$
$$\mathbf{X}, \mathbf{X}', \mathbf{X}'' \geqslant 0.$$

The given problem, thus, contains $3n$ variables and $m+2n$ constraint equations. However, this problem can be reduced in size by using a special technique which ultimately reduces the constraints to the form

$$\mathbf{AX} = \mathbf{b}.$$

First, considering the lower bound constraints, the equality constraint $\mathbf{X} - \mathbf{X}'' = \mathbf{L}$ could be written as $\mathbf{X} = \mathbf{L} + \mathbf{X}''$ and thus $\mathbf{X}$ can be eliminated from all the remaining constraints. The problem will now have $\mathbf{X}'$ and $\mathbf{X}''$ as variables and since both are non-negative, $\mathbf{X}$ will always be non-negative.

The real problem arises when dealing with upper bound variable, because the substitution $\mathbf{X} = \mathbf{U} - \mathbf{X}'$ does not guarantee that $\mathbf{X}$ will remain non-negative. This difficulty is overcome by using a special technique called *bounded variable simplex method*.

The upper bound problem can be written as

$$\text{maximize } \quad Z = \mathbf{CX},$$
$$\text{subject to } \mathbf{AX} = \mathbf{b},$$
$$\mathbf{X} + \mathbf{X}' = \mathbf{U},$$
$$\mathbf{X}, \mathbf{X}' \geqslant 0.$$

However, instead of considering the constraints $\mathbf{X}+\mathbf{X}'=\mathbf{U}$ in the simplex table, their effect can be accounted for by modifying the feasibility condition of the simplex method, the optimality condition remaining the same. The upper bound condition requires a basic variable to become non-basic at its upper bound (*cf.*, regular simplex method in which all non-basic variables are zero). Moreover, when a non-basic variable enters the solution, its value should not exceed its upper bound. Thus while developing the new feasibility condition, two things are to be taken care of :

(1) The non-negativity and upper bound constraints of the entering variable.

(2) The non-negativity and upper bound constraints of all those basic variables that are affected by the entering variable.

The procedure for obtaining an optimum basic feasible solution for a linear programming problem by bounded variable method consists of the following steps :

Step 1

If R.H.S. of any constraint is negative, make it positive by multiplying the constraint by -1.

Step 2

Convert the inequations of the constraints into equations by the addition of suitable slacks and/or surplus variables and obtain an initial basic feasible solution.

Step 3

If any variable is at a positive lower bound, it should be substituted at its lower bound.

Step 4

Calculate the net evaluation $c_j-\mathrm{E}_j$. For a maximization problem, if $c_j-\mathrm{E}_j \leqslant 0$ for the non-basic variables at their upper bound, optimum basic feasible solution is attained. If not, go to step 5. Reverse is true for a minimization problem.

Step 5

Select the most positive of $c_j-\mathrm{E}_j$.

Step 6

Let x_j be a nonbasic variable at zero level which is selected to enter the solution. Let $(\mathbf{X}_\mathrm{B})_i=(\mathbf{X}^*{}_\mathrm{B})_i$ be the ith variable of the current basic solution $\mathbf{X}_\mathrm{B}$. Then

$$(\mathbf{X}_\mathrm{B})_i=(\overset{*}{\mathbf{X}}_\mathrm{B})_i-a_{ij},$$

where a_{ij} is the ith element of $a_j=\mathbf{B}^{-1}\,\mathbf{b}_j$, where $\mathbf{b}_j$ is the vector of $\mathbf{A}$ corresponding to x_j.

Compute the quantities

$$\theta_1 = \min_i \left\{ \frac{(\overset{*}{\mathbf{X}_B})_i}{a_{ij}},\ a_{ij} > 0 \right\},$$

$$\theta_2 = \min_i \left\{ \frac{u_i - (X^*_B)_i}{-a_{ij}},\ a_{ij} < 0 \right\},$$

and $$\theta = \text{Min.}\ (\theta_1, \theta_2, u_j),$$

where u_j is the upper bound for the variable x_j . Let $(X_B)_r$ be the variable corresponding to $\theta = \min. (\theta_1, \theta_2, u_j)$. Then

(*i*) if $\theta = \theta_1$, $(\mathbf{X}_B)_r$ leaves the solution and x_j enters by using the regular row operations of the simplex method.

(*ii*) if $\theta = \theta_2$, $(\mathbf{X}_B)_r$ leaves the solution and x enters ; then $(X_B)_r$ being nonbasic at its upper bound must be substituted out by using

$$(\mathbf{X}_B)_r = u_r - (\mathbf{X}_B)'_r\ ,\ \text{where } 0 \leqslant (\mathbf{X}_B)_r' \leqslant u_r.$$

(*iii*) if $\theta = u_j$, x_j is substituted at its upper bound difference $u_j - x_j'$, while remaining nonbasic.

EXAMPLF 6·4·1

Using bounded variable simplex method, solve the L.P.P.,

$$\text{maximize } Z = 3x_1 + x_2 + x_3 + 7x_4,$$

subject to the constraints $2x_1 + 3x_2 - x_3 + 4x_4 \leqslant 40,$

$$-2x_1 + 2x_2 + 5x_3 - x_4 \leqslant 35,$$

$$x_1 + x_2 - 2x_3 + 3x_4 \leqslant 100,$$

$$x_1 \geqslant 2,\ x_2 \geqslant 1,\ x_3 \geqslant 3,\ x_4 \geqslant 4.$$

[*Meerut B. Sc. (Math.) 1972*]

Solution :

Since x_1, x_2, x_3 and x_4 have positive lower bounds, they must be substituted at their lower bounds.

Let $x_1 = 2 + y_1,$
$x_2 = 1 + y_2,$
$x_3 = 3 + y_3,$
$x_4 = 4 + y_4.$

The given problem becomes

maximize $Z = 3(2 + y_1) + (1 + y_2) + (3 + y_3) + 7(4 + x_4)$

$$= 38 + 3y_1 + y_2 + y_3 + 7y_4,$$

i.e., maximize $Y = (Z - 38) = 3y_1 + y_2 + y_3 + 7y_3,$

subject to $2(2 + y_1) + 3(1 + y_2) - (3 + y_3) + 4(4 + y_4) \leqslant 40$ i e., $2y_1 + 3y_2 - y_3 + 4y_4 \leqslant 20,$

$$-2(2 + y_1) + 2(1 + y_2) + 5(3 + y_3) - (4 + y_4) \leqslant 35$$

i.e., $-2y_1 + 2y_2 + 5y_3 - y_4 \leqslant 26,$

$(2 + y_1) + (1 + y_2) - 2(3 + y_3) + 3(4 + y_4)$
$\leqslant 100$ i.e., $y_1 + y_2 - 2y_3 + 3y_4 \leqslant 91,$

$$y_1, y_2, y_3, y_4 \geqslant 0.$$

This problem can now be solved by regular simplex method. The solution is left as an exercise for the students. The optimal solution is

$$y_1=63/4,\ y_2=0,\ y_3=23/2 \text{ and } y_4=0.$$

$$\therefore \quad x_1=63/4+2=71/4,$$
$$x_2=0+1=1,$$
$$x_3=23/2+3=29/2,$$
$$x_4=0+4=4.$$

$$Z_{max}=3\times\frac{71}{4}+1\times1+\frac{1\times29}{2}+4\times7=387/4.$$

EXAMPLE 6·4·2

Solve the following L.P.P. by the bounded variable simplex method :

Maximize $Z=3x_1+5x_2+3x_3$,
subject to $x_1+2x_2+2x_3\leqslant14$,
$2x_1+4x_2+3x_3\leqslant23$,
$0\leqslant x_1\leqslant4,\ 0\leqslant x_2\leqslant5$ and $0\leqslant x_3\leqslant3$.

Solution.

The given inequations can be converted into equations by introducing slack variables s_1 and s_2 and the problem becomes

maximize $Z=3x_1+5x_2+3x_3+0s_1+0s_2$,
subject to $x_1+2x_2+2x_3+s_1=14$,
$2x_1+4x_2+3x_3+s_2=23$,
$0\leqslant x_1\leqslant4,\ 0\leqslant x_2\leqslant5,\ 0\leqslant x_3\leqslant3,\ s_1\geqslant0,\ s_2\geqslant0.$

The basic variables are $s_1=14$ and $s_2=23$. Since no upper bounds are specified for these basic variables, we arbitrarily assume their upper bounds to be ∞ each. The above information can be put in the form of a table shown below.

Table 6·29

c_j		3	5	3	0	0			
e_i	variables in current solution	x_1	x_2	x_3	s_1	s_2	b	θ'	$u_i-(X^*_B)_i$
0	s_1	1	2	2	1	0	14	7	∞—14
0	s_2	2	4	3	0	1	23	23/4	∞—23
$E_j=\Sigma e_i a_{ij}$		0	0	0	0	0			
c_j-E_j		3	5	3	0	0			
			↑						
u_j		4	5	3	∞	∞			

First iteration. x_2 is the incoming variable.

Now $\theta_1=$ min. $(7,\ 23/4)=23/4$,

$\theta_2=\infty$, since all elements in 'x'_3-column are non-negative and $u_2=5$.

$\therefore \quad \theta=\min.\ (\theta_1,\ \theta_2,\ u_2)$
$=\min.\ (32/4,\ \infty,\ 5)$
$=5.$

Since $\theta=u_2$ x_2 is substituted at its upper bound difference *i.e.*, $x_2=5-x_2'$ but it remains non-basic. The new table becomes

Table 6·30

c_j		3	-5	3	0	0			
e_i	C.S.V.	x_1	x_2'	x_3	s_1	s_2	b	θ'	$u_i-(X^*_B)_i$
o	s_1	1	-2	2	1	0	4	4	$\infty-4$
o	s_2	(2)	-4	3	0	1	3	3/2 $\leftarrow$	$\infty-3$
$E_j=\Sigma e_i a_{ij}$		0	0	0	0	0			
c_j-E_j		3 $\uparrow$	-5	3	0	0			
	u_j	4	5	3	∞	∞			

Second iteration. Now let x_1 be the incoming variable.

Then $\theta_1=\min.\ (4,\ 3/2)=3/2$,

$\theta_2=\infty$, since all elements in 'x_1'-column are non-negative, and $u_1=4$.

$\therefore \quad \theta=\min.\ (3/2,\ \infty,\ 4)=3/2.$

since $\theta=\theta_1$, introduce x_1 and drop s_2. This yields

Table 6·31

c_j		3	-5	3	0	0		
e_i	C.S.V.	x_1	x_2'	x_3	s_1	s_2	b	$u_i-(X_B^*)_i$
o	s_1	0	0	1/2	1	$-1/2$	3/2	$\infty-5/2$
3	x_1	1	(-2)	3/2	0	1/2	3/2	$4-3/2=5/2\leftarrow$
$E_j=\Sigma e_i a_{ij}$		3	-6	9/2	0	3/2		
c_j-E_j		0	1 $\uparrow$	$-3/2$	0	$-3/2$		
u_j		4	5	3	∞	∞		

Third iteration. x'_2 is the entering variable.

$\theta_1=\infty,$

$$\theta_2=\frac{4-3/2}{-(-2)}=5/4, \text{ corresponding to } x_1,$$

$u_2=5,$

$\therefore \quad \theta = \min. \ (\infty, \ 5/4, \ 5) = 5/4$. Since $\theta = \theta_2$, introduce x'_2 into the basis and drop x_1 and then substitute it out at its upper bound $4 - x'_1$. Thus by removing x_1 and introducing x'_2, the table becomes

Table 6·32

e_i	C.S.V.	x_1	x'_2	x_3	s_1	s_2	b
0	s_1	0	0	1/2	1	—1/2	5/2
—5	x'_2	—1/2	1	—3/4	0	—1/4	—3/4

Now substituting $x_1 = 4 - x'_1$, the final table becomes

Table 6·33

c_j		—3	—5	3	0	0	
e_i	C.S.V.	x'_1	x'_2	x_3	s_1	s_2	b
0	s_1	0	0	1/2	1	—1/2	5/2
—5	x'_2	1/2	1	—3/4	0	—1/4	5/4
$E_j = \Sigma e_i \, a_{ij}$		—5/2	—5	15/4	0	5/4	
$c_j - E_j$		—1/2	0	—3/4	0	—5/4	*Optimal feasible solution*

$\therefore$ Optimal solution is $x'_1 = 0, \ x'_2 = 5/4, \ x_3 = 0$.

In terms of original variables, the solution is

$$x_1 = 4 - 0 = 4,$$
$$x_2 = 5 - 5/4 = 15/4,$$
$$x_3 = 0, \text{ and } Z_{max} = 3 \times 4 + 5 \times 15/4 + 3 \times 0 = 123/4.$$

6·5 Sensitivity Analysis

Once the optimal solution to a linear programming problem has been attained, it may be desirable to study how the current solution changes when the parameters of the problem are changed. In many practical problems this information is much more important than the single result provided by the optimal solution. Such an analysis converts the static linear programming solution into a dynamic tool to study the effect of changing conditions such as in business and industry.

The change in parameters of the problem may be discrete or continuous. The study of the effect of discrete changes in parameters on the optimal is called *sensitivity analysis* or *postoptimality analysis*, while that of continuous changes in parameters is called *parametric programming*. One way to determine the effects of parameter changes is to solve the problem anew, which may be computationally inefficient. Alternatively, the *current* optimal solution may be investigated, making use of the properties of the simplex criterion. The second method reduces additional computations considerably and hence forms the objective of present discussion.

The changes in the parameters of a linear programming problem include

1. Changes in the right-hand side of the constraints b_i.
2. Changes in the cost coefficients c_j.
3. Addition of new variables.
4. Changes in the coefficients of constraints a_{ij}.
5. Addition of new constraints.
6. Deletion of variables.
7. Deletion of constraints.

Generally, these parameter changes result in one of the following three cases :

1. The optimal solution remains unchanged *i.e.*, the basic variables and their values remain unchanged.
2. The basic variables remain unchanged but their values change.
3. The basic variables as well as their values are changed.

While dealing with these changes, one important objective is to find the maximum extent to which a parameter or a set of parameters can be changed so that the current optimal solution remains optimal. In other words, the objective is to determine how *sensitive* is the optimal solution to the changes in those parameters. Such an analysis is called *sensitivity analysis.*

6·5·1 Changes in the right Hand Side of the Constraints b_i

Suppose that an optimal solution to a linear programming problem has already been found and it is desired to find the effect of increasing or decreasing some resource. Clearly, this will affect not only the objective function but also the solution. Large changes in the limiting resource may even change the variables in the solution.

EXAMPLE 6·5·1

(*a*) Solve the problem

$$\text{maximize } Z=5x_1+12x_2+4x_3,$$
$$\text{subject to} \quad x_1+2x_2+x_3\leqslant 5,$$
$$2x_1-x_2+3x_3=2,$$
$$x_1, x_2, x_3\geqslant 0.$$

(*b*) Discuss the effect of changing the requirement vector from

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ on the optimum solution.

(*c*) Discuss the effect of changing the requirement vector from

$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ on the optimum solution.

(*d*) Which resource should be increased and how much to achieve the best marginal increase in the value of the objective function ?

(*e*) Which resource should be decreased and how much to achieve the best marginal increase in the value of the objective function ?.

Solution.

(*a*) The standard form of this problem is

maximize $Z=5x_1+12x_2+4x_3+os_1-MA_1$,

subject to $x_1+2x_2+x_3+s_1=5$,

$2x_1-x_2+3x_3+A_1=2$,

$x_1, x_2, x_3, s_1, A_1 \geqslant 0$.

Putting $x_1=x_2=x_3=0$ in the constraint equations, we get $s_1=5$ and $A_1=2$ as the initial basic feasible solution which can be expressed in the form of a simple matrix or table.

Table 6·34

Objective function c_j		5	12	4	0	$-M$	
c_B	variables in current solution	x_1	x_2	x_3	s_1	A_1	b
0	s_1	1	2	1	1	0	5
$-M$	A_1	2	-1	3	0	1	2

Initial basic feasible solution

First Iteration : (*i*) Perform optimality test.

Table 6·35

c_j		5	12	4	0	$-M$		
c_B	*c.s.v.*	x_1	x_2	x_3	s_1	A_1	b	θ
0	s_1	1	2	1	1	0	5	5
$-M$	A_1	2	-1	(3)	0	1	2	2/3←*key row*
$E_j=\Sigma c_B a_{ij}$		$-2M$	M	$-3M$	0	$-M$		
$\bar{c}_j=c_j-E_j$		$5+2M$	$12-M$	$4+3M$ ↑ k	0	0		

(*ii*) Make key element unity.

Table 6·36

c_B	*c.s.v.*	x_1	x_2	x_3	s_1	A_1	b
0	s_1	1	2	1	1	0	5
$-M$	A_1	$\frac{2}{3}$	$-\frac{1}{3}$	(1)	0	$\frac{1}{3}$	$\frac{2}{3}$

Key element unity

(*ii*) Replace A_1 by x_3.

Table 6·37

	c_j	5	12	4	0	$-M$			
c_B	*c.s.v.*	x_1	x_2	x_3	s_1	A_1	b	θ	
0	s_1	$\frac{1}{3}$	$\left(\frac{7}{3}\right)$	0	1	$-\frac{1}{3}$	$\frac{13}{3}$	$\frac{13}{7}$	←key row
4	x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{2}{3}$	-2	
$E_j=\Sigma c_B$	a_{ij}	$\frac{8}{3}$	$-\frac{4}{3}$	4	0	$\frac{4}{3}$			
$\bar{c}_j=c_j-E_j$		$\frac{7}{3}$	$\frac{40}{3}$	0	0	$-M-\frac{4}{3}$			
			$\uparrow k$						

Second feasible solution

Second Iteration. (*i*) Make key element unity.

Table 6·38

c_B	*c.s.v.*	x_1	x_2	x_3	s_1	A_1	b
0	s_1	$\frac{1}{7}$	(1)	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$
4	x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{2}{3}$

Key element unity

(*ii*) Replace s_1 by x_2.

Table 6·39

	c_j	5	12	4	0	$-M$			
c_B	*c.s.v.*	x_1	x_2	x_3	s_1	A_1	b	θ	
12	x_2	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$	13	
4	x_3	$\left(\frac{5}{7}\right)$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{9}{7}$	$\frac{9}{5}$	←key row
$E_j=\Sigma c_B$	a_{ij}	$\frac{32}{7}$	12	4	$\frac{40}{7}$	$-\frac{4}{7}$			
$\bar{c}_j=c_j-E_j$		$\frac{3}{7}$	0	0	$-\frac{40}{7}$	$-\frac{M-4}{7}$			
		$\uparrow k$							

Third feasible solution

Third Iteration. (*i*) Make Key Element Unity.

Table 6·40

c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
12	x_2	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{13}{7}$
4	x_3	(1)	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{9}{5}$

Key element unity

(*ii*) Replace x_3 by x_1.

Table 6·41

	c_j	5	12	4	0	$-M$	
c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
12	x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{8}{5}$
5	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{9}{5}$
$E_j=\Sigma c_B a_{ij}$		5	12	$\frac{23}{5}$	$\frac{29}{5}$	$-\frac{2}{5}$	
	$\bar{c}_j=c_j-E_j$	0	0	$-\frac{3}{5}$	$-\frac{29}{5}$	$-\frac{M-2}{5}$	

Optimal feasible solution

Thus the optimal solution is

$$x_1=\frac{9}{5},$$

$$x_2=\frac{8}{5},$$

$$x_3=0,$$

$$Z_{max}=5\times\frac{9}{5}+12\times\frac{8}{5}+0=\frac{141}{5}.$$

(*b*) New values of the current basic variables are given by

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}=\mathbf{B}^{-1}\mathbf{b}=\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{14}{5}-\frac{2}{5} \\ \frac{7}{5}+\frac{4}{5} \end{bmatrix}=\begin{bmatrix} \frac{12}{5} \\ \frac{11}{5} \end{bmatrix}.$$

Since both x_1 and x_2 are non-negative, the current basic solution consisting of x_1 and x_2 remains feasible and optimal at the new values $x_1=\frac{11}{5}$, $x_2=\frac{12}{5}$ and $x_3=0$. The new optimum value of Z is $5\times\frac{11}{5}+12\times\frac{12}{5}+4\times0=\frac{199}{5}$.

(*c*) New values of the current basic variables are

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}=\mathbf{B}^{-1}\mathbf{b}=\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{6}{5}-\frac{9}{5} \\ \frac{3}{5}+\frac{18}{5} \end{bmatrix}=\begin{bmatrix} -\frac{3}{5} \\ \frac{21}{5} \end{bmatrix}.$$

Since x_2 becomes $-$ve, the *current* optimal solution becomes infeasible. As discussed in section 6·2, dual simplex method may be used to clear infeasibility of the problem. Table 6·41 is modified and written as below.

Table 6·42

	c_j	5	12	4	0	$-$M		
c_B	*c.s.v.*	x_1	x_2	x_3	s_1	A_1	b	
12	x_2	0	1	$\left(-\frac{1}{5}\right)$	$\frac{2}{5}$	$-\frac{1}{5}$	$-\frac{3}{5}$	←Key row
5	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{21}{5}$	
$E_j=\Sigma c_B a_{ij}$		5	12	$\frac{23}{5}$	$\frac{29}{5}$	$-\frac{2}{5}$		
$\bar{c}_j=c_j-E_j$		0	0	$-\frac{3}{5}$ ↑k	$-\frac{29}{5}$	$-M+\frac{2}{5}$		

As $b_1=-\frac{3}{5}$, the first row is the key row and x_2 is the outing variable. Find the ratios of nonbasic elements of $\bar{c}_j$ row to the elements of key row. Neglect the ratios corresponding to positive or zero elements of key row. The desired ratio is $\frac{-\frac{3}{5}}{-\frac{1}{5}}=3$. Hence '$x_3$'-

column is the key column, x_3 is the incoming variable and $\left(-\frac{1}{5}\right)$ is the key element. Make the key element unity. This is shown in table 6·43.

Table 6·43

c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
12	x_2	0	-5	(1)	-2	1	3
5	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{21}{5}$

Key element unity

Replace x_2 by x_3. This is shown in table 6·44.

Table 6·44

	c_j	5	12	4	0	$-M$	
c_B	c.s.v.	x_1	x_2	x_3	s_1	A_1	b
4	x_3	0	-5	1	-2	1	3
5	x_1	1	7	0	3	-1	0
$E_j=\Sigma c_B a_{ij}$		5	15	4	7	-1	
$\bar{c}_j=c_j-E_j$		0	-3	0	-7	$-M+1$	

As all elements in $\bar{c}_j$ row are negative or zero and all b_i are positive, the solution given by table 6·44 is optimal. The optimal solution is

$$x_1=0,$$
$$x_2=0,$$
$$x_3=3,$$
$$Z_{max}=5\,(0)+12\,(0)+4\times 3=12.$$

(*d*) In order to find the resource that should be increased (or decreased), we shall write the dual objective function, which is

$$G=5y_1+2y_2,$$

where $y_1=29/5$ and $y_2=-2/5$ are the optimal dual variables. Thus the first resource should be increased as each additional unit of the first resource increases the objective function by 29/5. Next we are to find how much the first resource should be increased so that each additional unit continues to increase the objective function by 29/5. This requirement will be met so long as the primal problem remains feasible. If $\triangle$ be the increase in the first resource, it can be determined from the condition

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}=\mathbf{B}^{-1}\mathbf{b}=\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 5+\triangle \\ 2 \end{bmatrix}$$

$$=\begin{bmatrix} 10/5 & +2\triangle/5 & -2/5 \\ 5/5 & +\triangle/5 & +4/5 \end{bmatrix}=\begin{bmatrix} \dfrac{8+2\triangle}{5} \\ \dfrac{9+\triangle}{5} \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As x_1 and x_2 remain feasible ($\geqslant 0$) for all values of $\triangle > 0$, the first resource can be increased indefinitely while maintaining the condition that each additional unit will increase the objective function by 29/5.

(*e*) The second resource should be decreased as each additional unit of the second resource decreases the objective function by 2/5. Let $\triangle$ be the decrease in the second resource. To find its extent, we make use of the condition that the current solution remains feasible so long as

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}=\mathbf{B}^{-1}\mathbf{b}\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 5 \\ 2-\triangle \end{bmatrix}$$

$$\begin{bmatrix} 10/5 & -2/5 & +\triangle/5 \\ 5/5 & +4/5 & -2\triangle/5 \end{bmatrix}=\begin{bmatrix} \dfrac{8+\triangle}{5} \\ \dfrac{9-2\triangle}{5} \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Evidently x_1 remains positive only so long as $\dfrac{9-2\triangle}{5} \geqslant 0$ or $\triangle \leqslant 9/2$.

If $\triangle > 9/2$, x_1 becomes negative and must leave the solution.

6·5-2 Changes in the Cost Coefficients c_j

Changes in the coefficients of the objective function may take place due to a change in cost or profit of either basic variables or non-basic variables. Each of these two cases will first be considered separately. The discussion, will then, be followed by a combined case. All the three cases will be studied by considering an example.

EXAMPLE 6·5-2·1

A company wants to produce three products A, B and C. The unit profits on these products are Rs. 4, Rs. 6 and Rs. 2 respectively. These products require two types of resources—man-power and material. The following L.P. model is formulated for determining the optimal product mix :

Maximize $Z=4x_1+6x_2+2x_3$,

subject to

$x_1+x_2+x_3 \leqslant 3$, (man-power)

$x_1+4x_2+7x_3 \geqslant 9$, (material)

$$x_1, x_2, x_3 \geqslant 0,$$

where x_1, x_2, x_3 are the number of products A, B and C produced.

(*a*) Find the optimal product mix and the corresponding profit to the company.

(*b*) (*i*) Find the range on the values of non-basic variable coefficient c_3 such that the current optimal product mix remains optimal.

(*ii*) What happens if c_3 is increased to Rs. 12 ? What is the new optimal product mix in this case ?

(*c*) (*i*) Find the range on basic variable coefficient c_1 such that the current optimal product mix remains optimal.

(*ii*) Find the effect when $c_1 =$ Rs. 8 on the optimal product mix.

(*d*) Find the effect of changing the objective function to $Z = 2x_1 + 8x_2 + 4x_3$ on the current optimal product mix.

Solution. The standard form of the problem is

maximize $\quad Z = 4x_1 + 6x_2 + 2x_3 + 0s_1 + 0s_2,$

subject to $\quad x_1 + x_2 + x_3 + s_1 = 3,$

$x_1 + 4x_2 + 7x_3 + s_2 = 9,$

$x_1, x_2, x_3, s_1, s_2 \geqslant 0.$

Putting $x_1 = x_2 = x_3 = 0$ in the constraint equations, we get $s_1 = 3$ and $s_2 = 9$ as the initial basic feasible solution which can be expressed in the form of a simple matrix or table shown below.

Table 6·45

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
0	s_1	1	1	1	1	0	3
0	s_2	1	4	7	0	1	9

Initial basic feasible solution

First Iteration : Perform optimality test.

Table 6·46

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b	θ
0	s_1	1	1	1	1	0	3	3
0	s_2	1	(4)	7	0	1	9	9/4 $\leftarrow$ Key row
$E_j = \Sigma c_B a_{ij}$		0	0	0	0		0	
$\bar{c}_j = c_j - E_j$		4	6	2	0		0	
			$\uparrow k$					

(*ii*) Make key element unity.

Table 6·47

c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
0	s_1	1	1	1	1	0	3
0	s_2	1/4	(1)	7/4	0	1/4	9/4

Key element unity

(*iii*) Replace s_2 by x_2.

Table 6·48

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b	θ
0	s_1	$\left(\frac{3}{4}\right)$	0	$\frac{-3}{4}$	1	$\frac{-1}{4}$	$\frac{3}{4}$	1←Key row
6	x_2	$\frac{1}{4}$	1	$\frac{7}{4}$	0	$\frac{1}{4}$	$\frac{9}{4}$	9
$E_j=\Sigma c_B\ a_{ij}$		$\frac{3}{2}$	6	$\frac{21}{2}$	0	$\frac{3}{2}$		
$\bar{c}_j=\ c_j-E_j$		$\frac{5}{2}$ ↑ k	0	$\frac{-17}{2}$	0	$\frac{-3}{2}$	*Second feasible solution*	

Second Iteration : (*i*) Make key element unity.

Table 6·49

c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
0	s_1	(1)	0	-1	$\frac{4}{3}$	$\frac{-1}{3}$	1
6	x_2	$\frac{1}{4}$	1	$\frac{7}{4}$	0	$\frac{1}{4}$	$\frac{9}{4}$

Key element unity

(*ii*) Replace s_1 by x_1.

Table 6·50

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
4	x_1	1	0	-1	$\frac{4}{3}$	$\frac{-1}{3}$	1
6	x_2	0	1	2	$\frac{-1}{3}$	$\frac{1}{3}$	2
$E_j=\Sigma c_B\ a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$	
$\bar{c}_j=c_j-E_j$		0	0	-6	$\frac{-10}{3}$	$\frac{-2}{3}$	*Optimal feasible solution*

$\therefore$ Optimal solution is $x_1=1$, $x_2=2$, $x_3=0$ and $Z_{max}=$ Rs. $(4\times1+6\times2+2\times0)=$Rs. 16.

Effect of changing the objective function coefficient of a nonbasic variable

(*b*) (*i*) The coefficient c_3 corresponds to the non-basic variable x_3 for product C. In the optimal product mix shown in table 6.50, product C is not produced because of the low associated profit of Rs. 2 per unit (c_3). Clearly, if c_3 further decreases, it will have no effect on the current optimal product mix. However, if c_3 is increased beyond a certain value, it may become profitable to produce the product C.

As a rule, the sensitivity of the current optimal solution is determined by studying how the current opt mal solution given in table 6·50 changes as a result of changes in the input data. When value of c_3 changes, the value of net evalation (relative profit coefficient) of the non-basic variable x_3 *i.e.*, $\bar{c}_3$ in table 6·50 also changes. The table will remain optimal, as long as $\bar{c}_3$ remains nonpositive.

$\therefore$ For table 6·50 to remain optimal, $\bar{c}_3 \leqslant 0$

$$\text{or} \quad c_3-(4, 6)\begin{bmatrix} -1 \\ 2 \end{bmatrix} \leqslant 0,$$

$$\text{or} \quad c_3-(-4+12) \leqslant 0,$$

$$\text{or} \quad c_3 \leqslant 8.$$

This means that as long as the unit profit of product C is less than Rs. 8, it is not profitable to produce it.

(*ii*) If $c_3=12$, $\bar{c}_3=c_3-(4, 6)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$=12-(-4+12)$$

$$=12-8$$

$$=+4.$$

As $\bar{c}_3$ becomes positive, the current product mix given by table 6 50 does not remain optimal. The optimum profit can be increased further by producing product C. Non-basic variable x_3 can enter the solution to increase Z. This is shown in table 6·51.

Table 6·51

	c_j	4	6	12	0				
c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	b	θ	
4	x_1	1	0	-1	$\frac{4}{3}$	$\frac{-1}{3}$	1	-1	
6	x_2	0	1	(2)	$\frac{-1}{3}$	$\frac{1}{3}$	2	1	←Key row
$E_j=\Sigma c_B a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$			
$\bar{c}_j=c_j-E_j$		0	0	4	$\frac{-10}{3}$	$\frac{-2}{3}$			
				↑ k					

First Iteration. (*i*) Make key element unity.

Table 6·52

c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	b
4	x_1	1	0	-1	$\frac{4}{3}$	$\frac{-1}{3}$	1
6	x_2	0	$\frac{1}{2}$	(1)	$\frac{-1}{6}$	$\frac{1}{6}$	1

Key element unity

(*ii*) Replace x_2 by x_3.

Table 6·53

	c_j	4	6	12	0	0	
c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	b
4	x_1	1	$\frac{1}{2}$	0	$\frac{7}{6}$	$\frac{-1}{6}$	2
12	x_3	0	$\frac{1}{2}$	1	$\frac{-1}{6}$	$\frac{1}{6}$	1
$E_j=\Sigma c_B a_{ij}$		4	8	12	$\frac{8}{3}$	$\frac{4}{3}$	
$\bar{c}_j=c_j-E_j$		0	-2	0	$\frac{-8}{3}$	$\frac{-4}{3}$	

Optimal feasible solution

$\therefore$ New optimal product mix is $x_1=2$, $x_2=0$, $x_3=1$ and $Z_{max}=\text{Rs. }(4\times2+6\times0+12\times1)=\text{Rs. }20$.

Effect of changing the objective function coefficient of a basic variable

(*c*) *i*) Clearly, when c_1 decreases below a certain level, it may no

longer remain profitable to produce product A. On the other hand, if c_1 increases beyond a certain value, it may become so profitable that it is most paying to produce only product A. In either case the optimal product mix will change and hence there is lower as well as upper limit on c_1 within which the optimal product mix will not be affected.

Referring again to table 6·50, it can be seen that any variation in c_1 (and/or in c_2 also) will not change $\bar{c}_1$ and $\bar{c}_2$ (*i.e.*, they remain zero), while $\bar{c}_3$, $\bar{c}_4$, $\bar{c}_5$ will change. However, as long as $\bar{c}_j$ ($j=3, 4, 5$) remain non-positive, table 6·50 will remain optimal. $\bar{c}_3$, $\bar{c}_4$ and $\bar{c}_5$ can be expressed as functions of c_1 as follows :

$$\bar{c}_3=2-(c_1, 6)\begin{bmatrix} -1 \\ 2 \end{bmatrix}=2-(-c_1+12)=c_1-10,$$

$$\bar{c}_4=0-(c_1, 6)\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}=0-\left(\frac{4}{3}c_1-2\right)=-\frac{4}{3}c_1+2,$$

$$\bar{c}_5=0-(c_1, 6)\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}=0-\left(-\frac{1}{3}c_1+2\right)=\frac{1}{3}c_1-2.$$

For $\bar{c}_3$ to be $\leqslant 0$, $c_1-10\leqslant 0$ or $c_1\leqslant 10$,

for $\bar{c}_4$ to be $\leqslant 0$, $-\frac{4}{3}c_1+2\leqslant 0$ or $c_1\geqslant \frac{3}{2}$,

for $\bar{c}_5$ to be $\leqslant 0$, $\frac{1}{3}c_1-2\leqslant 0$ or $c_1\leqslant 6$.

$\therefore$ Range on c_1 for the optimal product mix to remain optimal is $\frac{3}{2}\leqslant c_1\leqslant 6$. Thus so long as c_1 lies within these limits, the optimal solution in table 6·50 viz., $x_1=1$, $x_2=2$, $x_3=0$ remains optimal. However, within this range, as the value of c_1 is changed, Z_{max} undergoes a change. For example, when $c_1=3$, $Z_{max}=$ Rs. $(3\times 1+6\times 2)=$ Rs. 15.

(*ii*) When $c_1=8$, $\bar{c}_3=c_1-10=8-10=-2$,

$$\bar{c}_4=-\frac{4}{3}c_1+2=-\frac{4}{3}\times 8+2=-\frac{26}{3},$$

$$\bar{c}_5=\frac{1}{3}c_1-2=\frac{8}{3}-2=+\frac{2}{3},$$

$$\bar{c}_1=\bar{c}_2=0.$$

As $\bar{c}_5$ becomes positive, the solution given in table 6·50 no longer

remains optimal. Slack variable s_2 enters the solution. This is shown in table 6 54.

Table 6·54

c_B	c_j / C.S.V.	8 x_1	6 x_2	2 x_3	0 s_1	0 s_2	b	θ
8	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1	-3
6	x_2	0	1	2	$-\frac{1}{3}$	$\left(\frac{1}{3}\right)$	2	6← Key row
	$E_j=\Sigma c_B a_{ij}$	8	6	4	$\frac{26}{3}$	$-\frac{2}{3}$		
	$\bar{c}_j=c_j-E_j$	0	0	-2	$-\frac{26}{3}$	$+\frac{2}{3}$ ↑ k		

First Iteration. (*i*) Make key element unity.

Table 6·55

c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	b
8	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
6	x_2	0	3	6	-1	(1)	6

Key element unity

(*ii*) Replace x_2 by s_2.

Table 6·56

c_B	c_j / C.S.V.	8 x_1	6 x_2	2 x_3	0 s_1	0 s_2	b
8	x_1	1	1	1	1	0	3
0	s_2	0	3	6	-1	1	6
	$E_j=\Sigma c_B a_{ij}$	8	8	8	8	0	
	$\bar{c}_j=c_j-E_j$	0	-2	-6	-8	0	

Optimal feasible solution

Thus the optimal product mix changes to $x_1=3$ units with $Z_{max}=$ Rs. 24.

Effect of changing the objective function coefficients of basic as well as non-basic variable.

(*d*) The effect on the optimal product mix can be determined by checking whether the $\bar{c}_j$ row in table 6·50 remains nonpositive.

$$\bar{c}_1=0,$$
$$\bar{c}_2=0,$$

$$\bar{c}_3=4-(2,\ 8)\begin{bmatrix}-1\\2\end{bmatrix}=4-(-2+16)=-10<0.$$

$$\bar{c}_4=0-(2,\ 8)\begin{bmatrix}\frac{4}{3}\\-\frac{1}{3}\end{bmatrix}=0-\left(\frac{8}{3}-\frac{8}{3}\right)=0,$$

$$\bar{c}_5=0-(2,\ 8)\begin{bmatrix}-\frac{1}{3}\\\frac{1}{3}\end{bmatrix}=0-(2/3+8/3)=-2<0.$$

Hence the optimal solution does not change. The optimal product mix remains $x_1=1$, $x_2=2$, $x_3=0$ and $\mathbf{Z}_{max}$ Rs. $(1\times2+2\times8+0\times4)=$ Rs. 18. There is indication of an alternate optimal solution since $\bar{c}_4=0$.

6·5·3. Addition of a New Variable

Referring to example 6·5·2·1, let us suppose that Research and Development department of the company has proposed a fourth product D which requires 1 unit of manpower and 1 unit of material and earns a unit profit of Rs. 3 when sold in the market. It is desired to find whether it is profitable to produce product D.

Addition of this product in the already existing product mix is equivalent to addition of a new variable (say x_4) and a column $\begin{bmatrix}1\\1\end{bmatrix}$ in the initial table (table 6·45). Now the present optional product mix given by table 6·50 remains optimal so long as the relative profit coefficient (net evaluation) of this new product, say $\bar{c}_6$ remains non-positive.

Now from the revised simplex method we know that

$$\bar{c}_6=c_6-\mathbf{c_B}\overline{\mathbf{P}_6}=c_6-\mathbf{c_B}\cdot\mathbf{B}^{-1}\,\mathbf{P}_6$$
$$=c_6-\pi\mathbf{P}_6,$$

where $c_6=$ Rs. 3, $\mathbf{P}_6=\begin{bmatrix}1\\1\end{bmatrix}$ and π is the simplex multiplier corresponding to the current optimal solution contained in 6 50 and is given by

$$\pi=\mathbf{c_B}\,\mathbf{B}^{-1}$$

$$=(4,\ 6)\begin{bmatrix}\frac{4}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}\end{bmatrix}=\left(\frac{10}{3}\ ,\ \frac{2}{3}\right).$$

$$\therefore \quad \bar{c}_6 = 3 - \left(\frac{10}{3}, \frac{2}{3}\right)\begin{bmatrix}1\\1\end{bmatrix} = 3 - \left(\frac{10}{3}+\frac{2}{3}\right) = -1.$$

As $\bar{c}_6$ is non-positive, the present optimal solution does not change even after the product D is introduced. As product D cannot improve the present value of the maximum profit, it should not be produced.

If, however, $\bar{c}_6$ turns out to be positive, it follows that product D can increase the value of maximum profit; simplex method can then be applied to find the new optimal solution.

EXAMPLE 6.5.3.1.

Consider the problem

$$\text{maximize } Z = 45x_1 + 100x_2 + 30x_3 + 50x_4,$$
$$\text{subject to} \quad 7x_1 + 10x_2 + 4x_3 + 9x_4 \leqslant 1{,}200$$
$$3x_1 + 40x_2 + x_3 + x_4 \leqslant 800$$
$$x_1, x_2, x_3, x_4 \geqslant 0.$$

The optimal table for this problem is given below.

Table 6·57

	c_j	45	100	30	50	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b
30	x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$\frac{1}{15}$	$\frac{800}{3}$
100	x_2	$\frac{1}{30}$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	$\frac{40}{3}$
$\bar{c}_j = c_j - E_j$		$-\frac{25}{3}$	0	0	$-\frac{50}{3}$	$-\frac{22}{3}$	$-\frac{2}{3}$	

If a new variable x_7 is added to this problem with a column $\begin{bmatrix}10\\10\end{bmatrix}$ and $c_7 = 120$, find the change in the optimal solution.

Solution

$$\bar{c}_7 = c_7 - \mathbf{c_B}\,\bar{\mathbf{P}}_7 = c_7 - \mathbf{c_B}.\mathbf{B}^{-1}.\mathbf{P}_7 = c_7 - \pi\mathbf{P}_7,$$

where $c_7 = 120$, $\mathbf{P}_7 = \begin{bmatrix}10\\10\end{bmatrix}$ and π, the simplex multiplier corresponding to the original optimal solution in table 6·57 is given by

$$\pi = (\pi_1, \pi_2) = \mathbf{c_B}\mathbf{B}^{-1} = (30, 100)\begin{bmatrix}\frac{4}{15} & -\frac{1}{15}\\[4pt] -\frac{1}{150} & \frac{2}{75}\end{bmatrix} = \left(\frac{22}{3}, \frac{2}{3}\right).$$

$$\therefore\ \bar{c}_7=c_7-\pi \mathbf{P}_7=120-\left(\frac{22}{3},\ \frac{2}{3}\right)\begin{bmatrix}10\\10\end{bmatrix}$$

$$=120-\left(\frac{220}{3}+\frac{20}{3}\right)=+40.$$

Sinee $\mathbf{c}_r$ is positive, the existing optimal solution can be improved.

$$\text{Now } \bar{\mathbf{P}}_7=\mathbf{B}^{-1}\mathbf{P}_r=\begin{bmatrix}\frac{4}{15} & -\frac{1}{15}\\ -\frac{1}{150} & \frac{2}{75}\end{bmatrix}\begin{bmatrix}10\\10\end{bmatrix}=\begin{bmatrix}2\\ \frac{1}{5}\end{bmatrix}.$$

Now we start with the original optimal table (table 6·57) and add entries corresponding to variable x_7 as follows :

Table 6·58

	c_j	45	100	30	50	0	0	120		
c_B	C.S.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	θ
30	x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	2	$\frac{800}{3}$	$\frac{400}{3}$
100	x_2	$\frac{1}{30}$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	$\left(\frac{1}{5}\right)$	$\frac{40}{3}$	$\frac{200}{3}\downarrow$ Key row
$\bar{c}_j=c_j-E_j$		$-\frac{25}{3}$	0	0	$-\frac{50}{3}$	$-\frac{22}{3}$	$-\frac{2}{3}$	$+40$ $\uparrow$ k		

Make key element unity.

Table 6·59

c_B	C.S.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
30	x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	2	$\frac{800}{3}$
100	x_2	$\frac{1}{6}$	5	0	$-\frac{1}{6}$	$-\frac{1}{30}$	$\frac{2}{15}$	(1)	$\frac{200}{3}$

Key element unity

Replace x_2 by x_7.

Table 6·60

	c_j	45	100	30	50	0	0	123	
c_B	C.S.V.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
30	x_3	$\frac{4}{3}$	-10	1	$\frac{8}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{400}{3}$
120	x_7	$\frac{1}{6}$	5	0	$-\frac{1}{6}$	$-\frac{1}{30}$	$\frac{2}{15}$	1	$\frac{200}{3}$
$E_j=\Sigma c_B a_{ij}$		60	300	30	60	6	6	120	
$\bar{c}_j=c_j-E_j$		-15	-200	0	-10	-6	-6	0	

Optimal feasible solution

Since $\bar{c}_j$ is negative, table 6·60 gives the optimal solution with $x_3=\frac{400}{3}$, $x_7=\frac{200}{3}$ (basic variables), $x_1=x_2=x_4=x_5=x_6=0$ (non-basic variables) and $Z_{max}=30\times\frac{400}{3}+120\times\frac{200}{3}=4{,}000+8{,}000$ $=12{,}000$.

6.5.4. Changes in the Coefficients of the Constraints a

When changes take place in the constraint coefficients of a *non-basic variable* in a current optimal solution, feasibility of the solution is not affected. The only effect, if any, may be on the optimality of the solution. This effect can be studied by following the steps given in section 6·5·3.

However, if the constraint coefficients of a *basic variable* get changed, things become more complicated since the feasibility of the current optimal solution may also be affected (lost). The basic matrix is affected, which, in turn, may affect all the quantities given in the current optimal table. Under such circumstances, it may be better to solve the problem over again.

EXAMPLE 6.5.4.1

Find the effect of the following changes in the original optimal table 6·57 of problem 6·5·3·1 :

(*a*) 'x_1'—column in the problem changes from $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$

(*b*) 'x_1'—column changes from $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

Solution

(*a*) x_1 is a nonbasic variable in the optimal solution.

$$\bar{c}_1=c_1-\mathbf{c_B}\bar{\mathbf{P}}_1=c_1-\mathbf{c_B}\mathbf{B}^{-1}\mathbf{P}_1$$

$$=c_1-\pi\mathbf{P}_1,\quad \text{where } c_1=45,\quad \mathbf{P}_1=\begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

and $$\pi=\mathbf{c_B}\mathbf{B}^{-1}=(30,100)\begin{bmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{150} & \frac{2}{75} \end{bmatrix}\left(\frac{22}{3},\ \frac{2}{3}\right).$$

$$\therefore\quad \bar{c}_1=45-\left(\frac{22}{3},\ \frac{2}{3}\right)\begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$=45-\left(\frac{154}{3}+\frac{10}{3}\right)=45-\frac{164}{3}=-\frac{29}{3}.$$

Since c_1 remains non-positive, the original optimum solution remains optimum for the new problem also.

(b) $\bar{c}_1=c_1-\mathbf{c}_B\bar{\mathbf{P}}_1=c_1-\mathbf{c}_B\mathbf{B}^{-1}\mathbf{P}_1$

$$=c_1-\pi\mathbf{P}_1=45-\left(\frac{22}{3},\ \frac{2}{3}\right)\begin{bmatrix}5\\8\end{bmatrix}$$

$$=45-\left(\frac{110}{3}+\frac{16}{3}\right)=+3.$$

As $\bar{c}_1$ is positive, the existing optimum solution can be improved.

$$\text{Now } \bar{\mathbf{P}}_1=\mathbf{B}^{-1}\mathbf{P}_1=\begin{bmatrix}\frac{4}{15} & -\frac{1}{15}\\ -\frac{1}{150} & \frac{2}{75}\end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix}=\begin{bmatrix}\frac{4}{5}\\ \frac{27}{150}\end{bmatrix}.$$

Now we start with the original optimal table (table 6·57) and incorporate the changes due to variable x_1.

Table 6·61

	c_j	45	100	30	50	0	0			
c_i	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b	θ	
30	x_3	$\frac{4}{5}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	$\frac{800}{3}$	$\frac{1000}{3}$	
100	x_2	$\left(\frac{27}{150}\right)$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	$\frac{40}{3}$	$\frac{2,000}{27}$	←key row
$\bar{c}_j=$	c_j-E_j	+3 ↑k	0	0	$-\frac{50}{3}$	$-\frac{22}{3}$	$-\frac{2}{3}$			

Make key element unity.

Table 6·62

c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b	
x_3	$\frac{4}{5}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	$\frac{800}{3}$	
x_2	(1)	$\frac{150}{27}$	0	$-\frac{5}{27}$	$-\frac{1}{27}$	$\frac{4}{27}$	$\frac{2,000}{27}$	*Key element unity*

Replace x_2 by x_1

Table 6·63

	c_j	45	100	30	50	0	0	
c_B	*c.s.v.*	x_1	x_2	x_3	x_4	x_5	x_6	b
30	x_3	0	$-\frac{40}{9}$	1	$\frac{67}{27}$	$\frac{8}{27}$	$-\frac{5}{27}$	$\frac{5,600}{27}$
45	x_1	1	$\frac{50}{9}$	0	$-\frac{5}{27}$	$-\frac{1}{27}$	$\frac{4}{27}$	$\frac{2,000}{27}$
$E_j=\Sigma c_B a_{ij}$		45	$\frac{350}{3}$	30	$\frac{1785}{27}$	$\frac{195}{27}$	$\frac{10}{9}$	
$\bar{c}_j=c_j-E_j$		0	$-\frac{50}{3}$	0	$-\frac{435}{27}$	$-\frac{195}{27}$	$-\frac{10}{9}$	

Optimal feasible solution

Since $\bar{c_j}$ is non-positive, table 6·63 gives the optimal solution with

$$x_1=\frac{2,000}{27}, \qquad x_3=\frac{5600}{27} \text{ basic variables}),$$

$$x_2=x_4=x_5=x_6=0 \text{ (non-basic variables)},$$

$$Z_{max}=\frac{2000}{27}\times 45+\frac{5,600}{27}\times 30$$

$$=\frac{10,000}{3}+\frac{56,000}{9}$$

$$=\frac{86,000}{9}$$

6·5·5. Addition of a New Constraint

Addition of a new constraint may or may not affect the feasibility of the current optimal solution. For this, it is sufficient to check whether new constraint is satisfied by the current optimal solution or not. If it is satisfied, the inclusion of the constraint has no effect on the current optimal solution *i.e.*, it remains feasible as well as optimal. If, however, the constraint is not satisfied, the current optimal solution becomes infeasible. Dual simplex method is then used to find the new optimal solution.

EXAMPLE 6.5-5.1

(*a*) In problem 6·5·2·1 an administrative constraint is added. Products A, B, and C require 2, 3 and 2 hours of administrative services, while the total available administrative hours are 10. How does the optimal solution given by table 6·50 change ?

(*b*) If the total available administrative time is 6 hours, find the new optimal solution.

Solution

(*a*) The optimal feasible solution given by table 6·50 is $x_1=1$, $x_2=2$, $x_3=0$; while the additional constraint is $2x_1+3x_2+2x_3 \leqslant 10$. As this constraint is satisfied by the optimal solution, the solution remains feasible and optimal for the modified problem.

(*b*) As the additional constraint $2x_1+3x_2+2x_3 \leqslant 6$ is not satisfied by the current optimal solution, table 6·50 is no longer optimal for the modified problem. In order to find the new optimal solution, we add the new constraint as the third row in table 6·64. Using s_3 as the slack variable for this constraint, the (modified) optimal table may be written as

Table 6.64

	c_j	4	6	2	0	0	0	
c_B	C.s.v	x_1	x_2	x_3	s_1	s_2	s_3	b
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	0	1
6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	2
0	s_3	2	3	2	0	0	1	6
$E_j=\Sigma c_B a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$	0	
$\bar{c}_j=c_j-E_j$		0	0	-6	$-\frac{10}{3}$	$-\frac{2}{3}$	0	

Since x_1 and x_2 are in the basic solution, their corresponding coefficients in the basic constraint must be zero. To eliminate the coefficients of x_1 and x_2, we multiply the first row by -2, the second row by -3 and add them to the third row. Table 6.65 represents the new canonical tableau after the row operations. Note that $\bar{c}_j$ row is not affected since the new basic variable s_3 is the slack variable.

Table 6.65

	c_i	4	6	2	0	0	0		
c_B	C.s.v.	x_1	x_2	x_3	s_1	s_2	s_3	b	
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	0	1	
6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	2	
0	s_3	0	0	(-6)	$-\frac{5}{3}$	$-\frac{1}{3}$	1	-4	←Key row
$E_j=\Sigma c_B a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$	0		
$\bar{c}_j=c_j-E_j$		0	0	-6 ↑	$-\frac{10}{3}$	$-\frac{2}{3}$	0		

In table 6·65, $\bar{c}_j$ row is optimal, but since b_3 is negative, the current basic solution is infeasible. In other words, table6·65 is dual feasible and, therefore, dual simplex method is applied to find the new optimal solution.

Evidently s_3 is the variable that leaves the basis. The ratios for the nonbasic variables are 1, 2, 2 respectively. The variable x_3 which corresponds to the minimum ratio is the entering variable. The key element, -6 has been shown bracketed. Regular simplex method is used to find the optimal solution. Key element is made unity in table 6.66.

Table 6·66

c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	0	1
6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	2
0	s_3	0	0	(1)	$\frac{5}{18}$	$\frac{1}{18}$	$-\frac{1}{6}$	$\frac{2}{3}$

Key element unity

Replace s_3 by x_3.

Table 6·67

	c_j	4	6	2	0	0	0	
c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b
4	x_1	1	0	0	$\frac{29}{18}$	$-\frac{5}{18}$	$-\frac{1}{6}$	$\frac{5}{3}$
6	x_2	0	1	0	$-\frac{8}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{2}{3}$
2	x_3	0	0	1	$\frac{5}{18}$	$\frac{1}{18}$	$-\frac{1}{6}$	$\frac{2}{3}$
	$E_j=\Sigma c_B a_{ij}$	4	6	2	$\frac{5}{3}$	$\frac{1}{3}$	1	
	$\bar{c}_j=c_j-E_j$	0	0	0	$-\frac{5}{3}$	$-\frac{1}{3}$	-1	

Optimal feasible solution

Table 6·67 is optimal and the optimal product mix is to produce 5/3 units of product A, 2/3 units of product B and 2/3 units of product C with the new maximum profit=Rs. $(4\times 5/3+6\times 2/3+2\times 2/3)$=Rs. 12. Thus the addition of a new constraint decreases the

optimum profit from Rs. 16 (table 6·50) to Rs. 12. This is true of every linear programming problem. In general, whenever a new constraint is added to a linear programming problem, the old optimal value will always be better or at least equal to the optimal value. In other words, addition of a new constraint cannot improve the optimal value of any linear programming problem.

The idea of adding new constraints can sometimes be used to reduce the computational time and hence cost of solving a linear programming problem. As the computational effort in solving a linear programming problem increases with the number of constraints, it will be advantageous to identify and delete the constraints that are not binding. Such constraints are called inactive or secondary constraints. These may pertain to resources which can be obtained easily or can be directly controlled. The new problem with fewer number of constraints is then solved. After the optimal solution is obtained, the secondary constraints are added to verify whether the optimal solution satisfies them or not. If not, the dual simplex method is applied to get the new optimal solution. No doubt, the overall saving in computational time and cost will depend on how accurately the initial judgements were made while identifying the secondary constraints.

6·5·6 Deletion of a Variable

Deletion of a *non-basic variable* is a totally superflnous operation and does not affect the feasibility and/or optimality of the current optimal solution. However, deletion of a *basic variable* may affect the optimality and a new optimum solution may have to be found out. For this, a heavy penalty $-M$ ($+M$ in case of minimization problems) is assigned to the variable under consideration and the new optimum solution is obtained by applying regular simplex method to the (modified) current optimum table.

EXAMPLE 6.5-6.1.

Consider the optimal table 6.67 of example 6.5-5.1. If product B is not to be produced, so that variable x_2 is to be deleted from this table, find the optimum solution to the resulting L.P. problem.

Solution. Since x_2 is a basic variable, we assign a penalty $-M$ to x_2 in table 6·67 since the given L.P. problem is of maximization type. The resulting modified table is shown below.

Table 6.68

	c_j	4	$-M$	2	0	0	0		
c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
4	x_1	1	0	0	$\frac{29}{18}$	$-\frac{5}{18}$	$-\frac{1}{6}$	$\frac{5}{3}$	-10
$-M$	x_2	0	1	0	$-\frac{8}{9}$	$\frac{2}{9}$	$\left(\frac{1}{3}\right)$	$\frac{2}{3}$	2←Key row
2	x_3	0	0	1	$\frac{5}{18}$	$\frac{1}{18}$	$-\frac{1}{6}$	$\frac{2}{3}$	-4
$E_j = \Sigma c_B a_{ij}$		4	$-M$	2	$7+\frac{8}{9}M$	$-1-\frac{2}{9}M$	$-1-\frac{M}{3}$		
$\bar{c}_j = c_j - E_j$		0	0	0	$-7-\frac{8}{9}M$	$1+\frac{2}{9}M$	$1+\frac{M}{3}$		
							↑ k		

Initial feasible solution

Make key element unity.

Table 6·69

c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b
4	x_1	1	0	0	$\frac{29}{18}$	$-\frac{5}{18}$	$-\frac{1}{6}$	$\frac{5}{3}$
$-M$	x_2	0	3	0	$-\frac{8}{3}$	$\frac{2}{3}$	(1)	2
2	x_3	0	0	1	$\frac{5}{18}$	$\frac{1}{18}$	$-\frac{1}{6}$	$\frac{2}{3}$

Key element unity

Replace x_2 by s_3.

Table 6.70

	c_j	4	$-M$	2	0	0	0		
c_B	C.S.V.	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
4	x_1	1	$\frac{1}{2}$	0	$\frac{7}{6}$	$-\frac{1}{6}$	0	2	-3
0	s_3	0	3	0	$-\frac{8}{3}$	$\left(\frac{2}{3}\right)$	1	2	3←key row
2	x_3	0	$\frac{1}{2}$	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	1	·6
$E_j = \Sigma c_B a_{ij}$		4	3	2	$\frac{13}{3}$	$-\frac{1}{3}$	0		
$\bar{c}_j = c_j - E_j$		0	$-M-3$	0	$-\frac{13}{3}$	$\frac{1}{3}$	0		
						↑ k			

2nd feasible solution

Make key element unity.

Table 6.71

c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	s_3	b
4	x_1	1	$\frac{1}{2}$	0	$\frac{7}{6}$	$-\frac{1}{6}$	0	2
0	s_3	0	$\frac{9}{2}$	0	-4	(1)	$\frac{3}{2}$	3
2	x_3	0	$\frac{1}{2}$	1	$-\frac{1}{6}$	$\frac{1}{6}$	1	

Key element unity

Replace s_3 by s_2.

Table 6.72

	c_j	4	$-M$	2	0	0	0	
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	s_3	b
4	x_1	1	$\frac{5}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{5}{2}$
0	s_2	0	$\frac{9}{2}$	0	-4	1	$\frac{3}{2}$	3
2	x_3	0	$-\frac{1}{4}$	1	$\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{2}$
$E_j=\Sigma c_B a_{ij}$		4	$\frac{9}{2}$	2	3	0	$\frac{1}{2}$	
$\bar{c}_j=c_j-E_j$		0	$-M-\frac{9}{2}$	0	-3	0	$-\frac{1}{2}$	

New optimal feasible solution

$\therefore$ New optimal solution is $x_1=\frac{5}{2}$, $x_2=0$, $x_3=\frac{1}{2}$; $Z_{max}=$ Rs. $\left(4\times\frac{5}{2}+2\times\frac{1}{2}\right)=$Rs. 11.

6.5-7. Deletion of a Constraint

The constraint to be deleted may be either binding or unbinding on the optimal solution. The deletion of an unbinding constraint can only enlarge the feasible region but will not affect the optimal solution This can be easily verified graphically. Moreover, if the constraint under consideration has a slack or surplus variable of zero value in the basis matrix, it cannot be binding and hence will not affect the optimal solution.

The deletion of a binding constraint will, however, cause post-opitmality problem. The simplest way to proceed in this case is via

the addition of one or two new variables. For example, the constraint

$$a_{i1}x_1+a_{i2}x_2+\ldots\ldots+a_{in}x_n=b_i$$

can be written as

$$a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n+x_{n+1}-x_{n+2}=b_i;\ x_{n+1},\ x_{n+2}\geqslant 0,$$

where x_{n+1} and x_{n+2} are slack and surplus variables respectively. The problem can then be solved by the procedure laid down in section 6·5·3 for addition of new variables.

6·6. Parametric Programming

Section 6·5 on sensitivity analysis discussed the effect of discrete changes in the input coefficients of the linear programming problem on its optimal solution. However, if there is continuous change in the values of these coefficients, none of the results derived in that section are applicable. *Parametric linear programming* investigates the effect of *predetermined continuous* variations of these coefficients on the optimal solution. It is simply an extension of sensitivity analysis and aims at finding the various basic solutions that become optimal, one after the other, as the coefficients of the problem change continuously. The coefficients change as a linear function of a single parameter, hence the name parametric linear programming for this computational technique. As in sensitivity analysis, the purpose of this technique is to reduce the additional computations required to obtain the changes in the optimal solution. Only two types of parametric problems will be considered here :

1. *Parametric cost problem*: in which cost vector c varies linearly as a function of parameter λ.

2. *Parametric rigth-hand-side problem*: in which the requirement vector b varies linearly as a function of parameter λ.

6·6·1. Parametric Cost Problem

Let the linear programming problem before parametrization be

minimize $Z=\mathbf{CX}$,

subject to $\mathbf{AX}=\mathbf{b}$,

$\mathbf{X}\geqslant 0$,

where $\mathbf{C}$ is the given cost vector.

Let this cost vector change to $\mathbf{C}+\lambda\mathbf{C}'$ so that the parametric cost problem becomes

minimize $Z=(\mathbf{C}+\lambda\mathbf{C}')\,\mathbf{X}$,

subject to $\mathbf{AX}=\mathbf{b}$,

$\mathbf{X}\geqslant 0$,

where $\mathbf{C}'$ is the given predetermined cost variation vector and λ is

an unknown (positive or negative) parameter. As λ changes, the cost coefficients of all variables also change. We wish to determine the family of optimal solutions as λ changes from $-\infty$ to $+\infty$.

This problem is solved by using the simplex method and sensitivity anaylsis. When $\lambda=0$, the parametric cost problem reduces to the original L.P. problem; simplex method is used to find its optimal solution. Let $\mathbf{B}$ and $\mathbf{X_B}$ represent the optimal basis matrix and the optimal basic feasible solution respectively for $\lambda=0$. The net evaluations or relative cost coefficients are all non-negative (minimization problem) and are given by

$$\begin{aligned}\bar{c}_j &= c_j - E_j\\ &= c_j - \Sigma \mathbf{c_B} a_{ij}\\ &= c_j - \mathbf{c_B}\overline{\mathbf{P}}_j,\end{aligned}$$

where $\mathbf{c_B}$ is the cost vector of the basic variables and $\overline{\mathbf{P}}_j$ is the jth column (corresponding to the variable x_j) in the optimal table.

As λ changes from zero to a positive or negative value, the feasible region and values of the basic variables $\mathbf{X_B}$ remain unaltered, but the relative cost coefficients change. For any variable x_j, the relative cost coefficient is given by

$$\begin{aligned}\bar{c}_j(\lambda) &= (c_j+\lambda c'_j)-(\mathbf{c_B}+\lambda \mathbf{c'_B})\overline{\mathbf{P}}_j\\ &= (c_j-\mathbf{c_B}\overline{\mathbf{P}}_j)+\lambda(c'_j-\mathbf{c'_B}\overline{\mathbf{P}}_j)\\ &= \bar{c}_j+\lambda\bar{c}'_j.\end{aligned}$$

Since vectors $\mathbf{C}$ and $\mathbf{C'}$ are known, $\bar{c}_j$ and $\bar{c}'_j$ can be determined. For the current minimization problem, $\bar{c}_j(\lambda)$ must be non-negative for the solution to be optimal [$\bar{c}_j(\lambda)$ must be non-positive for a maximization problem]. Thus

$$\bar{c}_j(\lambda)\geqslant 0,$$

or

$$\bar{c}_j+\lambda\bar{c}'_j\geqslant 0.$$

In other words, for a given solution, we can determine the range for λ within which the solution remains optimal.

EXAMPLE 6.6.1.1. Consider the linear programming problem

maximize $Z=4x_1+6x_2+2x_3$,

subject to $x_1+x_2+x_3\leqslant 3$,

$x_1+4x_2+7x_3\leqslant 9$,

$x_1, x_2, x_3\geqslant 0$.

The optimal solution to this problem is given by the following table :

Table 6.73

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	2
$E_j=\Sigma c_B a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$	
$\bar{c}_j=c_j-E_j$		0	0	-6	$-\frac{10}{3}$	$-\frac{2}{3}$	

Solve this problem if the variation cost vector $C'=(2, -2, 2, 0, 0)$. Identify all critical values of the parameter λ.

Solution. The given parametric cost problem is

$$\text{maximize } Z=(4+2\lambda)\ x_1+(6-2\lambda)\ x_2+(2+2\lambda)x_3+0s_1+0s_2,$$

$$\text{subject to } x_1+x_2+x_3+s_1=3,$$

$$x_1+4x_2+7x_3+s_2=9,$$

$$x_1,\ x_2,\ x_3,\ s_1,\ s_2 \geqslant 0.$$

When $\lambda=0$, the problem reduces to the L.P. problem, whose optimal solution is given by table 6·73. The relative profit coefficients in this optimal table are all non-positive. For values of λ other than zero, the relative profit coefficients became linear functions of λ. To compute them we, first, add a new relative profit row called $\overline{c'_j}$ row to table 6·73. This shown in table 6·74.

Table 6·74

		c'_j	2	-2	2	0	0	
		c_j	4	6	2	0	0	
c'_B	c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
2	4	x_1	0	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
-2	6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	2
	$\overline{c_j}$		0	0	-6	$-\frac{10}{3}$	$-\frac{2}{3}$	$Z=16$
	$\overline{c'_j}$		0	0	8	$-\frac{10}{3}$	$\frac{4}{3}$	$Z'=-2$

In table 6.74, $\overline{c'_j}$ is calculated just as $\overline{c_j}$ row except that vector $\mathbf{C}$ is replaced by $\mathbf{C}'$. For example,

$$\bar{c}_2 = c_2 - E_2$$
$$= c_2 - \Sigma \mathbf{c}_B a_{i2}$$
$$= c_2 - \mathbf{c}_B \overline{\mathbf{P}}_2$$
$$= 6 - (4,\ 6)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 6 - 6 = 0.$$

$$\therefore \quad \overline{c'_1} = c'_1 - \mathbf{c}'_B \overline{\mathbf{P}_1}$$
$$= 2 - (2,\ -2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$
$$\overline{c'_2} = -2 - (2,\ -2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0,$$
$$\overline{c'_3} = 2 - (2,\ -2)\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2 - (-2 - 4) = 8,$$
$$\overline{c'_4} = 0 - (2,\ -2)\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix} = -\left(\frac{8}{3} + \frac{2}{3}\right) = -\left(\frac{10}{3}\right)'$$
$$\overline{c'_5} = 0 - (2,\ -2)\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = -\left(-\frac{2}{3} - \frac{2}{3}\right) = \frac{4}{3},$$
$$Z' = 1 \times 2 - 2 \times 2 = -2.$$

Table 6.74 represents a basic feasible solution for the given parametric cost problem. It is give by

$$x_1 = 1,\ x_2 = 2,\ x_3 = s_1 = s_2 = 0.$$

Value of the objective function, $Z(\lambda) = Z + \lambda Z' = 16 - 2\lambda$.

The relative profit coefficients, which are linear functions of λ, are given by

$$\overline{c_j}(\lambda) = \overline{c_j} + \lambda\, \overline{c'_j},\ j = 1, 2, 3, 4, 5.$$

Table 6.74 will be optimal if $\overline{c_j}(\lambda) \leqslant 0$ for $j = 3, 4, 5$. Thus we can determine the range of λ for which table 6.74 remains optimal as follows :

$$\bar{c}_3(\lambda) = \bar{c}_3 + \lambda \bar{c}'_3 = -6 + 8\lambda \leqslant 0 \text{ or } \lambda \leqslant 3/4,$$
$$\overline{c_4}(\lambda) = \overline{c_4} + \lambda\, \overline{c'_4} = -\frac{10}{3} - \frac{10}{3}\lambda \leqslant 0 \text{ or } \lambda \geqslant -1,$$

$$\bar{c}_5(\lambda) = \bar{c}_5 + \lambda \bar{c}'_5 = -\frac{2}{3} + \frac{4}{3}\lambda \leqslant 0 \quad \text{or } \lambda \leqslant \frac{1}{2}.$$

Thus $x_1 = 1$, $x_2 = 2$, $x_3 = s_1 = s_2 = 0$ is an optimal solution for the given parametric problem for all values of λ between -1 and $\frac{1}{2}$ and $Z_{max} = 16 - 2\lambda$.

For $\lambda > \frac{1}{2}$, the relative profit coefficient of the non-basic variable s_2, namely $\bar{c}_5$ (λ) becomes positive and table 6.74 no longer remains optimal. Regular simplex method is used to iterate towards optimality. s_2 is the entering variable and computation of 'θ' —column indicates x_2 to be the variable that leaves the basis matrix so that the key element is $\frac{1}{3}$. The key element is made unity in table 6.75.

Table 6·75

c'_B	c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
2	4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1
-2	6	x_2	0	3	6	-1	(1)	6

Key element unity

Replace x_2 by s_2.

Table 6·76

	c'_j		2	-2	2	0	0	
	c_j		4	6	2	0	0	
c'_B	c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
2	4	x_1	1	1	1	1	0	3
0	0	s_2	0	3	6	-1	1	6
		$\bar{c}_j$	0	2	-2	-4	0	$Z=12$
		$\bar{c}'_j$	0	-4	0	-2	0	$Z'=6$

Table 6·76 will be optimal if $\bar{c}_j$ $(\lambda) \leqslant 0$, for $j = 2, 3, 4$.

Now $\bar{c}_2$ $(\lambda) = \bar{c}_2 + \lambda\, \bar{c}'_2 = 2 - 4\lambda \leqslant 0 \quad \therefore \quad \lambda \leqslant 1/2$,

$$\bar{c}_3\,(\lambda) = \bar{c}_3 + \lambda\, \bar{c}'_3 = -2 \leqslant 0, \text{ which is true,}$$

$$\bar{c}_4\,(\lambda) = \bar{c}_4 + \lambda \bar{c}'_4 = -4 - 2\lambda \leqslant 0\,, \quad \therefore \ \lambda \geqslant -2.$$

$\therefore$ For all $\lambda \geqslant \frac{1}{2}$, the optimal solution is given by

$$x_1 = 3,\ x_2 = x_3 = s_1 = 0,\ s_2 = 6 \text{ and } Z_{max} = 12 + 6\lambda.$$

For $\lambda < -1$, the relative profit coefficient of the non-basic variable s_1, namely $\overline{c_4}\,(\lambda)$ becomes positive and again table 6.74 no longer remains optimal. s_1 becomes the entering variable and x_1 the leaving variable. Key element is $\frac{4}{3}$ This element is made unity in table 6.77.

Table 6.77

c'_B	c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
2	4	x_1	$\frac{3}{4}$	0	$-\frac{3}{4}$	(1)	$-\frac{1}{4}$	$\frac{3}{4}$
-2	6	x_2	0	1	2	$-\frac{1}{3}$	$\frac{1}{3}$	2

Key element unity

Replace x_1 by s_1.

Table 6.78

	c_j'		2	-2	2	0	0	
	c_j		4	6	2	0	0	
c'_B	c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
0	0	s_1	$\frac{3}{4}$	0	$-\frac{3}{4}$	1	$-\frac{1}{4}$	$\frac{3}{4}$
-2	6	x_2	$\frac{1}{4}$	1	$\frac{7}{4}$	0	$\frac{1}{4}$	$\frac{9}{4}$
		$\overline{c_j}$	$\frac{5}{2}$	0	$-\frac{17}{2}$	0	$-\frac{3}{2}$	$Z=\frac{27}{2}$
		$\overline{c'_j}$	$\frac{5}{2}$	0	$\frac{11}{2}$	0	$\frac{1}{2}$	$Z'=-\frac{9}{2}$

Table 6.78 will be optimal if $\overline{c_j}\,(\lambda) \leqslant 0$ for $j=1, 3, 5$.

Now $\overline{c_1}\,(\lambda)=\overline{c_1}+\lambda\,\overline{c'_1}=\frac{5}{2}+\frac{5}{2}\lambda \leqslant 0 \quad \therefore \lambda \leqslant -1,$

$$\overline{c_3}\,(\lambda)=\overline{c_3}+\lambda\,\overline{c'_3}=-\frac{17}{2}+\frac{11}{2}\lambda \leqslant 0 \;\therefore\; \lambda \leqslant \frac{17}{11},$$

$$\overline{c_5}\,(\lambda)=\overline{c_5}+\lambda\,\overline{c'_5}=-\frac{3}{2}+\frac{1}{2}\lambda \leqslant 0 \;\therefore\; \lambda \leqslant 3.$$

$\therefore$ For all $\lambda \leqslant -1$, the optimal solution is given by

$x_1=0,\ x_2=\frac{9}{4},\ x_3=0,\ s_1=\frac{3}{4},\ s_5=0$ and $Z_{max}=\frac{27}{2}-\frac{9}{2}\lambda.$

Thus tables 6·74, 6·76 and 6.78 give families of optimal solutions for $-1 \leqslant \lambda \leqslant \frac{1}{2}$, $\lambda \geqslant \frac{1}{2}$, and $\lambda \leqslant -1$ respectively.

6.6-2. Parametric Right-Hand-Side Problem

The right-hand-side constants in a linear programming problem represent the limits in the resources and the outputs. In some practical problems all the resources are not independent of one another. A shortage of one resource may cause shortage of other resources at varying levels. Same is true for outputs also. For example, consider a firm manufacturing electrical appliances. A shortage in electric power will decrease the demand of all the electric items produced in varying degrees depending upon the electric energy consumed by them. In all such problems, we are to consider simultaneous changes in the right-hand-side constants, which are functions of one parameter and study how the optimal solution is affected by these changes.

Let the linear programming problem before parameterization be

$$\text{maximize} \quad Z = \mathbf{cX},$$
$$\text{subject to } \mathbf{AX} = \mathbf{b},$$
$$\mathbf{X} = 0.$$

where $\mathbf{b}$ is the known requirement (right-hand-side) vector. Let this requirement vector $\mathbf{b}$ change to $\mathbf{b} + \lambda \mathbf{b}'$ so that parametric right-hand-side problem becomes

$$\text{maximize} \quad Z = \mathbf{cX},$$
$$\text{subject to } \mathbf{AX} = \mathbf{b} + \lambda \mathbf{b}',$$
$$\mathbf{X} \geqslant 0,$$

where $\mathbf{b}'$ is the given and predetermined variation vector and λ is an unknown parameter. As λ changes, the right-hand-constants also change. We wish to determine the family of optimal solutions as λ changes from $-\infty$ to $+\infty$.

When $\lambda = 0$, the parametric problem reduces to the original L.P. problem ; simplex method is used to find its optimal solution.

Let $\mathbf{B}$ and $\mathbf{X_B}$ represent the optimal basis matrix and the optimal basic feasible solution respectively for $\lambda = 0$. Then $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$. As λ changes from zero to a positive or negative value, the values of the basic variables change and the new values are given by

$$\mathbf{X_B} = \mathbf{B}^{-1}(\mathbf{b} + \lambda\, \mathbf{b}') = \mathbf{B}^{-1}\mathbf{b} + \lambda\, \mathbf{B}^{-1}\, \mathbf{b}'$$

$$= \bar{\mathbf{b}} + \lambda\, \bar{\mathbf{b}}'.$$

A change in λ has no effect on the values of relative profit coefficients $\bar{c}_j$ *i.e.*, $\bar{c}_j$ values remain non-positive (maximization problem).

For a given basis matrix $\mathbf{B}$, values of $\bar{\mathbf{b}}$ and $\bar{\mathbf{b}}'$ can be calculated. The solution $\mathbf{X_B}=\bar{\mathbf{b}}+\lambda\,\bar{\mathbf{b}}'$ is feasible and optimal as long as $\bar{\mathbf{b}}+\lambda\,\bar{\mathbf{b}}'$ is $\geqslant 0$. In other words, for a given solution we can determine the range for λ within which the solution remains optimal.

EXAMPLE 6.6-2.1. Consider the linear programming problem

$$\text{maximize } Z=4x_1+6x_2+2x_3$$
$$\text{subject to } x_1+x_2+x_3\leqslant 3,$$
$$x_1+4x_2+7x_3\leqslant 9,$$
$$x_1,\ x_2,\ x_3\geqslant 0.$$

The optimal solution to this problem is given by

Table 6.79

c_j		4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b
4	x_1	1	0	-1	$\frac{4}{3}$	$\frac{-1}{3}$	1
6	x_2	0	1	2	$\frac{-1}{3}$	$\frac{1}{3}$	2
$E_j=\Sigma c_B a_{ij}$		4	6	8	$\frac{10}{3}$	$\frac{2}{3}$	
$\bar{c}_j=c_j-E_j$		0	0	-6	$\frac{-10}{3}$	$\frac{-2}{3}$	

Solve the problem if the variation right-hand-side vector $\mathbf{b}'=\begin{bmatrix}3\\-3\end{bmatrix}$. Perform complete parametric analysis and identify all critical values of parameter λ.

Solution. The given parametric right-hand-side problem is

$$\text{maximize } Z=4x_1+6x_2+2x_3+0\ s_1+0\ s_2,$$
$$\text{subject to } x_1+x_2+x_3+s_1=3+3\lambda,$$
$$x_1+4x_2+7x_3+s_2=9-3\lambda,$$
$$x_1,\ x_2,\ x_3,\ s_1,\ s_2\geqslant 0.$$

When $\lambda=0$, the problem reduces to the L.P. problem whose optimal solution is given by table 6.79. For values of λ other than zero, the values of right-hand-constants change because of the variation vector $\mathbf{b}'$. This is shown in the expanded table 6.80.

Table 6.80

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	$\bar{\mathbf{b}}$	$\bar{\mathbf{b}}'$
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1	5
6	x_2	0	1	2	$\left(-\frac{1}{3}\right)$	$\frac{1}{3}$	2	-2←Key row
	$\bar{c}_j$	0	0	-6	$-\frac{10}{3}$ ↑k	$-\frac{2}{3}$	$Z=16$	$Z'=8$

The vectors $\bar{\mathbf{b}}$ and $\bar{\mathbf{b}}$, are computed as follows :

$$\bar{\mathbf{b}}=\mathbf{B}^{-1}\mathbf{b}=\begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 \\ 9 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\bar{\mathbf{b}}=\mathbf{B}^{-1}\mathbf{b}'=\begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix}=\begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

For a fixed λ, the values of basic variables in table 6.80 are given by

$$x_1=\bar{b}_1+\lambda\,\bar{b}_1'=1+5\lambda,$$
$$x_2=\bar{b}_2+\lambda\,\bar{b}'_2=2-2\lambda.$$

$\bar{c}_j$ values are not affected as long as the basis consists of variables x_1 and x_2. As λ changes, values of basic variables x_1 and x_2 change and table 6.80 remains optimal as long as the basis (x_1, x_2) remains feasible. In other words, table 6.80 remains optimal as long as

$$x_1=1+5\,\lambda \geqslant 0 \quad \text{or} \quad \lambda \geqslant -\frac{1}{5},$$

$$x_2=2-2\,\lambda \geqslant 0 \quad \text{or} \quad \lambda \leqslant 1.$$

Therefore, table 6·80 remains optimal as λ varies from $-1/5$ to 1. Thus for all $-1/5 \leqslant \lambda \leqslant 1$, the optimal solution is given by

$$x_1=1+5\lambda,\ x_2=2-2\lambda,\ x_3=s_1=s_2=0,\ Z_{max}=16+8\lambda.$$

For $\lambda>1$, the basic variable x_2 becomes negative. Although this makes table 6·80 infeasible for the primal, it remains feasible for the dual since all $\bar{c}_j$ coefficients are non-positive. Dual simplex method can, therefore, be applied to find the new optimal solution for $\lambda>1$. Evidently x_2 is the variable that leaves the basis. The ratios of the nonbasic variables are -3, 10, -2. Thus variable s_1 is the entering variable. The key element $-1/3$ has been shown bracketed. Regular simplex method is now used to find the new optimal solution. In

table 6·81, the key element has been made unity.

Table 6·81

c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	$\bar{b}$	$\bar{b}'$
4	x_1	1	0	-1	$\frac{4}{3}$	$-\frac{1}{3}$	1	5
6	x_2	0	-3	-6	(1)	-1	-6	6

Replace x_2 by s_1.

Table 6·82

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	$\bar{b}$	$\bar{b}'$
4	x_1	1	4	7	0	1	9	-3
0	s_1	0	-3	-6	1	-1	-6	6
$E_j=\Sigma c_B a_{ij}$		4	16	28	0	4		
$\bar{c}_j=c_j-E_j$		0	-10	-26	0	-4		

The basic solution given by table 6·82 is

$$x_1=9-3\lambda, \quad x_2=0, \quad x_3=0, \quad s_1=-6+6\lambda, \quad s_2=0$$

$$Z_{max}=36-12\lambda.$$

This solution is optimal as long as the basic variables x_1 and s_1 remain non-negative *i.e.*, as long as

$$x_1=9-3\lambda \geqslant 0 \quad \text{or} \quad \lambda \leqslant 3,$$

$$s_1=-6+6\lambda \geqslant 0 \quad \text{or} \quad \lambda \geqslant 1.$$

Thus the above solution is optimal for all $1 \leqslant \lambda \leqslant 3$.

For $\lambda > 3$, the basic variable x_1 becomes negative. As there is no negative coefficient in the first row, the primal solution is infeasible. Hence there exists no optimal solution to the problem for all $\lambda > 3$.

For $\lambda \leqslant -\frac{1}{5}$, the basic variable x_1 in table 6·80 becomes negative. Although this makes table 6·80 infeasible for the primal, it remains feasible for the dual, since all c_j coefficients are nonpositive. Dual simplex method can, therefore, be applied to find the new optimal solution for $\lambda \leqslant -\frac{1}{5}$. Evidently x_1 is the variable that leaves the basis. The ratios of nonbasic variables are 6, $-\frac{5}{2}$, 2. Thus variable s_2 is the entering variable and $-\frac{1}{3}$ is the key element. This element is made unity in table 6·83.

Table 6·83

c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	$\bar{b}$	$\bar{b}'$
4	x_1	-3	0	3	-4	(1)	-3	-15
6	x_2	0	1	2	$-\frac{1}{2}$	$\frac{1}{2}$	2	-2
						Key element unity		

Replace x_1 by s_2.

Table 6·84

	c_j	4	6	2	0	0		
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	$\bar{b}$	$\bar{b}'$
0	s_2	-3	0	3	-4	1	-3	-15
6	x_2	1	1	1	1	0	3	3
$E_j=\Sigma c_B a_{ij}$		6	6	6	6	0		
$\bar{c}_j=c_j-E_j$		-2	0	-4	-6	0		

The basic solution given by table 6·84 is

$$x_1=0,\quad x_2=3+3\lambda,\quad x_3=0,\quad s_1=0,\quad s_2=-3-15\lambda$$

and $\quad Z_{max}=18+18\lambda.$

This solution is optimal so long as

$$x_2=3+3\lambda\geqslant 0 \qquad \text{or} \quad \lambda\geqslant -1,$$

$$s_2=-3-15\lambda\geqslant 0 \qquad \text{or} \quad \lambda\leqslant -\frac{1}{5}.$$

Thus the above solution is optimal for all $-1\leqslant\lambda\leqslant-\frac{1}{5}$.

For $\lambda<-1$, the basic variable x_2 in table 6·84 becomes negative. As there is no negative coefficient in the second row, the primal solution is infeasible. Hence there exists no optimal solution to the problem for all $\lambda<-1$.

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EXERCISES

Section 6·1·1.

1. (*a*) Explain duality theory of linear programming.

[*Bangalore Univ. B.E. July, 1978*]

(*b*) Write down the dual of the following linear programming problem :

$$\begin{aligned} &\text{minimize } Z=7x_1+3x_2+8x_3,\\ &\text{subject to} \quad 8x_1+2x_2+x_3\geqslant 3,\\ &\qquad 3x_1+6x_2+4x_3\geqslant 4,\\ &\qquad 4x_1+x_2+5x_3\geqslant 1,\\ &\qquad x_1+5x_2+2x_3\geqslant 7,\\ &\qquad x_1, x_2, x_3\geqslant 0. \end{aligned}$$

[*Delhi B.Sc. (Math.) 1974*]

(*Ans.* Maximize $W=3y_1+4y_2+y_3+7y_4$,

$$\begin{aligned} \text{subject to} \quad &8y_1+3y_2+4y_3+y_4\leqslant 7,\\ &2y_1+6y_2+y_3+5y_4\leqslant 3,\\ &y_1+4y_2+5y_3+2y_4\leqslant 8,\\ &y_1, y_2, y_3, y_4, \text{ all } \geqslant 0.) \end{aligned}$$

2. Write the dual of the problem

$$\begin{aligned} &\text{maximize } Z=2x_1+5x_2+3x_3,\\ &\text{subject to} \quad 2x_1+4x_2-x_3\leqslant 8,\\ &\qquad -2x_1-2x_2+3x_3\geqslant -7,\\ &\qquad x_1+3x_2-5x_3\geqslant -2,\\ &\qquad 4x_1+x_2+3x_3\geqslant 4,\\ &\qquad x_1, x_2, x_3\geqslant 0. \end{aligned}$$

(*Ans.* Minimize $W=8y_1+7y_2+2y_3+4y_4$,

$$\begin{aligned} \text{subject to} \quad &2y_1+2y_2-y_3+4y_4\geqslant 2,\\ &4y_1+2y_2-3y_3+y_4\geqslant 5,\\ &-y_1-3y_2+5y_3+3y_4\geqslant 3,\\ &y_1, y_2, y_3, y_4\geqslant 0.) \end{aligned}$$

Section 6·1·2.

3. Construct the dual of the problem

$$\text{maximize } Z=4x_1+5x_2+12x_3,$$
$$\text{subject to} \quad 2x_1+x_2+x_3\leqslant 4,$$
$$3x_1-2x_2+x_3=3,$$
$$x_1, x_2, x_3, \geqslant 0.$$

(*Ans.* Minimize $W=4y_1+3y_2$,
subject to $2y_1+3y_2\geqslant 4$,
$y_1-2y_2\geqslant 5$,
$y_1+y_2\geqslant 12$,
$y_1\geqslant 0$, y_2 unrestricted in sign.)

4. Construct the dual of the linear programming problem

$$\text{minimize } Z=10x_1-6x_2-8x_3,$$
$$\text{subject to} \quad x_1-3x_2+x_3=5,$$
$$-2x_1+x_2+3x_3=8,$$
$$x_1, x_2, x_3\geqslant 0.$$

(*Ans.* Maximize $W=5y_1+8y_2$,
subject to $y_1-2y_2\leqslant 10$,
$-3y_1+y_2\leqslant -6$,
$y_1+3y_2\leqslant -8$,
y_1, y_2 unrestricted in sign.)

5. Obtain the dual of the problem

$$\text{minimize } Z=x_3+x_4+x_5,$$
$$\text{subject to} \quad x_1-x_3+x_4-x_5=-2,$$
$$x_2-x_3-x_4+x_5=1,$$
$$x_1, x_2, x_3, x_4, x_5\geqslant 0.$$

[*Gauhati M.Sc.* (*Stat.*) *1975*]

(*Ans.* Maximize $W=-2y_1+y_2$,
subject to $-y_1-y_2\leqslant 1$,
$y_1-y_2\leqslant 1$,
$-y_1+y_2\leqslant 1$,
y_1, y_2 unrestricted in sign.)

6. Construct the dual of the problem

$$\text{maximize } Z=6x_1+4x_2+6x_3+x_4,$$

subject to the constraints

$$4x_1+4x_2+4x_3+8x_4=21,$$
$$3x_1+17x_2+80x_3+2x_4\leqslant 48,$$
$$x_1, x_2\geqslant 0,$$
$$x_3, x_4 \text{ are unrestricted.}$$

[*Delhi M.Sc.* (*Math.*) *1972*]

(*Ans.* Minimize $W=21y_1+48y_2$,
subject to
$$4y_1+3y_2 \geqslant 6,$$
$$4y_1+17y_2 \geqslant 4,$$
$$4y_1+80y_2=6,$$
$$8y_1+2y_2=1,$$
y_1 unrestricted in sign, $y_2 \geqslant 0$)

7. Construct the dual of the problem

Maximize $Z=4x_1+2x_2$,
subject to
$$x_1-2x_2 \geqslant 2,$$
$$x_1+2x_2=8,$$
$$x_1-x_2 \leqslant 10,$$
$x_1 \geqslant 0$, x_2 unrestricted in sign.

(*Ans.* Minimize $W=-2y_1+8y_2+10y_3$,
subject to
$$-y_1+y_2+y_3 \geqslant 4,$$
$$-2y_1+2y_2-y_3=10,$$
$y_1, y_3 \geqslant 0$, y_2 unrestricted in sign)

Section 6·1-4.

8. Construct the dual of the following L.P.P. and solve both the primal and the dual :

Minimize $Z=4x_1+2x_2+3x_3$,

subject to the constraints
$$2x_1 \quad +4x_3 \geqslant 5,$$
$$2x_1+3x_2+x_3 \geqslant 4,$$
$$x_1, x_2, x_3 \geqslant 0.$$

$\left(\textit{Ans.}\right.$ Primal : $x_1=0,\ x_2=\frac{11}{12},\ x_3=\frac{5}{4};\ Z_{min}=\frac{67}{12};$

Dual : $y_1=\frac{7}{12},\ y_2=\frac{2}{3};\ W_{max}=\frac{67}{12}$ $\left.\right)$

9. Construct the dual of the following L.P.P. and solve both the primal and the dual :

Maximize $Z=5x_1+12x_2+4x_3$,
subject to
$$x_1+2x_2+x_3 \leqslant 5,$$
$$2x_1-x_2+3x_3=2,$$
$$x_1, x_2, x_3 \geqslant 0.$$

$\left(\textit{Ans.}\right.$ Primal : $x_1=\frac{9}{5},\ x_2=\frac{8}{5},\ x_3=0;\ Z_{max}=\frac{141}{5};$

Dual : $y_1=\frac{29}{5},\ y_2=y_2'-y_2''=0-\frac{2}{5}$
$$=-\frac{2}{5};\ W_{min}=\frac{141}{5} \quad \left.\right)$$

10. Find the dual of the following set of inequations and solve it :

$$2x_1+3x_2\leqslant 12,$$
$$-3x_1+2x_2\leqslant -4,$$
$$3x_1-5x_2\leqslant 2,$$
$$x_1 \text{ unrestricted}, \; x_2\geqslant 0.$$

$$\left(\textit{Ans.} \quad x_1=\frac{36}{13},\; x_2=\frac{28}{13}\right)$$

[*Hint* : Add the objective function
maximize $Z=0x_1+0x_2$.]

11. Using duality, find the optimal solution to the problem

maximize $Z=3x_1-2x_2$
subject to $x_1+x_2\leqslant 5$,
$-x_2\leqslant -1$,
$0\leqslant x_1\leqslant 4$,
$0\leqslant x_2\leqslant 6$.

[*Meerut M.Sc. (Math.) 1973*]

$$\left(\textit{Ans.} \quad x_1=\frac{5}{19},\; x_2=\frac{16}{19};\; Z_{min}=\frac{235}{19}\right)$$

12. Apply the simplex method to solve the following :

maximize $Z=30x_1+23x_2+29x_3$,
subject to $6x_1+5x_2+3x_3\leqslant 26$,
$4x_1+2x_2+5x_3\leqslant 7$,
every $x_j\geqslant 0$.

Also read the solution to the dual of the above problem from the final table.

[*Agra M. Stat. 1973*]

$$\left(\textit{Ans.} \quad \text{Primal}: x_1=0,\; x_2=\frac{7}{2},\; x_3=0;\; Z_{max}=\frac{161}{2};\right.$$
$$\left.\text{Dual} \;: y_1=0,\; y_2=\frac{23}{2};\; W_{min}=\frac{161}{2}\right)$$

13. Solve the following primal. Also find its dual and solve it.

maximize $Z=10y_1-y_2-9y_3+8y_4$,
subject to $2y_1-y_2-3y_3-y_4+2=0$,
$5y_1-2y_2-3y_4+5=0$,
$-7y_1+4y_2-y_3-4y_4+1\geqslant 0$,
$-3y_1-2y_2-5y_3-6y_4+10\geqslant 0$,
y_1, y_2, y_3, y_4 all $\geqslant 0$.

$$\left(\textit{Ans.} \quad \text{Primal}: y_1=\frac{1}{7},\; y_2=\frac{8}{7},\; y_3=0,\; y_4=\frac{8}{7};\; Z_{max}=-\frac{62}{7};\right.$$
$$\text{Dual} \;: x_1=\frac{111}{7},\; x_2=-\frac{57}{7},\; x_3=\frac{1}{7},\; x_4=0;$$
$$\left.W_{min}=-\frac{62}{7}.\right)$$

14. (*a*) What is duality ? What is the significance of dual variables in simplex solution ?

(*b*) In order to produce 1,000 tonnes of non-oxidising steel for engine valves, at least the following quantities of manganese (Mn), chromium (Cr) and molybdenum (Mb) are needed :

Mn 50 kg, Cr 60 kg and Mb 70 kg.

These metals are available in packages A, B and C having different proportions of Mn, Cr and Mb and also differing in prices as under :

Contents in kg. per packet

Package	*Mn*	*Cr*	*Mb*	*Price per packet (Rs.)*
A	10	10	5	45
B	10	15	5	60
C	5	5	25	75

How many packets of packages A, B and C should be purchased to minimize cost ? What is the possible cost ? [Solve through **DUAL** only].

[*Gujarat Univ. B.E. April, 1976*]

15. (*a*) What is meant by dual problem of L.P. model ?

(*b*) Consider the problem

$$\text{maximize } Z=5x_1+8x_2,$$

$$\text{subject to } \quad x_1+x_2\leqslant 2,$$
$$x_1-2x_2\leqslant 0,$$
$$-x_1+4x_2\leqslant 1,$$
$$x_1, x_2\geqslant 0.$$

What is the dual of the above problem ? Find the solution of the primal problem by solving its dual.

[*P.U. Prod. Engg. April, 1979*]

Section 6.2.

16. Solve the following linear programming problem using dual simplex method :

$$\text{maximize } Z=-3x_1-x_2,$$

$$\text{subject to } \quad x_1+x_2\geqslant 1,$$
$$2x_1+3x_2\geqslant 2,$$
$$x_1, x_2\geqslant 0.$$

[*Bangalore Univ. B.E. July, 1978*]

17. Use dual simplex method to

$$\text{minimize } Z=2x_1+x_2,$$
$$\text{subject to } \quad 3x_1+x_2\geqslant 3,$$
$$4x_1+3x_2\geqslant 6,$$
$$x_1+2x_2\leqslant 3,$$
$$x_1,\ x_2\geqslant 0.$$

$$\left(\textit{Ans.}\quad x_1=\frac{3}{5},\ x_2=\frac{6}{5};\ Z_{min}=\frac{12}{5}\right)$$

18. Solve the following problem by dual simplex method :

$$\text{Minimize } Z=20x_1+16x_2,$$
$$\text{subject to } \quad x_1+x_2\geqslant 12,$$
$$2x_1+x_2\geqslant 17,$$
$$x_1\geqslant 2\cdot 5,$$
$$x_2\geqslant 6,$$
$$x_1,\ x_2\geqslant 0.$$

(*Ans.* $x_1=5,\ x_2=7;\ Z_{min}=212$)

19. Solve by dual simplex method the problem

$$\text{minimize } Z=10x_1+6x_2+2x_3,$$
$$\text{subject to } \quad -x_1+x_2+x_3\geqslant 1,$$
$$3x_1+x_2-x_3\geqslant 2,$$
$$x_1,\ x_2,\ x_3\geqslant 0.$$

[*Roorkee B.E. 1973*]

$$\left(\textit{Ans.}\quad x_1=\frac{1}{4},\ x_2=\frac{5}{4},\ x_3=0;\ Z_{min}=10\right)$$

20. Use dual simplex method to solve

$$\text{minimize } Z=5x_1+6x_2+3x_3,$$
$$\text{subject to } \quad 5x_1+5x_2+3x_3\geqslant 50,$$
$$x_1+x_2-x_3\geqslant 20,$$
$$7x_1+6x_2-9x_3\geqslant 30,$$
$$5x_1+5x_2+5x_3\geqslant 35,$$
$$2x_1+4x_2-15x_3\geqslant 10,$$
$$12x_1+10x_2\geqslant 90,$$
$$x_2-10x_3\geqslant 20,$$
$$x_1,\ x_2,\ x_3\geqslant 0.$$

Section 6.3.

21. Use revised simplex method to solve the following problem :

$$\text{Minimize } Z=2x_1+x_2,$$
$$\text{subject to } 3x_1+x_2=3,$$
$$4x_1+3x_2\geqslant 6,$$
$$x_1+2x_2\leqslant 3,$$
$$x_1, x_2\geqslant 0.$$

(*Ans.* $x_1=3/5$, $x_2=6/5$, $Z_{min}=12/5$)

22. Solve the following problem by the revised simplex method :

$$\text{Maximize } x_0=6x_1-2x_2+3x_3,$$
$$\text{subject to } 2x_1-x_2+2x_3\leqslant 2,$$
$$x_1+4x_3\leqslant 4,$$
$$x_1, x_2, x_3\geqslant 0.$$

[*Roorkee M.E. (Elect.) 1977*]

(*Ans.* $x_1=4$, $x_2=6$, $x_3=0$; $x_{0max}=12$)

23. Solve by the revised simplex method, the problem

$$\text{maximize } Z=x_1+x_2+3x_3,$$
$$\text{subject to constraints } 3x_1+2x_2+x_3\leqslant 3,$$
$$2x_1+x_2+2x_3\leqslant 2,$$
$$x_1, x_2, x_3\geqslant 0.$$

[*Meerut M. Sc. (Math.) 1975, 1977*]

(*Ans.* $x_1=0$, $x_2=0$, $x_3=1$; $Z_{max}=3$)

24. Use the revised simplex method to solve the problem

$$\text{maximize } Z=30x_1+23x_2+29x_3,$$
$$\text{subject to } \quad 6x_1+5x_2+3x_3\leqslant 26,$$
$$4x_1+2x_2+5x_3\leqslant 7,$$
$$x_1, x_2, x_3\geqslant 0.$$

[*Agra M. Stat. 1973*]

(*Ans.* $x_1=0$, $x_2=7/2$, $x_3=0$; $Z_{max}=161/2$)

25. Use the revised simplex method to

$$\text{minimize } Z=2x_1+3x_2+2x_3-x_4+x_5,$$
$$\text{subject to } \quad 3x_1-3x_2+4x_3+2x_4-x_5=0,$$
$$x_1+x_2+x_3+3x_4+x_5=2,$$
$$x_1, \ldots\ldots, x_5\geqslant 0.$$

[*Ans.* $x_1 = x_2 = x_3 = 0$ (non-basic variables),
$x_4 = 2/5$, $x_5 = 4/5$ basic variables),
$Z_{min} \quad = 2/5$]

Section 6.4.

26. Solve the following problem by lower and upper bounding technique.

$$\text{maximize } Z = 4y_1 + 4y_2 + 3y_3,$$
$$\text{subject to } -y_1 + 2y_2 + 3y_3 \leqslant 15,$$
$$-y_2 + y_3 \leqslant 4,$$
$$2y_1 + y_2 - y_3 \leqslant 6,$$
$$y_1 - y_2 + 2y_3 \leqslant 10,$$
$$0 \leqslant y_1 \leqslant 8,\ 0 \leqslant y_2 \leqslant 4,\ 0 \leqslant y_3 \leqslant 4.$$

(*Ans.* $y_1 = 17/5$, $y_2 = 16/5$, $y_3 = 4$; $Z_{max} = 192/5$)

27. Solve the following linear programming problem by using bounded variable simplex method :

$$\text{maximize } Z = 4y_1 + 2y_2 + 6y_3,$$
$$\text{subject to } \quad 4y_1 - y_2 \leqslant 9,$$
$$-y_1 + y_2 + 2y_3 \leqslant 8,$$
$$-3y_1 + y_2 + 4y_3 \leqslant 12,$$
$$1 \leqslant y_1 \leqslant 3,\ 0 \leqslant y_2 \leqslant 5,\ 0 \leqslant y_3 \leqslant 2.$$

(*Ans.* $y_1 = 3$, $y_2 = 5$, $y_3 = 2$; $Z_{max} = 34$).

Section 6.5

28. What do you understand by the term sensitivity analysis ? Discuss briefly the effect of

(*i*) variation of the b_i,

(*ii*) variation of the c_j.

[*Delhi B.Sc.* (*Math.*) *1977*]

Section 6.5-1

29. Consider the L.P.P.

$$\text{maximize } Z = 2y_2 - 5y_3,$$
$$\text{subject to } \quad y_1 + y_3 \geqslant 2,$$
$$2y_1 + y_2 + 6y_3 \leqslant 6,$$
$$y_1 - y_2 + 3y_3 = 0,$$
$$y_1, y_2, y_3 \geqslant 0.$$

(*a*) Solve the linear programming problem.

(*b*) If the right-hand-side of the primal is changed from [2, 6, 0] to [2, 10, 5], find the new optimal solution.

[*Pb. Univ. M B.A. 1976*]

(*Ans.* (a) $y_1=0$, $y_2=2$, $y_3=0$; $Z_{max}=4$;
(b) $y_1=0$, $y_2=0$, $y_3=0$, $Z_{max}=0$)

30. (a) Describe the role of duality for sensitivity analysis of an L.P. problem.

(b) Consider the problem

$$\text{maximize } Z=5x_1+3x_2+7x_3,$$

$$\text{subject to } \quad x_1+x_2+2x_3\leqslant 22,$$
$$3x_1+2x_2+x_3\leqslant 26,$$
$$x_1+x_2+x_3\leqslant 18,$$
$$x_1, x_2, x_3 \geqslant 0.$$

What will be the solution if the first constraint changes to

$$x_1+x_2+2x_3\leqslant 26\ ?$$

(*P.U. Prod. Engg. Nov., 1977*)

31. Consider the problem

$$\text{maximize } Z=5x_1+2x_2+3x_3,$$

$$\text{subject to } \quad x_1+5x_2+2x_3\leqslant b_1,$$
$$x_1-5x_2-6x_3\leqslant b_2,$$
$$x_1, x_2, x_3 \geqslant 0,$$

where b_1 and b_2 are constants. For specific values of b_1 and b_2, the optimal solution is

Table 6.85

	c_j	5	2	3	0	0	
c_B	c.s.v.	x_1	x_2	x_3	s_1	s_2	b_i
5	x_1	1	b	2	1	0	30
0	s_1	0	c	-8	-1	1	10
$\bar{c}_j=c_j-E_j$		0	$-a$	-7	$-d$	$-e$	

where a, b, c, d and e are constants. Determine

(a) The values of b_1 and b_2 that yield the given optimal solution.

(b) The optimal dual solution.

(c) The values of a, b and c in the optimal table.

(d) If it is required to increase optimum Z, should b_1 or b_2 be increased and by how much ?

(*Pb. Univ. M.Sc. Engg. 1977*)

(*Ans.* (*a*) $b_1=30$, $b_2=40$.

(*b*) $y=d=5$, $y_2=e=0$.

(*c*) $a=23$, $b=5$, $c=-10$.

(*d*) b_1 to be increased up to 10 units.)

Section 6.5-2

32. (*a*) Solve the problem

$$\begin{aligned}
\text{maximize } & Z=x_1+1.5x_2,\\
\text{subject to } & 2x_1+2x_2 \leqslant 160,\\
& x_1+2x_2 \leqslant 120,\\
& 4x_1+2x_2 \leqslant 280,\\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

(*b*) Over what values of profit for x_2 will the present solution be still optimal ?

(*c*) Determine the optimal range for c_1.

(*Ans.* (*a*) $x_1=40$, $x_2=40$; $Z_{max}=100$.

(*b*) $1 \leqslant c_2 \leqslant 2$.

(*c*) $\frac{3}{4} \leqslant c_1 \leqslant \frac{3}{2}$)

33. Solve the problem

(*a*) maximize $Z=45x_1+100x_2+30x_3+50x_4$,

subject to $7x_1+10x_2+4x_3+9x_4 \leqslant 1{,}200$,

$3x_1+40x_2+x_3+x_4 \leqslant 800$,

$x_1, x_2, x_3, x_4 \geqslant 0$.

(*b*) Find the effect of

(*i*) changing the cost coefficients c_1 and c_4 from 45 and 50 to 40 and 60 respectively.

(*ii*) changing c_1 to 40 and c_2 to 90.

(*iii*) changing c_3 from 30 to 24.

(*Ans.* (*a*) $x_1=0$, $x_2=40/3$, $x_3=800/3$, $x_4=0$;

$$Z_{max}=\frac{28{,}000}{3}.$$

(*b*) (*i*) Same as in part (*a*).

(*ii*) $x_1=0$, $x_2=40/3$, $x_3=800/3$, $x_4=0$;

$$Z_{max}=\frac{27{,}600}{3}$$

(*iii*) $x_1=160$, $x_2=8$, $x_3=0$, $x_4=0$; $Z_{max}=8{,}000$)

34. Consider the problem

$$\begin{aligned} \text{maximize } Z &= 2x_2 - 5x_3, \\ \text{subject to } x_1 \quad\; + x_3 &\geqslant 2, \\ 2x_1 + x_2 + 6x_3 &\leqslant 6, \\ x_1 - x_2 + 3x_3 &= 0, \\ x_1, x_2, x_3 &\geqslant 0. \end{aligned}$$

(*a*) Write the dual from the standard form.

(*b*) Solve the primal and hence find the solution to the dual.

(*c*) Suppose that the coefficients of x_2 and x_3 in the objective function are changed from (2, −5) to (1, 1), find the new solution.

(*Ans.* (*a*) Minimize $W = 2y_1 + 6y_2$,

$$\begin{aligned} \text{subject to } y_1 + 2y_2 + y_3 &\geqslant 0, \\ y_2 - y_3 &\geqslant 2, \\ y_1 + 6y_2 + 3y_3 &\geqslant -5, \\ y_1, y_2 \geqslant 0, \; y_3 &\text{ unrestricted.} \end{aligned}$$

(*b*) $y_1 = 0, y_2 = 2/3, y_3 = -4/3, W_{min} = 4$.

(*c*) Unbounded.)

35. (*a*) Solve the problem

$$\begin{aligned} \text{maximize } Z &= x_1 + 5x_2 + 3x_3, \\ \text{subject to } x_1 + 2x_2 + x_3 &= 3, \\ 2x_1 - x_2 \qquad &= 4, \\ x_1, x_2, x_3 &\geqslant 0. \end{aligned}$$

(*b*) If the objective funct on is changed to

maximize $Z = 2x_1 + 5x_2 + 2x_3$, find the new optimal solution.

(*Ans.* (*a*) $x_1 = 2, x_2 = 0, x_3 = 1$; $Z_{max} = 5$.

(*b*) $x_1 = 11/5, x_2 = 2/5, x_3 = 0$; $Z_{max} = 32/5$)

36. For the problem

$$\begin{aligned} \text{minimize } Z &= x_2 - 3x_3 + 2x_5, \\ \text{subject to } 3x_2 - x_3 + 2x_5 &\leqslant 7, \\ -2x_2 + 4x_3 &\leqslant 12, \\ -4x_2 + 3x_3 + 8x_5 &\leqslant 10, \\ x_2, x_3, x_5 &\geqslant 0, \end{aligned}$$

the optimal table is

Table 6·86

	c_j	0	1	-3	0	2	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	x_6	b
1	x_2	$\frac{2}{5}$	1	0	$\frac{1}{10}$	$\frac{4}{5}$	0	4
-3	x_3	$\frac{1}{5}$	0	1	$\frac{3}{10}$	$\frac{2}{5}$	0	5
0	x_6	1	0	0	$-\frac{1}{2}$	10	1	11
	$\bar{c}_j$	$\frac{1}{5}$	0	0	$\frac{4}{5}$	$\frac{12}{5}$	0	

(*a*) Formulate the dual problem for this primal problem.

(*b*) What are the optimal values of dual variables ?

(*c*) How much must c_5 be decreased before x_5 goes into solution ?

(*d*) How much can the 7 in first constraint be increased before the basis would change ?

[*Dibrugarh M.Sc.* (*Stat.*) *1976* ; *Roorkee M.E.* (*Mech.*) *1977*]

[*Ans.* (*a*) Maximize $W=7y_1+12y_2+10y_3$,

subject to $-3y_1+2y_2+4y_3\leqslant 1$,

$-y_1+4y_2+3y_3\geqslant 3$,

$-2y_1-8y_3\leqslant 2$,

$y_1, y_2\geqslant 0$.

(*b*) $y_1=\frac{1}{5}, y_2=\frac{4}{5}, y_3=0$.

(*c*) $\triangle c_5> \frac{12}{5}$

(*d*) $\triangle b_1 <-1$]

Sections 6.5-3 to 6.5-4 :

37. Consider the problem

maximize $Z=3x_1+2x_2+5x_3$,

subject to $x_1+2x_2+x_3\leqslant 430$,

$3x_1+2x_3\leqslant 460$,

$x_1+4x_2\leqslant 420$,

$x_1, x_2, x_3\geqslant 0$.

The optimal solution to this problem is given by the following table :

Table 6·87

c_B	c_j / c.s.v.	3 / x_1	2 / x_2	5 / x_3	0 / s_1	0 / s_2	0 / s_3	b
2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	s_3	2	0	0	-2	1	1	
$E_j=\Sigma c_B a_{ij}$		7	2	5	1	2	0	
$\bar{c}_j=c_j-E_j$		-4	0	0	-1	-2	0	

Suppose that the constraint coefficients of 'x_1'-column are changed from (1, 3, 1) to (1, 1, 6) in the starting matrix and profit coefficients of x_2 and x_3 are changed from (2, 5) to (1, 3), find the new optimal solution.

[*Ans.* $x_1=4,\ x_2=99,\ x_3=228;\ Z_{max}=795$]

Section 6.5-5 :

38. A firm produces three items A, B and C and requires two types of resources—manhours and raw material. The following L. P. problem has been formulated to determine the optimum production schedule that maximizes the total profit :

Maximize $Z=3y_1+y_2+5y_3$,

subject to $6y_1+3y_2+5y_3 \leqslant 45$ (manhours),

$3y_1+4y_2+5y_3 \leqslant 30$ (raw material)

$y_1, y_2, y_3 \geqslant 0$,

where y_1, y_2, y_3 are the number of items A, B and C. The optimal solution with y_4 and y_5 as slack variables is

Table 6.88

c_B	c_j / c.s.v.	3 / y_1	1 / y_2	5 / y_3	0 / y_4	0 / y_5	b
3	y_1	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	5
5	y_3	0	1	1	$-\frac{1}{5}$	$\frac{2}{5}$	3
$\bar{c}_j=c_j-E_j$		0	-3	0	0	-1	

(*a*) Find the range on the unit profit of product *A*. If $c_1=4$, what is the optimal solution ?

(*b*) If additional 10 units of raw material can be obtained at a cost of Rs. 12, is it profitable to do so ?

(*c*) If the available raw material is increased to 50 units, what is the optimal solution ?

(*d*) Due to 'technological breakthrough' the raw material required by item B is reduced to 2 units. Will it affect the optimal solution ?

(*e*) If a supervision constraint, $2y_1+y_2+3y_3 \leqslant 20$ is added to the original problem, how is the optimal solution affected ?

39. Consider the following table which presents an optimum solution to some linear programming problem.

Table 6·89

	c_j	2	4	1	3	2	0	0	0	
c_B	*c.s.v.*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
2	x_1	1	0	0	-1	0	$\frac{1}{2}$	$\frac{1}{5}$	-1	3
4	x_2	0	1	0	2	1	-1	0	$\frac{1}{2}$	1
1	x_3	0	0	1	-1	-2	5	$-\frac{3}{10}$	2	7
$\bar{c}_j=c_j-E_j$		0	0	0	-2	0	-2	$-\frac{1}{10}$	-2	

If an additional constraint $2x_1+3x_2-x_3+2x_4+4x_5 \leqslant 5$ were annexed to the system, will there be any change in the optimal solution ? Justify your answer.

[*Meerut B.Sc.* (*Math.*) *1972*]
(*Ans.* No)

40. A manufacturer produces four products A, B, C and D by using two types of machines (lathes and milling machines). The times required on the two machines to manufacture one unit of each of the four products, the profit per unit of the product and the total time available on the two types of machines per day are given below.

Table 6·90

Machines	*Product time required per unit (minutes)*				*Total time available per day (minutes)*
	A	B	C	D	
Lathe	4	9	7	10	5,500
Milling machine	2	1	3	20	3,500
Profit/unit (Rs.)	15	25	25	65	

(*a*) Find the number of units of the various products to be produced for maximizing profit.

(*b*) Find the effect of increasing the profit per unit of product C to Rs. 30.

(*c*) Find the effect of changing the profit per unit of product of A and B to Rs. 10 and Rs. 30 respectively.

(*d*) Find the effect of changing the total time available per day on the two machines to 3,500 and 5,500 minutes respectively.

(*e*) If a new product E, which requires 7 minutes/unit on lathe and 4 minutes/unit on milling machine can also be produced, will it be worthwhile to produce it if it brings a profit of Rs. 30 per unit ?

(*f*) If products A, B, C, D require 3, 4, 5 and 2 minutes/unit respectively on grinding machine in addition to the present operations, find the optimal solution. The total time available per day on grinding machine is 3,000 minutes.

(*g*) If product A requires 3 minutes on lathe and 3 minutes on milling machine (instead of 4 and 2 minutes respectively) per unit, find the new optimum solution.

41. Consider the linear programming model

$$\text{maximize } Z=\sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}\, x_{ij},$$

$$\text{subject to} \quad \sum_{i=1}^{4} x_{ij}=b_j \text{ for } j=1, 2, 3, 4,$$

$$\sum_{j=1}^{4} x_{ij}=b_i \text{ for } i=1, 2, 3, 4.$$

Expand the objective function and the constraints and write them in detail. Write the dual of the above problem. Also describe how the dual problem can be used for sensitivity analysis.

[*P.U. Prod. Engg. April, 1977*)

42. (*a*) Describe the role of duality for sensitivity analysis of an L.P. problem.

(*b*) Consider the problem

$$\text{maximize } Z = 5x_1 + 3x_2 + 7x_3,$$
$$\text{subject to} \quad x_1 + x_2 + 2x_3 \leqslant 22,$$
$$x_1, x_2, x_3 \geqslant 0.$$

Section 6.6-1.

43. Consider the parametric linear programming problem

$$\text{maximize } Z = (\theta - 1)x_1 + x_2,$$
$$\text{subject to} \quad x_1 + 2x_2 \leqslant 10,$$
$$2x_1 + x_2 \leqslant 11,$$
$$x_1 - 2x_2 \leqslant 3,$$
$$x_1, x_2 \geqslant 0.$$

Perform a complete parametric programming analysis. Identify all critical values of the parameter θ and all optimal basic solutions.

[*Sambalpur M.Sc. (Math.) 1977*]

(*Ans.* $x_1 = 0,\ x_2 = 5$ for $0 \leqslant \theta \leqslant \frac{3}{2}$;

$x_1 = 4,\ x_2 = 3$ for $\frac{3}{2} \leqslant \theta \leqslant 3$;

$x_1 = 5,\ x_2 = 1$ for $3 \leqslant \theta$.)

44. Consider the parametric problem

$$\text{maximize } Z = (3 + 3\theta)\, x_1 + 2x_2 + (5 - 6\theta)\, x_3,$$
$$\text{subject to } x_1 + 2x_2 + x_3 \leqslant 430,$$
$$3x_1 \qquad + 2x_3 \leqslant 460,$$
$$x_1 + 4x_2 \qquad \leqslant 420,$$
$$x_1, x_2, x_3 \geqslant 0,$$

where θ is a non-negative parameter.

[*Ans.* For $0 \leqslant \theta \leqslant \frac{1}{3}$, $(x_1, x_2, x_3) = (0, 100, 230)$;

$Z_{max} = 1,350 - 1,380\ \theta$;

for $\frac{1}{3} \leqslant \theta \leqslant \frac{5}{12}$, $(x_1, x_2, x_3) = (10, 102.5, 215)$;

$Z_{max} = 1310 - 1260\ \theta$;

for $\theta \geqslant \frac{5}{12}$, $(x_1, x_2, x_3) = \left(\frac{460}{3}, \frac{200}{3}, 0\right)$;

$Z_{max} = \frac{1,780}{3} + 460\ \theta.$]

45. Solve the parametric cost problem

$$\text{minimize } Z=(2+\lambda)x_1+(1+4\lambda)x_2,$$
$$\text{subject to} \quad 3x_1+x_2 \geqslant 3,$$
$$4x_1+3x_2 \geqslant 6,$$
$$x_1+2x_2 \leqslant 3,$$
$$x_1, x_2 \geqslant 0,$$

ane λ is a non-negative parameter.

46. Perform a complete parametric programming analysis of the following L.P. problem :

$$\text{minimize } Z=\lambda x-y,$$
$$\text{subject to } 3x-y \geqslant 5,$$
$$2x+y \leqslant 3,$$
$$-\infty \leqslant \lambda \leqslant \infty.$$

[*Meerut M.Sc. (Math.) 1973*]

(*Ans.* For $-2 \leqslant \lambda \leqslant 3$, $x=\frac{8}{5}$, $y=-\frac{1}{5}$; $Z_{min}=\frac{1}{5}+\frac{8}{5}\lambda$;

for $\lambda=3$, a multiple solution exists).

47. The following table gives an optimal solution to a linear programming problem :

Table 6.91

	c_j	4	6	2	0	0	
c_B	c.s.v.	x_1	x_2	x_3	x_4	x_5	b
4	x_1	1	0	1	3	-1	1
6	x_2	0	1	1	-1	2	2
	$\bar{c}_j$	0	0	-8	-6	-8	$Z=16$

where x_4 and x_5 are slack variables.

(*a*) How much can c_3 be increased before the current solution becomes non-optimal ? Find an optimal solution when $c_3=12$.

(*b*) Find the range on c_1 for the given basis to be optimal.

(*c*) Find the range on b_2 for the given basis to be optimal.

(*d*) Find the optimal solution by dual simple method when b_2 is increased by 2 units.

(*e*) Find the range on λ for which the given solution is still optimum if $\mathbf{C}$ is replaced by $\mathbf{C}+\lambda\mathbf{C}'$, where $\mathbf{C}'=(0, 0, 1, -1, 2)$ and $-\infty \leqslant \lambda \leqslant \infty$.

Section 6·6-2

48. Solve the problem

$$\begin{aligned}
\text{maximize } Z &= 3x_1+2x_2+5x_3,\\
\text{subject to } \quad & x_1+2x_2+x_3 \leqslant 430+500\,\lambda,\\
& 3x_1 \quad +2x_3 \leqslant 460+100\,\lambda,\\
& x_1+4x_2 \quad \leqslant 420-200\,\lambda,\\
& x_1,\, x_2,\, x_3 \geqslant 0,
\end{aligned}$$

where λ is a non-negative parameter.

(*Ans.* For $0 \leqslant \lambda \leqslant \frac{1}{55}$, $(x_1, x_2, x_3)=(0, 100+225\,\lambda, 230+50\,\lambda)$ and $Z_{max}=1350+700\lambda$;

for $\frac{1}{55} \leqslant \lambda \leqslant 2.1$, $(x_1, x_2, x_3)=(0, 105-50\,\lambda, 230+50\,\lambda)$ and $Z_{max}=1360+150\,\lambda$;

for $\lambda > 2.1$, no feasible solution exists.)

49. Minimize $Z=4y_1+y_2$,

$$\begin{aligned}
\text{subject to } \quad & 3y_1+y_2=3+3\theta,\\
& 4y_1+3y_2 \geqslant 6+2\theta,\\
& y_1+2y_2 \leqslant 3+4\theta,\\
& y_1,\, y_2,\, \theta \geqslant 0.
\end{aligned}$$

50. Given the L.P. problem

$$\begin{aligned}
\text{maximize } Z &= 7y_1+4y_2+6y_3+5y_4,\\
\text{subject to } \quad & 2y_1+y_2+2y_3+y_4 \leqslant 6+\theta,\\
& y_1-2y_2+2y_3+4y_4 \leqslant 20-\theta,\\
& 3y_1+y_2-3y_3+2y_4 \leqslant 40-\theta,\\
& y_1,\, y_2,\, y_3,\, y_4 \geqslant 0,
\end{aligned}$$

perform a complete parametric programming analysis and identify all the critical values of the parameter θ.

51. For problem 47, find the range on λ for which the given basis (x_1, x_2) is still optimal if the original **b** vector is replaced by $\mathbf{b}+\lambda\,\mathbf{b}'$, where $\mathbf{b}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $-\infty<\lambda<\infty$. Also find the optimal solution when $\lambda=\frac{1}{2}$. (Assume that (x_4, x_5) forms the initial basis.)

PART II

COMPETITIVE MODELS

A *competitive* or *strategic* model represents situations in which a person making a decision has relatively little information about the parameters in question. *In deterministic models* the parameters are constant and their values are known to the person taking the decision. In *probabilistic models* the parameters are random variables with known distributions *i.e.*, the probabilities that a certain parameter will assume some specific values are known to the decision maker. Moreover, these models deal with conflicting interests, *internal to* the organization, such as conflicting interests of minimizing production cost and minimizing inventory cost. Competitive models, on the other hand, are characterised by conflicts *external* to the organization or at least that form of external conflicts called, 'Competition'. Competitive models deal with the situations where two or more persons are making decisions in situations involving conflicting interests so that outcome of the decisions made by one depends upon the decision made by other(s). A large number of such situations with conflicting interests are seen in social, political, economic and military sphere. For example, candidates fighting an election where each tries to secure votes more than the others ; advertisement, and marketing campaigns by competing firms ; two enemy forces planning war tactics, etc.

Competitive models deal with two types of problems : games and bidding, of which only the former will be presented here.

PART II

Competitive Models

7

The Theory of Games

It was in 1928 when von Neumann (called the father of game theory) developed the theory of games. However, it was only after 1944, when von Neumann and Morgenstren published their now well known '*Theory of Games and Economic Behaviour*' that the theory received the proper attention. The theory of games (or game theory) deals with mathematical analysis of competitive problems and is based on the *minimax principle* put forward by von Neumann which implies that each competitor will act so as to minimize his maximum loss (or maximize his minimum gain.)

Everyone is interested in games and in learning how to win. Therefore, the game theory has received considerable popular attention. However, so far only simple competitive problems have been analysed by this mathematical theory. This theory does not describe how a game should be played. It describes only the procedure and principles by which plays should be selected. It is, therefore, a decision theory applicable to competitive situations. A few situations where game theory has been successfully applied are given below.

7·1. Examples on the Applications of the Theory of Games

EXAMPLE 7·1·1. (Two-Person Zero-Sum Game with Saddle Point).

In a certain game, player A has three possible choices L, M and N, while player B has two possible choices P and Q. Payments are to be made according to the choices made.

Table 7·1

Choices	*Payment*
L, P	A pays B Rs. 3
L, Q	B pays A Rs. 3
M, P	A pays B Rs. 2
M, Q	B pays A Rs. 4
N, P	B pays A Rs. 2
N, Q	B pays A Rs. 3

What are the best strategies for players A and B in this game ? What is the value of the game for A and B ?

EXAMPLE 7·1·2 (Two-Person Zero-Sum Game without Saddle Point) :

In a game of matching coins, player A wins Rs. 2 if there are two heads, wins nothing if there are two tails and loses Re. 1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of game to A.

EXAMPLE 7·1-3. (Two-Person Zero-Sum Game without Saddle Point) :

The two armies are at war. Army A has two air-bases, one of which is thrice as valuable as the other. Army B can destroy an undefended air-base, but it can destroy only one of them. Army A can also defend only one of them. Find the best strategy for A to minimize its losses.

EXAMPLE 7·1·4 (3×3 Game, Matrix Reduction by Dominance) :

In an election for M.L.A., two political parties A and B are thinking of nominating a candidate in a closed session, whose results are to be announced simultaneously. The following odds are offered for the various possible combinations of candidates :

Table 7·2

Party A	*Odds*	*Party B*
Sharma	3 : 1	Singh
Sharma	4 : 1	Gill
Sharma	1 : 3	Bajwa
Goel	3 : 7	Singh
Goel	3 : 2	Gill
Goel	1 : 4	Bajwa
Kapoor	4 : 1	Singh
Kapoor	1 : 4	Gill
Kapoor	3 : 1	Bajwa

The parties want to select candidates in accordance with standard minimax criterion. What are the optimal strategies for parties A and B ?

EXAMPLE 7·1·5 (3×3 Game, Matrix Reduction by Dominance) :

Two players P and Q play a game. Each of them has to choose one of the three colours, white (W), black (B) and Red (R) independently of the other. Thereafter the colours are compared. If both P and Q have chosen white (W, W), neither wins anything. If player P selects white and player Q black (W, B), player P loses Rs. 2 or player Q wins the same amount and so on. The complete payoff table is shown below (Table 7·3). Find the optimum strategies for P and Q and the value of the game.

Table 7·3

		Colour chosen by Q		
		W	B	R
Colour chosen by P	W	0	−2	7
	B	2	5	6
	R	3	−3	8

EXAMPLE 7·1·6 (2×3 Game, No Dominance)

Two airlines operate the same air-route, both trying to get as large a market as possible. Based on a certain market, daily gains and losses in rupees are shown in table 7·4 in which positive values favour airline A and negative values favour airline B. Find the solution for the game.

Table 7·4

		Airline B		
		Does nothing	*Advertises special rates*	*Advertises special features (i.e. movies, fine food)*
Airline A	*Advertises special rates*	275	−50	−75
	Advertises special features (i.e., movies, fine food)	125	130	150

EXAMPLE 7.1-7 (3×3 Game, No Dominance) :

Two oil companies, Indian Oil Co. and Caltex, operating in a city, are trying to increase their market at the expense of the other. The Indian Oil Co. is considering possibilities of decreasing price, giving free soft drinks on Rs. 40 purchases of oil or giving away a drinking glass with each 40 litre purchase. Obviously, Caltex cannot ignore this and comes out with its own programme to increase its share in the market. The payoff matrix from the viewpoints of increasing or decreasing market shares is given in table 7·5 below.

Table 7·5

CALTEX

		Decrease price	*Free soft drinks on Rs. 40 purchase*	*Free drinking glass on 40 litres or more*
	Decrease price	4%	1%	–3%
INDIAN OIL CO.	*Free soft drinks on Rs. 40 purchase*	3	1	6
	Free drinking glass on 40 litres or so	–3	4	–2

Determine the optimum strategies for the two oil companies.

7.2. Competitive Games

A competitive situation will be called a *competitive game* if it has the following six properties :

(*a*) there are *finite* number of participants. The number of participants is $n \geqslant 2$. If $n=2$, the game is called a two-person game; if $n>2$, it is called n-person game.

(*b*) each participant has a *finite* number of possible courses of action.

(*c*) each participant must know all the courses of action available to others but must not know which of these will be chosen.

(*d*) a play of the game is said to *occur* when each player chooses one of his courses of action. The choices are assumed to be made simultaneously, so that no participant knows the choice of other until he has decided his own.

(*e*) after all participants have chosen a course of action, their respective gains are finite.

(*f*) the gain of the participant depends upon his own actions as well as those of others.

In the present discussion only games which involve competition, actions and counteractions will be dealt with. Henceforth the simple term *game* will be used in place of competitive games.

7 3. Useful Terminology

The terminology often used in the theory of games is given below.

(*a*) each participant (interested party) is called a *player.*

(*b*) a *play* of the game results when each player has chosen a course of action.

(*c*) after *each play* of the game, one player pays the other an amount determined by the courses of action chosen.

(*d*) the decision rule by which a player determines his course of action is called a *strategy*. To reach the decision regarding which strategy to use, neither player needs to know the other's strategy.

(*e*) if a player decides to use only one particular course of action during every play, he is said to use a *pure strategy*. A pure strategy is usually represented by a number with which the course of action is associated.

(*f*) if a player decides in advance, to use all or some of his available courses of action in some fixed proportion, he is said to use *mixed strategy*. Thus a mixed strategy is a selection among pure strategies with some fixed probabilities (proportions). The advantage of a mixed strategy over a pure strategy, fter the pattern of play has become evident, is that the opponents are kept guessing as to what a player's course of action will be. A mixed strategy of a player with m possible courses of action is denoted by a set X of m non-negative numbers. The sum of these numbers is unity and each number represents the probability with which each course of action is chosen. Thus if x_i is the probability of choosing course i, we have

$$X = (x_1, x_2, x_3, ..., x_m),$$

where $\sum_{i=1}^{m} x_i = 1$ and $x_i \geqslant 0$; $i = 1, 2, 3, ..., m$.

It is evident that a pure strategy is a special case of a mixed strategy, where all but one x_i are zero. A player may be able to choose only m pure strategies, but he has an infinite number of mixed strategies to choose them.

(*g*) a game with two players, where a gain of one player *equals* the loss to the other is known as a *two-person zero-sum game*. In such a game interests of the two players are opposed so that the sum of their net gains are (sum of the game is) zero. If there are n players and sum of the game is zero, it is called *n-person zero-sum game*.

Two-person zero-sum games are also called *rectangular games* because their payoff matrix is in the rectangular form. In this chapter, we are primarily concerned with two-person zero-sum games only. The characteristics of such games are :

1. Only two players participate.
2. Each player has finite number of strategies to use.
3. Each specific strategy results in a payoff.
4. Total payoff to the two players at the end of each play is zero.

(*h*) pay off is the outcome of playing the game. A *pay* off (*gain or game*) *matrix* is a table showing the amounts received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

If a player A has m-courses of action and player B has n-courses, then a pay off matrix may be constructed by the following steps :

(*i*) row designations for each matrix are the courses of action available to A.

(*ii*) column designations for each matrix are the courses of action available to B.

(*iii*) with a two-person zero-sum game, the cell entries in B's payoff matrix will be the negative of the corresponding entries in A's payoff matrix and the matrices will appear as follows :

Table 7·6

Player B

		1	2	3	...	j ...	n
Player A	1	a_{11}	a_{12}	a_{13}	...	a_{1j}...	a_{1n}
	2	a_{21}	a_{22}	a_{23}	...	a_{2j}...	a_{2n}
	3	a_{31}	a_{32}	a_{33}	...	a_{3j}...	a_{3n}
	⋮	⋮	⋮	⋮		⋮	⋮
	i	a_{i1}	a_{i2}	a_{i3}	...	a_{ij}...	a_{in}
	⋮	⋮	⋮	⋮	...	⋮ ...	⋮
	m	a_{m1}	a_{m2}	a_{m3}	...	a_{mj}...	a_{mn}

A's payoff matrix

Table 7·7

Player B

		1	2	3	...j	...n
Player A	1	$-a_{11}$	$-a_{12}$	$-a_{13}$	$...-a_{1j}$	$...-a_{1n}$
	2	$-a_{21}$	$-a_{22}$	$-a_{23}$	$...-a_{2j}$	$...-a_{2n}$
	3	$-a_{31}$	$-a_{32}$	$-a_{33}$	$...-a_{3j}$	$...-a_{3n}$
	⋮	⋮	⋮	⋮	⋮	⋮
	i	$-a_{i1}$	$-a_{i2}$	$-a_{i3}$	$...-a_{ij}$	$...-a_{in}$
	⋮	⋮	⋮	⋮	...	⋮
	m	$-a_{m1}$	$-a_{m2}$	$-a_{m3}$	$...-a_{mj}$	$...-a_{mn}$

B's payoff matrix

Thus the sum of payoff matrices for A and B is a null matrix. Here, the objective is to determine the optimum strategies of both the players that result in optimum payoff to each, irrespective of the strategy used by the other.

Henceforth, we shall usally omit B's payoff matrix, keeping in mind that it is just the negative of A's payoff matrix.

To explain the above concepts let us consider the following example :

EXAMPLE 7.3-1

Table 7·8 illustrates a game, where competitors A and B are assumed to be equal in ability and intelligence. A has a choice of strategy 1 or strategy 2, while B can select strategy 3 or 4.

Table 7·8

Competitor B

		Strategy 3	*Strategy 4*	*Minimum of row*
Competitor A	*Strategy 1*	+4	+6	4
	Strategy 2	+3	+5	3
Maximum of column		4	6	

Both competitors know the payoffs for every possible strategy. It should be noted that the game favours competitor A since all values are positive. Values that favour B would be negative. Based upon these conditions, game is biased against B. However, since B must play the game he will play to minimize his losses.

The various possible strategies for the two competitors are

(1) A wins the highest game value if he plays strategy 1 all the time since it has higher values than strategy 2.

(2) B realizes this situation and plays strategy 3 in order to minimize his losses since the value of 4 in strategy 3 is lower than the value of 6 in strategy 4.

The game value must be 4 since A wins 4 points while B loses 4 points each time the game is played. The *'game value'* is the average winnings per play over a long number of plays. The game illustrated in table 7·6 is a two-person zero-sum game since A wins 4 points in each play while B loses the same amount. A game is solved when the following has been determined:

(*a*) the average amount per play that A will win in the long run if A and B use their best strategies. As explained earlier, it is called the value of the game.

(*b*) the strategy that A should use to ensure that his average gain per play is at least equal to the value of the game.

(*c*) the strategy that B should use to ensure that his average loss per play is no more than the value of the game.

7·4. Rules for Game Theory

The preceding two-person zero-sum game could be easily solved because of the distribution of values within the game matrix. However, some specific rules must be employed to solve other two-person zero-sum games containing the same number or larger number of rows and columns. The basic rules employed in solving such game are described below.

7·5. Rule 1. Look for a Pure Strategy (Saddle Point)

Let us consider example 7-1.1 which has already been enunciated. It is easy to arrange the payments in a matrix form. Let positive number represent a payment from B to A and negative number a payment from A to B. We then, have the payoff matrix shown in table 7·9.

Minimax and maximin values are also shown on the matrix. When player A plays his first strategy (namely L), he may gain -3 or 3 depending upon players B's selected strategy. He can guarantee, however, a gain of at least $\min\{-3, 3\} = -3$ regardless of B's selected strategy. Similarly if A plays his second strategy (namely M), he guarantees an income of at least min. $\{-2, 4\} = -2$; if he plays his third strategy (namely N) he guarantees an income of at least $\min\{2, 3\} = 2$. Thus the minimum value in each row represents the minimum gain guaranteed to A if he plays his *pure (grand) strategies.*

These values are indicated in the matrix under 'Minimum of row'. Now, player A, by selecting his third strategy (N), is maximizing his minimum gain. This gain is given by max. $\{-3, -2, 2\}=2$. This selection of player A is called the *maximin strategy* and his corresponding gain is called the *maximin or lower value* of the game.

Table 7·9

Player B

Plans (choices)

		P	Q	*Minimum of row*
Player A *Plans (choices)*	L	-3	3	-3
	M	-2	4	-2
	N	2	3	(2) maximin
Maximum of column		(2) minimax	4	

Player B, on the other hand, wants to minimize his losses. He realizes that if he plays his first pure strategy (namely P), he can lose no more than max. $\{-3, -2, 2\}=2$, regardless of A's selections. Similarly, if he plays his second pure strategy (Q), the maximum he loses is max. $\{3, 4, 3\}=4$. These values are indicated in the above matrix by 'Maximum of column'. Player B will select the strategy that minimizes his maximum loss. This is given by strategy P and his corresponding loss is given by min. $\{2, 4\}=2$. Player B's selection is called the *minimax strategy* and his corresponding loss is called the *minimax (or upper) value* of the game.

It is seen from the conditions governing the minimax criterion that the minimax (upper) value is *greater than or equal to* the maximin (lower) value. When the two are equal (minimax value = maximin value), the corresponding pure strategies are called *optimal strategies* and the game is said to have a *saddle point or equilibrium point.* The value of the game is given by the saddle point and is equal to the maximin and minimax values. Thus the saddle point is the point of intersection of the two courses of action and the gain at this point is the value of the game. The game is said to be *fair* if maximin value = minimax value = 0, and is said to be *strictly determinable* if maximin value = minimax value $\neq 0$. Note that neither player can improve his position by selecting any other strategy. Saddle point is the number which is *lowest in its row and highest in its column.*

In the above example, minimax value = maximin value = 2. The value of the game is thus equal to 2. The game has a saddle point given by the entry (N, P) of the matrix. As the game value is 2, (and not zero), the game is not fair, though it is strictly determinable.

We summarise below the steps required to detect a saddle point :

(1) At the right of each row, write the row minimum and ring the largest of them.

(2) At the bottom of each column, write the column maximum and ring the smallest of them.

(3) If these two elements are same, the cell where the corresponding row and column meet is a saddle point and the element in that cell is the value of the game.

(4) If the two ringed elements are unequal, there is no saddle point, and the value of the game lies between these two values.

(5) If there are more than one saddle points then there will be more than one solutions, each solution corresponding to each saddle point.

We give below a few more examples of games. Saddle points, if they exist, have been ringed. Optimum strategies and game values are also indicated.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -4 & 3 \\ -3 & -7 \end{bmatrix} \end{array}$$

No saddle point exists since there is no element which is both the lowest in its row and highest in its column.

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 3 & 2 \\ -2 & -3 \\ -4 & -5 \end{bmatrix} & \begin{array}{c} (1) \\ -3 \\ -5 \end{array} \\ & \begin{array}{cc} 3 & (2) \end{array} & \end{array}$$

Strategies : A, row 1 and B, column 2.

Saddle point : (1, 2).

Game value : +2.

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 1 & 13 & 11 \\ -9 & 5 & -11 \\ 0 & -3 & 13 \end{bmatrix} & \begin{array}{c} (1) \\ -11 \\ -3 \end{array} \\ & \begin{array}{ccc} (1) & 13 & 13 \end{array} & \end{array}$$

Saddle point : (1, 1).

Strategies : A, row 1; B, column 1.

Game value : +1.

$$\begin{array}{cc}
 & \text{B} \\
\text{A} & \left[\begin{array}{ccccc} 16 & 4 & 0 & 14 & -2 \\ 10 & 8 & 6 & 10 & 12 \\ 2 & 6 & 4 & 8 & 14 \\ 8 & 10 & 2 & 2 & 0 \end{array}\right]
\begin{array}{c} -2 \\ (6) \\ 2 \\ 0 \end{array} \\
 & \begin{array}{ccccc} 16 & 10 & (6) & 14 & 14 \end{array}
\end{array}$$

Saddle point : (2, 3).

Strategies : A, row 2 ; B, column 3.

Game value : +6.

If there is no saddle point, neither player can optimize his chances by using a pure strategy ; they must mix some or all of their courses of action, resulting in *mixed strategies.* Methods of finding mixed strategies will be discussed later in this chapter.

Note. *Always look for a saddle point before attempting to solve a game.*

7.6. Rule 2. Reduce Game by Dominance

If no pure strategies exist, the next step is to eliminate certain strategies (rows and/or columns) by dominance. The resulting game can be solved by some mixed strategy. To explain how dominance reduces the matrix, let us consider example 7.1-5. Table 7-3 is again written below as table 7.10.

Table 7.10

Colour chosen by Q

Colour chosen by P		W	B	R
	W	0	−2	7
	B	2	5	6
	R	3	−3	8

This matrix has no saddle point. Evidently, player Q will not play strategy R since this will result in heaviest losses to him and highest gains to player P. He can do better by playing columns W or B. Thus column R is to be deleted and strategy R is called dominated strategy.

The dominance rule for columns is : Every value in the dominating column (*s*) must be less than or equal to the corresponding value of the dominated column. The resulting matrix is

Table 7.11

		Q: W	Q: B
P	W	0	−2
	B	2	5
	R	3	−3

From table 7.11 it is clear that player P will not play row W since it will give him returns lower than given by row B. Hence row W is dominated by row B and can be deleted.

The dominance rule for rows is : Every value in the dominating row (*s*) must be greater than or equal to the corresponding value of the dominated row. The resulting matrix is

Table 7.12

		Player Q: W	Player Q: B
Player P	B	2	5
	R	3	−3

This 2×2 matrix can be easily solved as discussed later.

Dominance need not be based on the superiority of *pure strategies* only. A given strategy can be dominated if it is inferior to an average of two or more other pure strategies. To illustrate this let us consider the following game.

Table 7.13

		B: 1	B: 2	B: 3
A	1	6	1	3
	2	0	9	7
	3	2	3	4

This game has no saddle point. Further, none of the pure strategies of A is inferior to any of his other pure strategies. However, average of A's first and second pure strategies gives us

$$\left(\frac{6+0}{2}, \frac{1+9}{2}, \frac{3+7}{2}\right)=(3, 5, 5).$$

This is obviously superior to A's third pure strategy. Therefore, the third strategy may be deleted from the matrix. The resulting matrix becomes

Table 7.14

		B 1	2	3
A	1	6	1	3
	2	0	9	7

It should be noted that a game reduced by dominance may disclose a saddle point which was not found in the original matrix under rule 1 (look for a pure strategy or saddle point). This is not necessarily a true saddle point since it may not be the least value in its row and the highest value its column per the original matrix. Therefore, this pseudo-saddle point is ignored.

Note : *Always look for dominance when solving a game.*

7.7. Rule 3. Solve for a Mixed Strategy

In cases where there is no saddle point and dominance has been used to reduce the game matrix, players will resort to mixed strategies. A few different methods will be described to optimize the winning of each player and to solve the game. One of the players must determine what proportion of time to play each row while the other must know what portion of the time to play each column. The payoffs obtained will be the *expected payoffs* and the value of the game will be the *expected value* of the game.

7.8. Mixed Strategies (2×2 Games)

Arithmetic and algebraic methods are used for finding optimum strategies as well as game value for a 2×2 game. Each of these methods will be described in some details now.

7.8-1. Arithmetic Method for Finding Optimum Strategies and Game Value

It provides an easy method for finding the optimum strategies for each player in a 2×2 game. It consists of the following steps :

(*i*) subtract the two digits in column 1 and write them under column 2, ignoring sign.

(*ii*) subtract the two digits in column 2 and write them under column 1, ignoring sign.

(*iii*) similarly proceed for the two rows.

These values are called *oddments*. They are the frequencies with which the players must use their courses of action in their optimum strategies. Let us consider example 7.1-2 and explain these steps with the help of it. The payoff matrix for A is seen to be

Table 7.15

		Player B	
		H	T
Player A	H	2	−1
	T	−1	0

Since there is no saddle point, the optimal strategies will be mixed strategies. Using the steps described above we get

Table 7.16

		Player B			
		H	T		
Player A	H	2	−1	1	$\frac{1}{3+1}=0{\cdot}25$
	T	−1	0	3	$\frac{3}{3+1}=0{\cdot}75$
		1	3		
		0·25	0·75		

Thus for optimum gains, player A should use strategy H for 25% of the time and strategy T for 75% of the time, while player B should use strategy H 25% of the time and strategy T 75% of the time.

To obtain the value of the game any of the following expressions may be used :

Using A's oddments

B plays H; value of the game, $V = \text{Rs.}\left(\dfrac{1\times2-3\times1}{3+1}\right)$

$$= \text{Rs.}\left(\frac{2-3}{4}\right) = \text{Rs.}(-1/4).$$

B plays T; value of the game, $V = \text{Rs.}\left(\dfrac{1\times-1+3\times0}{3+1}\right) = \text{Rs.}(-1/4).$

Using B's oddments

A plays H; value of the game, $V = \text{Rs}\left(\dfrac{1\times2-1\times3}{1+3}\right) = \text{Rs.}(-\tfrac{1}{4}).$

A plays T; value of the game, $V = \text{Rs.}\left(\dfrac{-1\times1+0\times3}{1+3}\right) = \text{Rs.}(-\tfrac{1}{4}).$

The above values of V are equal only if sum of the oddments vertically and horizontally are equal. Cases in which it is not so are treated later.

Thus the full solution of the game is

$$\left.\begin{array}{l} A\ (1, 3), \\ B\ (1, 3), \\ V = \text{Rs.}\ (-\tfrac{1}{4}). \end{array}\right\}\text{Ans.}$$

This is the value of the game to A *i.e.*, A gains Rs. $-1/4$ *i.e.*, he loses Rs. 1/4 which B, in turn, gets. Arithmetic method is easier than algebraic method but it cannot be applied to larger games.

EXAMPLE 7·8·1 1

Reduce the following game by dominance and find the game value :

Table 7·17

		Player A			
		I	II	III	IV
	I	3	2	4	0
	II	3	4	2	4
Player A	III	4	2	4	0
	IV	0	4	0	8

[*Delhi M.Sc. (Math.) 1975, Kuru. M.Sc. (Math.) 1975*]

Solution

This matrix has no saddle point. We now try to reduce the size of the given pay off matrix by using the concept of dominance.

From player A's point of view, row I is dominated by row III. Row I is, therefore, deleted resulting in the following reduced matrix:

Table 7.18

Player B

		I	II	III	IV
	II	3	4	2	4
Player A	III	4	2	4	0
	IV	0	4	0	8

From player B's point of view, column I is dominated by III. Column I is, therefore, deleted and the following payoff matrix results :

Table 7.19

Player B

		II	III	IV
	II	4	2	4
Player A	III	2	4	0
	IV	4	0	8

In the above matrix no single row (or column) dominates another row (column). However, column II is dominated by the average of columns III and IV, which is

$$\begin{bmatrix} \frac{2+4}{2} \\ \frac{4+0}{2} \\ \frac{0+8}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Hence column II is deleted, resulting in the following matrix :

Table 7·20

		Player B III	IV
	II	2	4
Player A	III	4	0
	IV	0	8

Again, row II is dominated by the average of III and IV rows, which gives $\left(\frac{4+0}{2}, \frac{0+8}{2}\right) = (2, 4)$. Therefore row II is deleted and 2×2 game matrix results.

Table 7·21

		Player B III	IV
Player A	III	4	0
	IV	0	8

The above 2×2 matrix has no saddle point. It can be solved by arithmetic method which consists of the following steps :

(*i*) subtract the two digits in column III and write them under column IV, ignoring sign.

(*ii*) subtract the two digits in column IV and write them under column III, ignoring sign.

(*iii*) similarly proceed for the two rows.

These values are called oddments and they give the frequencies with which the players must use their courses of action in their optimal strategies. Proceeding along these steps we get

Table 7·22

		Player B		
		III	IV	
Player A	III	4	0	$8 \frac{2}{3}$
	IV	0	8	$4 \frac{1}{3}$
		$8 \frac{2}{3}$	$4 \frac{1}{3}$	

Thus the complete solution to the given problem is

Optimal strategy for player A : $\left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$,

Optimal strategy for player B : $\left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$,

Value of the game (for A) : $\frac{8\times4+0\times4}{8+4}=\frac{8}{3}$.

EXAMPLE 7·8-1·2

Reduce the following game by dominance property and solve it.

Table 7·23

		Player B				
		1	2	3	4	5
	I	1	3	2	7	4
	II	3	4	1	5	6
Player A	III	6	5	7	6	5
	IV	2	0	6	3	1

Solution

The above matrix has no saddle point. Row IV is dominated by row III. Deleting row IV we get

Table 7.24

Player B

		1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5

Now, column 4 is dominated by columns 1 and 2, also column 5 is dominated by column 2. Therefore, deleting columns 4 and 5 we have

Table 7·25

Player B

		1	2	3
Player A	I	1	3	2
	II	3	4	1
	III	6	5	7

In the above matrix, row I as well as II are dominated by row III. Therefore, we delete rows I and II and get

Table 7·26

Player B

		1	2	3
Player A	III	6	5	7

Out of the three strategies available to player B, he will use No. 2 in order to minimize his losses. Therefore, the solution to the problem is

Optimal strategy for A : III,
Optimal strategy for B : 2,
Game value (for A) : 5.

EXAMPLE 7·8·1·3

Solve the following game by using the principle of dominance :

Table 7.27

		Player B					
		I	II	III	IV	V	VI
Player A	1	4	2	0	2	1	1
	2	4	3	1	3	2	2
	3	4	3	7	–5	1	2
	4	4	3	4	–1	2	2
	5	4	3	3	–2	2	2

[*Delhi M.Sc. (Stat.) 1968*]

Solution

The above payoff matrix has no saddle point. From player A's point of view, row 1 is dominated by row 2 and row 5 is dominated by row 4. Accordingly, rows 1 and 5 are deleted. The following reduced matrix results.

Table 7·28

		Player B					
		I	II	III	IV	V	VI
Player A	2	4	3	1	3	2	2
	3	4	3	7	–5	1	2
	4	4	3	4	–1	2	2

From player B's point of view, columns I and II are dominated by columns IV, V and VI, also column VI is dominated by column V. Therefore, columns I, II and VI are deleted, resulting in

Table 7.29

		Player B		
		III	IV	V
	2	1	3	2
Player A	3	7	—5	1
	4	4	—1	2

Now none of single row (or column) dominates another row (or column). However, column V is dominated by the average of columns III and IV, which is

$$\begin{bmatrix} \frac{1+3}{2} \\ \frac{7-5}{2} \\ \frac{4-1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$$

Accordingly, column V is deleted and the following matrix is obtained :

Table 7·30

		Player B	
		III	IV
	2	1	3
Player A	3	7	—5
	4	4	—1

Further, row 4 is dominated by the average of row 2 and 3. Hence row 4 is deleted. The resulting 2×2 game is shown below.

Table 7.31

		Player B III	*Player B* IV
Player A	2	1	3
	3	7	−5

This 2×2 game has *no saddle point.*

Solving this game by arithmetic method we have

Table 7·32

		Player B III	*Player B* IV			
Player A	2	1	3	12	6	6/7
	3	7	−5	2	1	1/7
		8	6			
		4	3			
		4/7	3/7			

Therefore, optimal strategy for A : (0, 6/7, 1/7, 0, 0),

Optimal strategy for B : (0, 0, 4/7, 3/7, 0, 0),

Game value : $\dfrac{1 \times 8 + 3 \times 6}{8+6} = \dfrac{26}{14} = \dfrac{13}{7}$.

7·8-2. Algebraic Method for Finding Optimum Strategies and Game Value

While applying this method it is assumed that x represents the fraction of time (frequency) for which player A uses strategy 1 and $(1-x)$ represents the fraction of time (frequency) for which he uses strategy 2. Similarly y and $(1-y)$ represent the fraction of time for which player B uses strategies 1 and 2 respectively.

Let us consider example 7·1-3 to explain this method.

Since both armies have only two possible courses of action, the gain matrix for army A is

Table 7·33

		Army B	
		1 *Attack the smaller air-base*	2 *Attack the larger air-base*
Army A	*Defend smaller air base* 1	0 *Both survive*	−3 *The larger one destroyed*
	Defend larger air-base 2	−1 *The smaller one destroyed*	0 *Both survive*

There is no saddle point. Under this method, army A wants to divide its plays between the two rows so that the expected winnings by playing the first row are exactly equal to the expected winnings by playing the second row irrespective of what army B does. In order to arrive at the optimum strategies for Army A, it is necessary to equate its expected winnings when army B plays column 1 to its expected winnings when army B plays column 2.

i.e, when $0x+(-1)(1-x)=-3x+0(1-x)$

or when $-1+x=-3x$ *i.e.*, $4x=1$ $\therefore$ $x=1/4$.

Thus army A should play first row 1/4th of the time and second row 3/4th $(=1-x)$ of the time.

Similarly, army B wants to divide its time between columns 1 and 2 so that the expected winnings are same by playing each column. no matter what army A does. Optimum strategies for army B will be found by equating its expected winnings when army A plays row 1 to its expected winnings when army A plays row 2.

i.e., when $0 \cdot y-3(1-y)=-1.y+0(1-y)$

or when $-3+3y=-y$

or when $4y=3$, or when $y=3/4$.

Thus army B should play first column 3/4th of the time and second column 1/4th $(=1-y)$ of the time. These optimum strategies can be shown on the gain-matrix, which becomes

Table 7·34

Army B

		1	2	
Army A	1	0	−3	$\frac{1}{4}$
	2	−1	0	$\frac{3}{4}$
		$\frac{3}{4}$	$\frac{1}{4}$	

The game value can be found either for army A or for army B.

Game Value for army A : While army B plays column 1, 3/4 of time, army A wins zero for 1/4 time and —1 for 3/4 time; also while army B plays column 2, 1/4 of time, army A wins —3 for 1/4 time and zero for 3/4 time.

∴ Total expected winnings for army A are

$$\text{game value} = \tfrac{3}{4}\left(0\times\tfrac{1}{4}-1\times\tfrac{3}{4}\right)+\tfrac{1}{4}\left(-3\times\tfrac{1}{4}+0\times\tfrac{3}{4}\right)$$
$$=-9/16-316=-12/16=-3/4.$$

Game value for army B : While army A plays row 1, 1/4 of time, army B wins zero for 3/4 of time and —3 for 1/4 of time ; also while army A plays row 2, 3/4 of time, army B wins —1 for 3/4 of time and zero for 1/4 of time.

$$\therefore \text{ Game value for army B}=\tfrac{1}{4}\left(\tfrac{3}{4}\times 0-3\times\tfrac{1}{4}\right)-\tfrac{3}{4}\left[\tfrac{3}{4}\times(-1)+0\times\tfrac{1}{4}\right]$$
$$=\tfrac{1}{4}\left(-\tfrac{3}{4}\right)+\tfrac{3}{4}\left(-\tfrac{3}{4}\right)$$
$$=-3/16-9/16=-12/16=-\tfrac{3}{4}.$$

Thus the full solution of the game is

army A : $(\frac{1}{4}, \frac{3}{4})$,

army B : $(\frac{3}{4}, \frac{1}{4})$,

game value : $-\frac{3}{4}$.

7·9. Mixed Strategies ($2\times n$ Games or $m\times 2$ Games) :

These are the games in which one player has only two courses of action open to him while his opponent may have any number. To solve such games, the first step is to look for a saddle point; if there is one, the game is readily solved. If not, the next step is to reduce the given matrix to 2×2 size matrix by the rules of dominance. If the matrix can be reduced to 2×2 size, it can be easily solved by the arithmetic or other methods described in section 7·8. If, however, the given matrix cannot be reduced to 2×2 size matrix, it can be still solved by algebraic method, method of subgames and graphic method. Each of these methods will now be explained to some length.

7·9-1. Algebraic Method for 2×n or m×2 games :

The payoffs for a rectangular game can always be given in $m \times n$ matrix, where player A has m possible courses of action and player B has n possible courses of action and the payoff matrix is (a_{ij}). This is shown in table 7·35 below.

Table 7·35

		B					
		1	2	...	j	...	n
	1	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}
	2	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}
	.	.	.	...	.	...	
	.	.	.	...	.	...	
	.	.	.	...	.	...	
A	i	a_{i1}	a_{i2}	...	a_{ij}	...	a_{in}
	.	.	.	...	.	...	.
	.	.	.	...	.	...	.
	.	.	.	...	.	...	.
	m	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}

It can be shown mathematically that

1. Each rectangular game has a specific value V. This value is unique.
2. There exists for player A a best strategy X *i.e.*, there exist frequencies $x_1, x_2, \ldots, x_m$ such that $x_1+x_2+\ldots+x_m=1$ and such that if he plays strategy 1 with frequency x_1, strategy 2 with frequency x_2, ..., strategy m with frequency x_m, then he can assure himself *at least* an expected gain of V, where V is the value of the game.
3. Similarly for player B, there exists a best strategy,

$$Y=(y_1, y_2, \ldots, y_n), \quad \sum_{j=1}^{n} y_j=1,$$

such that if he plays strategies 1, 2, ..., n, with frequencies $y_1, y_2, \ldots, y_n$ respectively, he can assure himself *at most* a loss of V.

It can be shown that the unknown (frequencies) $x_1, x_2, \ldots, x_m$ $y_1, y_2, \ldots, y_n$ and V can be found from the following relations :

$$x_1+x_2+\dots+x_m=1,\ x_i\geqslant 0, \qquad \dots(7\cdot1)$$

$$y_1+y_2+\dots+y_n=1,\ y_i\geqslant 0, \qquad \dots(7\cdot2)$$

$$x_1a_{1j}+x_2a_{2j}+\dots+x_ma_{mj}\geqslant V,\ \text{for } j=1,2,\dots,n, \qquad \dots(7\cdot3)$$

$$y_1a_{i1}+y_2a_{i2}+\dots+y_na_{in}\leqslant V,\ \text{for } i=1,2,\dots,m. \qquad \dots(7\cdot4)$$

Relation (7·3) actually represents n inequations, one inequation for each j. Similarly, relation (7·4) represents m inequations. We, thus, have $m+n-1$ unknowns with $m+n+2$ relations (with added restrictions $x_i\geqslant 0$, $y_j\geqslant 0$ since negative frequencies have no meaning).

The *algebraic method* is a direct method to solve for the unknowns from relations (7·1), (7·2), (7·3) and (7·4). It must be borne in mind that each rectangular game has a value, namely V, which exists and is unique. Therefore, the idea is to find such value V which satisfies all the four relations. The first obvious step is to assume that relations (7·3) and (7·4) are equalities. We shall consider an example to describe how algebraic method can help to solve $2\times n$ or $m\times 2$ games.

EXAMPLE 7-9-1 :

Solve the game for which the payoff matrix is given in table 7·36.

Table 7·36

		B 1	2
	1	-2	-4
A	2	-1	3
	3	1	2

From the above discussion, we get the following relations for the unknowns x_1, x_2, x_3; y_1, y_2 and V :

$$x_1+x_2+x_3=1, \qquad \dots(7\cdot5)$$

$$y_1+y_2=1, \qquad \dots(7\cdot6)$$

$$x_1(-2)+x_2(-1)+x_3(1)\geqslant V, \qquad \dots(7\cdot7)$$

$$x_1(-4)+x_2(3)+x_3(2)\geqslant V, \qquad \dots(7\cdot8)$$

$$y_1(-2)+y_2(-4)\leqslant V, \qquad \dots(7\cdot9)$$

$$y_1(-1)+y_2(3)\leqslant V, \qquad \dots(7\cdot10)$$

$$y_1(1)+y_2(2)\leqslant V. \qquad \dots(7\cdot11)$$

Thus we have six unknowns x_1, x_2, x_3 ; y_1, y_2 and V and seven relations.

Now the first step towards solution is to assume all the inequalities as equalities and we get

$$x_1+x_2+x_3=1, \qquad \ldots(7 \cdot 12)$$
$$y_1+y_2=1, \qquad \ldots(7 \cdot 13)$$
$$-2x_1-x_2+x_3=V \qquad \ldots(7 \cdot 14)$$
$$-4x_1+3x_2+2x_3=V, \qquad \ldots(7 \cdot 15)$$
$$-2y_1-4y_2=V, \qquad \ldots(7 \cdot 16)$$
$$-y_1+3y_2=V, \qquad \ldots(7 \cdot 17)$$
$$y_1+2y_2=V. \qquad \ldots(7 \cdot 18)$$

Now considering equations (7·13), (7·16) and (7·17),

$$y_2=1-y_1,$$
$$-2y_1-4(1-y_1)=V \text{ or } -2y_1-4+4y_1=V \text{ or } 2y_1-4=V,$$
and
$$-y_1+3(1-y_1)=V \text{ or } -y_1+3-3y_1=V \text{ or } -4y_1+3=V.$$
$$\therefore\ 2y_1-4=-4y_1+3 \quad i.e., \quad 6y_1=7 \quad \text{or} \quad y_1=7/6.$$

This is unacceptable, since frequency y_1 cannot be greater than 1 $(0 \leqslant y_1 \leqslant 1)$. Therefore, if we assume relations (7·9) and (7·10) as equalities, we do not get a solution.

A general procedure is, then, to assume one inequality and the other relations as equalities. If we again meet with a contradiction, we assume other inequalities until a solution is finally reached.

However, these computations can be made simpler with the help of the following theorems :

Theorem 1 states that : if $x_1a_{1j}+x_2a_{2j}+\ldots+x_ma_{mj}>V$, then $y_j=0$.

Theorem 2 states that : if $y_1a_{i1}+y_2a_{i2}+\ldots+y_na_{in}<V$, then, $x_i=0$.

Let us apply these theorems to our problem. We, therefore, proceed as follows :

(1) Thus if $-2x_1-x_2+x_3>V$, then $y_1=0$. Therefore, equations (7·16), (7·17) and (7·18) can be true only if $y_2=0$.

$\therefore\ y_1+y_2=0$, which is contradicted by equation (7·13).

(2) Similarly, a contradiction is obtained if $-4x_1+3x_2+2x_3>V$.

(3) If $-2y_1-4y_2<V$, then $x_1=0$ and equations (7·12), (7·14) and (7·15) will lead to contradiction. Proceeding in this manner, we finally find that only the following relations lead to a solution :

$$x_1+x_2+x_3=1, \qquad \text{...(7·12)}$$

$$y_1+y_2 \quad =1, \qquad \text{...(7·13)}$$

$$-2x_1-x_2+x_3=V, \qquad \text{...(7·14)}$$

$$-4x_1+3x_2+2x_3>V, \qquad \text{...(7·15}a\text{)}$$

$$-2y_1-4y_2<V, \qquad \text{...(7·16}a\text{)}$$

$$-y_1+3y_2<V, \qquad \text{...(7·17}a\text{)}$$

$$y_1+2y_2=V. \qquad \text{...(7·18)}$$

From relation (7·15*a*), $y_2=0$, so that $y_1=1$. Further, from relation (7·16*a*), $x_1=0$ and from relation (7·17*a*), $x_2=0$, so that $x_3=1$. Substituting the values of y_1 and y_2 in (7·18), $V=1$. Therefore, the final solution is given by

$x_1=0,\ x_2=0,\ x_3=1,\ y_1=1,\ y_2=0$ and $V=1$.

i.e., optimum strategy for A is, A (0, 0, 1)

optimum strategy for B is, B (1, 0),

game value $=+1$.

Evidently, the algebraic solution of the game is very lengthy and the computations become much more complex as the number of possible choices for A & B increases.

Remark. The above problem can be easily solved by locating the saddle point. However, it has been solved by algebraic method just to illustrate the method.

7·9·2. Method of Subgames for 2×n or m×2 Games

We shall explain the method with the help of example 7·1-6, already enunciated. Table 7·4 for the problem is written again as table 7·37 below.

Table 7·37

		Airline B		
		1	2	3
Airline A	1	275	—50	—75
	2	125	130	150

We see that the game has no saddle point, nor it can be reduced by dominance. This game can be solved by algebraic method described in section 7·9-1, but we shall solve this game here by the method of subgames.

This 2×3 game can be thought of as three 2×2 subgames :

***Subgame* 1 :**

		B	
		1	2
A	1	275	−50
	2	125	130

(Ignoring column 3)

***Subgame* 2 :**

		B	
		1	3
A	1	275	−75
	2	125	150

(Ignoring column 2)

***Subgame* 3 :**

		B	
		2	3
A	1	−50	−75
	2	130	150

(Ignoring column 1)

Airline B which has more number of columns (than the number of rows for A), has more flexibility, generally resulting in a better strategy. In order to find optimum strategy for airline B, all the above three 2×2 subgames must be solved for their strategies and game values. We shall solve them by arithmetic method.

***Subgame* 1 :**

		B				
		1	2			
A	1	275	−50	5	1	1/66
	2	125	130	325	65	65/66
		180	150			
		6	5			
		36	30			
		36/66	30/66			

There is no saddle point.

Thus strategy for A is, A (1/66, 65/66),

,, ,, B ,, B (36/66, 30/66, 0),

$$\text{game value} = \text{Rs.} \left(\frac{275 \times 1 + 125 \times 65}{66}\right)$$

$$= \text{Rs. } 127{\cdot}30.$$

Subgame 2 :

		B: 1	B: 3			
A	1	275	−75	25	1	1/15
	2	125	150	350	14	14/15

There is no saddle point.

225	150
3	2
3/5	2/5
9/15	6/15

Thus strategies are : A (1/15, 14/15),

B (9/15, 0, 6/15),

$$\text{value of the game, } V = \text{Rs.} \left(\frac{275 \times 3 - 75 \times 2}{5}\right) = \text{Rs.}\left(\frac{825 - 150}{5}\right)$$

$$= \text{Rs.} \left(\frac{675}{5}\right) = \text{Rs. } 135.$$

Subgame 3 :

		B: 2	B: 3	*Row minimum*
A	1	−50	−75	−75
	2	130	150	(130)
Column maximum		(130)	150	

Thus this subgame has a saddle point (2, 2). Thus solution is

A (0, 1),

B (0, 1, 0),

V = Rs. 130.

Now, since airline B has the flexibility to play any two out of the courses of action available to it, it will play those strategies for which the loss occurring to the airline is minimum. As all the

values for the subgames are positive, airline A is the winner. Hence airline B will play subgame 1 for which the loss is minimum *i.e.* Rs. 127.30.

Hence the complete solution to the problem is

strategies : A (1/66, 65/66),

B (36/66, 30/66, 0),

game value V = Rs. 127·30.

Remark : *Subgame 3 has a saddle point, hence arithmetic method should not be applied to solve it. If it is applied, the resulting solution will be incorrect.*

7.9-3. Graphic Method for 2×n or m×2 Games

Graphic method is applicable to only those games in which one of the players has two strategies only. The advantage of this method is that it is relatively fast. Consider the following $2 \times n$ game :

Table 7·38

		B			
		y_1	y_2	...	y_n
A	x_1	a_{11}	a_{12}	...	a_{1n}
	$x_2=1-x_1$	a_{21}	a_{22}	...	a_{2n}

It is assumed that the game has no saddle point.

Player A has two strategies x_1 and x_2 $(=1-x_1)$, where $x_1 \geqslant 0$, $x_2 \geqslant 0$. Expected payoffs for A corresponding to the pure strategies of B are given below.

Table 7·39

B's pure strategies	*A's expected payoff*
1	$a_{11}x_1+a_{21}(1-x_1)=(a_{11}-a_{21})x_1+a_{21}$
2	$a_{12}x_1+a_{22}(1-x_1)=(a_{12}-a_{22})x_1+a_{22}$
⋮	⋮
n	$a_{1n}x_1+a_{2n}(1-x_1)=(a_{1n}-a_{2n})x_1+a_{2n}$

Thus A's expected payoff varies linearly with x_1. Now, according to the maximin criterion for mixed strategy, A should select that value of x_1 which maximizes his minimum expected payoff. This may be done by plotting the above lines as a function of x_1. This is

shown in figure 7.1. A number is allotted to each line corresponding to B's pure strategy. The lower boundary of these lines (shown by heavy lines) gives the minimum expected payoff to A as a function of x_1. The highest point on this lower boundary (shown by a dot), then gives the maximin expected payoff to A and hence the optimum value $x_1 = (x_1^*)$.

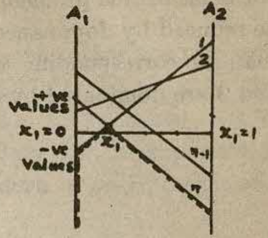

Fig. 7.1

The two optimum strategies for B are given by the two lines which pass through this *maximin point*. Thus the $2 \times n$ game is reduced to 2×2 game which can be easily solved by the methods already described.

The importance of the maximin point is that it is the highest level, on the average, at which B can hold A's winnings. Similarly it is the level at which A can hold B to minimize his losses. In other words, it just indicates how far one player can go before he is restrained by his opponent's defensive strategy. It is the average payoff around which the game revolves.

In the same way we can treat $(m \times 2)$ games. For these $(m \times 2)$ games we shall get *minimax point* which will be the lowest point on the upper boundary (instead of highest point on the lower boundary). Thus we conclude that any $(2 \times n)$ or $(m \times 2)$ game can be reduced to (2×2) game. To explain the graphic method we shall consider a few examples.

EXAMPLE 7.9-3.1

Solve the game given in table 7·40 by graphic method.

Table 7·40

		B			
		y_1	y_2	y_3	y_4
	x_1	19	6	7	5
A	x_2	7	3	14	6
	x_3	12	8	18	4
	x_4	8	7	13	−1

Solution. The first step is to look for a saddle point. It does not exist in this problem. The second step is to see if the game can be reduced by dominance. All the cell values in column 2 are less than the corresponding values in columns 1 and 3. Hence columns 1 and 3 are dominated by column 2 and the reduced matrix becomes

Table 7.41

		B: y_2	B: y_4
A	x_1	6	5
	x_2	3	6
	x_3	8	4
	x_4	7	-1

Again all the cell values for row 3 are higher than those for row 4. Hence row 3 dominates row 4 and the matrix is reduced to

Table 7.42

		B: y_2	B: $y_4=1-y_2$
A	x_1	6	5
	x_2	3	6
	x_3	8	4

This matrix can be solved now by graphic method. B's expected payoffs corresponding to A's pure strategies are given below.

A's pure strategies	B's expected payoffs
1	$6y_2+5(1-y_2)=y_2+5$
2	$3y_2+6(1-y_2)=-3y_2+6$
3	$8y_2+4(1-y_2)=4y_2+4$

These three straight lines can be plotted as functions of y_2 as follows :

Draw two lines B_2 and B_4 parallel to each other one unit apart and mark a scale on each of them (Figure 7.2). These two lines represent the two strategies available to B. To represent A's first strategy, join mark 6 on B_2 with mark 5 on B_4 ; to represent *A*'s second strategy, join mark 3 on B_2 with mark 6 on B_4 ; and so on and bound the figure from above as shown.

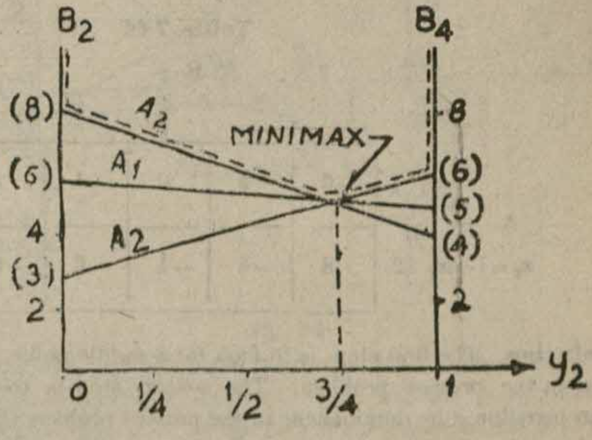

Fig. 7.2

Since player B wishes to minimize his maximum expected losses, the two lines which intersect at the lowest point of the upper bound show the two courses of action A should choose in his best strategy *i.e.*, A_1 and A_2. We can, thus, immediately reduce the 3×2 game to 2×2 game which can be easily solved by arithmetic method. The resulting 2×2 game is shown in table 7·43 below.

Table 7.43

		B			
		2	4		
A	1	6	5	3	3/4
	2	3	6	1	1/4
		1	3		
		1/4	3/4		

$\therefore$ **Optimum strategies are**

$$\left.\begin{array}{l} A\,(\tfrac{3}{4}, \tfrac{1}{4}, 0, 0), \\ B\,(0, \tfrac{1}{4}, 0, 3/4), \\ \text{value of the game is, } V = \dfrac{6\times 1+3\times 5}{1+3} = \dfrac{21}{4} = 5\,\dfrac{1}{4}. \end{array}\right\}\text{Ans.}$$

EXAMPLE 7.9.3-2

Solve the following 2×5 game graphic method :

Table 7·44

		B 1	2	3	4	5
A	x_1 1	-5	5	0	-1	8
	$x_2=1-x_1$ 2	8	-4	-1	6	-5

Solution. The first step is to look for a saddle point. It does not exist in the present problem. The second step is to see if the game can be reduced by dominance. In the present problem the matrix cannot be reduced by dominance. So, let us solve the matrix by graphic method. A's expected payoffs corresponding to B's pure strategies are

B's pure strategies	A's expected payoffs
1	$-5x_1+8(1-x_1)=-13x_1+8$
2	$5x_1-4(1-x_1)=9x_1-4$
3	$0.x_1-1(1-x_1)=x_1-1$
4	$-1x_1+6(1-x_1)=-7x_1+6$
5	$8x_1-5(1-x_1)=13x_1-5$

These five lines can be plotted as functions of x_1 as follows :

Draw two parallel lines A_1 and A_2 one unit apart and mark a scale on each of them. These two lines represent the two strategies available to A. To represent B's first strategy, join mark -5 on A_1 with mark 8 on A_2; to represent B's second strategy, join mark 5 on A_1 with mark -4 on A_2 and so on (for the remaining three strategies) and bound the figure from below.

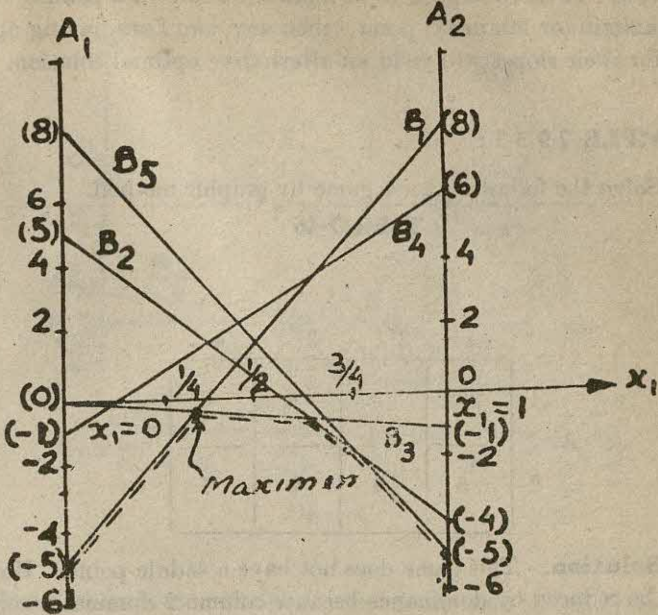

Fig. 7·3

Since player A wishes to maximize his minimum expected pay-off, the two lines which intersect at the highest point of the bound show the two courses of action B should choose in his best strategy, which are B_1 and B_3. We can thus immediately reduce the 2×5 game to 2×2 game which can be easily solved, say, by arithmetic method. The resulting 2×2 game is shown in table 7·45.

Table 7·45

		B: 1	B: 3		
A	1	−5	0	9	9/14
	2	8	−1	5	5/14
		1	13		
		1/14	13/14		

∴ Optimum strategies are

A (9/14, 5/14),
B (1/14, 0, 13/14, 0, 0),

value of the game is, $V = \dfrac{-5 \times 1 + 0 \times 13}{1 + 13} = \dfrac{-5}{14}$ **Ans.**

Note : If there happen to be more than two lines passing through the maximin (or minimax) point, then any two lines having opposite signs for their slopes will yield an alternative optimal solution.

EXAMPLE 7·9·3.3 :

Solve the following 2×4 game by graphic method.

Table 7·46

		B			
		1	2	3	4
A	1	3	3	4	0
	2	5	4	3	7

Solution. This game does not have a saddle point. However, it can be reduced by dominance because column 2 dominates column 1 resulting in the reduced matrix shown in table 7·47.

Table 7·47

			B		
			2	3	4
A	x_1	1	3	4	0
	$x_2=1-x_1$	2	4	3	7

This matrix can be solved by graphic method. A's expected payoffs corresponding to B's pure strategies are

B's pure strategies	A's expected payoffs
1	$3x_1+4(1-x_1)=-x_1+4$
2	$4x_1+3(1-x_1)=x_1+3$
3	$0x_1+7(1-x_1)=-7x_1+7$

These three straight lines can be plotted as functions of x_1. The method of plotting them has already been described; and they appear as shown in figure 7,4,

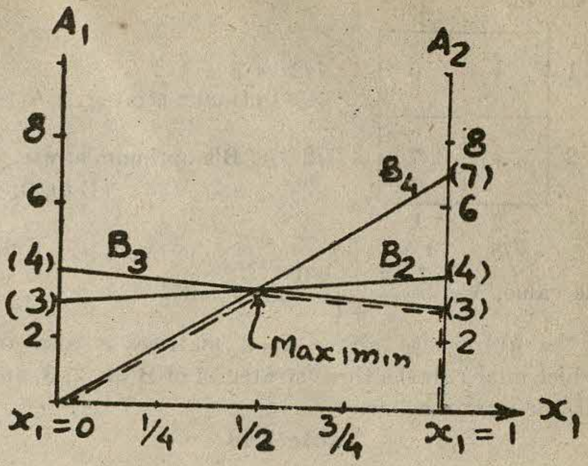

Fig. 7·4

Figure is to be bound from below as shown. All the three lines pass through the maximin point. As mentioned earlier, any two lines having opposite signs for their slopes will yield an alternative optimal solution. This means that the combination of B_2 and B_4 must be excluded as both have the same sign for their slopes. So the game reduces to two 2×2 games which can be easily solved, say, by arithmetic method.

(*i*)

		B			
		2	3		
A	1	3	4	1	1/2
	2	4	3	1	1/2
		1	1		
		1/2	1/2		

A's optimum strategy : A $(\frac{1}{2}, \frac{1}{2})$,

B's optimum strategy : B $(0, \frac{1}{2}, \frac{1}{2}, 0)$,

game value, $V=\dfrac{3\times 1+4\times 1}{1+1}=\dfrac{7}{2}=3\frac{1}{2}$.

(*ii*)

		B 3	4	
A	1	4	0	4 1/2 4/8
	2	3	7	4 1/2 4/8
		7	1	
		7/8	1/8	

A's optimum strategy : A (4/8, 4/8),

B's optimum strategy : B (0, 0, 7/8, 1/8),

game value, $V = \dfrac{7 \times 4 + 1 \times 0}{7+1} = \dfrac{7}{2} = 3\frac{1}{2}$.

Note that the average of above 2×2 matrices is also an optimal solution which mixes all the three strategies of B *i.e.* 2, 3, and 4. This is shown in the matrix below.

Table 7·48

		B: 2+3	3+4	
A	1	$\dfrac{3+4}{2} = \dfrac{7}{2}$	$\dfrac{4+0}{2} = 2$	2
	2	$\dfrac{4+3}{2} = \dfrac{7}{2}$	$\dfrac{3+7}{2} = 5$	(7/2)
		(7/2)	5	

This matrix has a saddle point. Value of the game is 7/2. A should use strategy 2 and B should use a combination of strategies 2 and 3.

Note . *Since the above matrix has a saddle point, it should not be solved by arithmetic or other method. If these methods are applied, the solution obtained will be incorrect.*

7·10. Mixed Strategies (3×3 Games)

To solve 3×3 or higher games also, the first step is to look for a saddle point, if there is one, the game is readily solved. If not, the next step is to reduce the given matrix to 2×2 size matrix or $2 \times n$ (or $m \times 2$) matrix by applying the rules of dominance. If the matrix can be reduced to 2×2 size, it can be easily solved by the methods described under section 7·8. If the matrix can only be reduced to $n \times 2$ (or $m \times 2$) size matrix, it can still be solved by applying the methods described under section 7·9. However, if the final matrix is of 3×3 size, it can be solved by algebraic method, method of matrices, method of linear programming and by iterative method of approximate solution.

Algebraic method has already been described for $2 \times n$ (or $m \times 2$) games under section 7·9. The method can be extended to 3×3 games and will not be discussed here. The remaining three methods will now be described in some details.

7·10-1. Method of Matrices (or 3×3 Games)

This method will be illustrated with the help of an example.

EXAMPLE 7·10-1.

Solve the game given by table 7·49.

Table 7·49

	B 1	B 2	B 3	
A 1	7	1	7	1
A 2	9	−1	1	−1
A 3	5	7	6	(5)
	9	(7)	(7)	

Solution. The first step is to look for a saddle point. It does not exist in this problem. The game value lies between 5 and 7. The next step is to see if the given matrix can be reduced by dominance. We see that the given matrix cannot be reduced. So let us solve this matrix by the method of matrices. Subtract each row from the row above (*i.e.* subtract 2nd row from the first and third row from the second) and write down the values below the matrix. Similarly subtract each column from the column to its left (*i.e.*, subtract second column from the first and third column from the second) and write down the results to the right of the matrix. Thus we get table 7·50.

Table 7·50

	B 1	B 2	B 3		
A 1	7	1	7	6	−6
A 2	9	−1	1	10	−2
A 3	5	7	6	−2	1
	−2	2	6		
	4	−8	−5		

Next, calculate the oddments for A_1, A_2, A_3 and B_1, B_2, B_3.

Oddment for $A_1 = \text{determinant} \begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix} = 10-4 = 6,$

" " $A_2 = \text{determinant} \begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix} = 12-6 = 6,$

" " $A_3 = \text{determinant} \begin{vmatrix} 6 & -6 \\ 10 & -2 \end{vmatrix} = -12+60 = 48,$

" " $B_1 = \text{determinant} \begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix} = -10+48 = 38,$

" " $B_2 = \text{determinant} \begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix} = 24-10 = 14,$

" " $B_3 = \text{determinant} \begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix} = 16-8 = 8.$

Next, write down these oddments (as shown in table 7·51) neglecting their signs. Since both the sums of oddments are same (60 each), this is a solution to the game. If the sums are different, both players do not use all of their courses of actions in their best strategies and this method fails.

Table 7·51

		B 1	B 2	B 3				
	1	7	1	7	6	1	1/10	3/30
A	2	9	−1	1	6	1	1/10	3/30
	3	5	7	6	48	8	8/10	24/30
		38	14	8	60			
		19	7	4				
		19/30	7/30	4/30				

Thus optimum strategies are

$$\left.\begin{array}{l} A\ (3/30,\ 3/30,\ 24/30), \\ B\ (19/30,\ 7/30,\ 4/30), \\ \text{game value, } V = \dfrac{7\times1+9\times1+5\times8}{1+1+8} = \dfrac{7+9+40}{10} = \dfrac{56}{10} \\ \qquad = \dfrac{28}{5} = 5\,\dfrac{3}{5}. \end{array}\right] \textbf{Ans.}$$

Note. The above method can be applied only when sum of vertical oddments is equal to the sum of horizontal oddments *i.e.*,

if both players use all their plays in their best strategies. The method breaks down when the players do not use all their courses of action in their best strategies. In such a case the method of linear programming may be applied.

7·10·2. Method of Linear Programming :

Game theory bears a strong relationship to linear programming, since every finite two person zero-sum game can be expressed as a linear programme and conversely every linear programme can be represented as a game. Linear programming is the most general method of solving any two-person-zero-sum game. If there is no saddle point, dominance is unsuccessful in reducing the game and the method of matrices also fails, then linear programming offers the best method of solution. We shall describe this method with the help of two examples.

EXAMPLE 7.10-2.1.

Solve **example** 7·1-7.

Solution. Table 7.5 for this example is expressed again as table 7.52 below.

Table 7.52

Caltex
B

		y_1	y_2	y_3	
Indian Oil Co A	x_1	4	1	−3	−3
	x_2	3	1	6	(1)
	x_3	−3	4	−2	−3
		(4)	(4)	6	

Let us denote the Indian Oil Co. by A and Caltex by B. Let x_1, x_2, x_3 and y_1, y_2, y_3 be the probabilities by which A and B respectively, select their pure strategies. The first step is to look for saddle point. It does not exist in this problem. The value of the game lies between 1 and 4. The next step is to see if the given matrix can be reduced by dominance We find that it cannot be reduced. Solving it by the method of matrices we get table 7.53.

Table 7.53

				B				
			y_1	y_2	y_3			
			1	2	3			
	x_1	1	4	1	-3	3	4	47
A	x_2	2	3	1	6	2	-5	46
	x_3	3	-3	4	-2	-7	6	23
			1	0	-9			
			6	-3	8			
			27	62	3			92/116

Since the sums of the oddments of A and B are not equal, the problem cannot be solved by the method of matrices. The method of linear programming will be used to solve it.

Let the value of the game (to A) be V. Consider the game from B's point of view. B is trying to minimize V. Then,

$$\text{against } A_1 \quad 4y_1+y_2-3y_3 \leqslant V,$$
$$\text{against } A_2, \quad 3y_1+y_2+6y_3 \leqslant V,$$
$$\text{against } A_3, -3y_1+4y_2-2y_3 \leqslant V,$$
$$y_1+y_2+y_3=1, \text{ (Sum of probabilities must be equal to 1)}$$

where $y_1, y_2, y_3 \geqslant 0$.

Divide each of above relations by V. Note that this division is valid only if $V>0$. If, however, $V<0$, the direction of inequality constraints must be reversed, and if $V=0$, division would be meaningless. However, both these cases can be easily solved by adding a positive constant K (where K $\geqslant$ the negative game value) to all the entries of the matrix, thus ensuring that the game value for the *revised matrix is greater than zero.* After obtaining the optimal solution, the true value of the game can be obtained by subtracting K from the game value so obtained.

In general, if the maximin value of this game is non-negative, then the value of the game is greater than zero, provided the game does not have a saddle point.

Since maximin value is 1 in the present example, we get the

following relations by dividing by V :

$$4.\ \frac{y_1}{V}+\ \frac{y_2}{V}-3.\ \frac{y_3}{V}\leqslant 1,$$

$$3.\ \frac{y_1}{V}+\ \frac{y_2}{V}+6.\ \frac{y_3}{V}\leqslant 1,$$

$$-3.\ \frac{y_1}{V}+4.\ \frac{y_2}{V}-2.\ \frac{y_3}{V}\leqslant 1,$$

$$\frac{y_1}{V}+\ \frac{y_2}{V}+\ \frac{y_3}{V}=\frac{1}{V}.$$

Putting $\frac{y_j}{V}=Y_j,\ j=1,\ 2,\ 3,$ we get

$$\left.\begin{aligned}4Y_1+\ Y_2-3Y_3&\leqslant 1,\\3Y_1+\ Y_2+6Y_3&\leqslant 1,\\-3Y_1+4Y_2-2Y_3&\leqslant 1,\end{aligned}\right] \qquad \ldots(7.19)$$

$$Y_1+Y_2+Y_3=\frac{1}{V}, \qquad \ldots(7.20)$$

where $Y_1,\ Y_2,\ Y_3\geqslant 0.$

Since B is trying to minimize V, he must maximize $\frac{1}{V}$. Thus the problem is to maximize objective function (equation 7·20) subject to constraints (7.19) which can be done by simplex method under the following steps :

Step 1 : Make the Problem as N+S Co-ordinate Problem

In order to solve the problem by simplex method, the set of inequalities (7.19) are converted into equalities by introducing new non-negative variables (slack variables) S_1, S_2 and S_3 in them. The slacks contribute zero to the objective function and the following equations result.

$$4Y_1+Y_2-3Y_3+S_1=1,$$
$$3Y_1+Y_2+6Y_3+S_2=1,$$
$$-3Y_1+4Y_2-2Y_3+S_3=1,$$

and it is desired to maximize $1/V=Y_1+Y_2+Y_3+0S_1+0S_2+0S_3,$

where $Y_1,\ Y_2,\ Y_3,\ S_1,\ S_2,\ S_3\geqslant 0.$

Step 2 : Make N-Co-ordinates Assume Zero Value

The initial feasible solution is obtained by putting decision variables Y_1, Y_2, Y_3, each equal to zero ; and we get $S_1=1$, $S_2=1$ and $S_3=1$ as the first feasible solution. The above information is expressed in the form of simplex matrix or table 7.54.

Table 7.54

Objective function c_j	1	1	1	0	0	0	
e_i variable in current solution	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
0 S_1	4	1	-3	1	0	0	1
0 S_2	3	1	6	0	1	0	1
0 S_3	-3	4	-2	0	0	1	1

Step 3 : Perform Optimality Test

Find $c_j - E_j$, where $E_j = \Sigma e_i a_{ij}$. If $c_j - E_j$ is positive under any column, at least one better solution is possible. Inserting the values of E_j and $c_j - E_j$ in table 7.54 gives table 7.55.

Table 7.55

Objective function c_j	1	1	1	0	0	0		
e_i variables in current solution	Y_1	Y_2	Y_3	S_1	S_2	S_3	b	θ
0 S_1	(4)	1	-3	1	0	0	1	$\frac{1}{4}$ ←key row
0 S_2	3	1	6	0	1	0		1/3
0 S_3	-3	4	-2	0	0	1	1	$-1/3$
$E_j = \Sigma e_i a_{ij}$	0	0	0	0	0	0		
$c_j - E_j$	1	1	1	0	0	0		*First feasible solution*
	↑ K							

Since there *is tie* in the $c_j - E_j$ row, column Y_1 is arbitrarily selected to be the key column.

Step 4 : Iterate Towards an Optimal Solution

'Y_1'-column is the key column. Mark it 'K'. Thus Y_1 is the incoming variable. Find the 'θ'-column. The minimum positive ratio is 1/4 and the row containing it is marked as key row. The key element is (4). Make this key element as (1). This is shown in table 7.56.

Table 7.56

c_j	1	1	1	0	0	0	
e_i current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
0 S_1	(1)	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$
0 S_2	3	1	6	0	1	0	1
0 S_3	-3	4	-2	0	0	1	1

Make all the elements in key column zero except the key element which is unity. This is shown in table 7.57. The new table will have S_1 replaced by Y_1. This completes the first step. The resulting table 7.57 is shown below.

Table 7.57

c_j		1	1	1	0	0	0		
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b	θ
1	Y_1	1	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$-1/3$
0	S_2	0	$\frac{1}{4}$	(33/4)	$-\frac{3}{4}$	1	0	$\frac{1}{4}$	$1/33 \leftarrow$ K
0	S_3	0	19/4	$-17/4$	$\frac{3}{4}$	0	1	7/4	$-7/17$
$E_j=\Sigma e_i a_{ij}$		1	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	0	0		
c_j-E_j		0	$\frac{3}{4}$	7/3	$-\frac{1}{4}$	0	0		
				$\uparrow$ K			*2nd feasible solution*		

Step 5

Repeat step 3 for table 7·57. On finding the value of $c_j - E_j$ for various columns we find that it is positive for some of them. Hence at least one better solution exists. The key column and key row have also been marked in table 7·57. Key element is (33/4). In table 7·58 the key element has been made unity.

Table 7·58

c_j		1	1	1	0	0	0	
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
1	Y_1	1	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$
0	S_2	0	1/33	(1)	$-1/11$	4/33	0	1/33
0	S_3	0	19/4	$-17/4$	$\frac{3}{4}$	0	1	7/4

Step 6

Make all the elements in key column zero except the key element which is unity. The resulting matrix will have S_2 replaced by Y_3. This is shown in table 7·59.

Table 7·59

c_j		1	1	1	0	0	0		
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b	θ
1	Y_1	1	3/11	0	2/11	1/11	0	3/11	1
1	Y_3	0	1/33	1	$-1/11$	4/33	0	1/33	1
0	S_3	0	$\left(\frac{161}{33}\right)$	0	4/11	17/33	1	62/33	$\frac{62}{161} \leftarrow$ K
$E_j=\Sigma e_i a_{ij}$		1	10/33	1	1/11	7/33	0		
c_j-E_j		0	23/33	0	$-1/11$	$-7/33$	0		
			$\uparrow$ K				*Third feasible solution*		

Step 7

Repeat step 3 for table 7·59. On finding the value of $c_j - E_j$ for various columns we find that it is positive for 'Y_2'-column. Hence at least one better solution exists. Key column and key row have also been marked in table 7·59. Key element is (161/33). In table 7·60 the key element has been made unity.

Table 7·60

	c_j	1	1	1	0	0	0	
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
1	Y_1	1	3/11	0	2/11	1/11	0	3/11
1	Y_3	0	1/33	1	−1/11	4/33	0	1/33
0	S_3	0	(1)	0	12/161	17/161	33/161	62/161

Step 8 :

Make all the elements in key column zero except the key element which is unity. The resulting matrix will have S_3 replaced by Y_2. This is shown in table 7·61.

Table 7·61

	c_j	1	1	1	0	0	0	
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
1	Y_1	1	0	0	26/161	10/161	−9/161	27/161
1	Y_3	0	0	1	−15/161	19/161	−1/161	3/161
1	Y_2	0	1	0	12/161	17/161	33/161	62/161
$E_j = \Sigma e_i a_{ij}$		1	1	1	23/161	46/161	23/161	
$c_j - E_j$		0	0	0	−23/161	−46/161	−23/161	

Optimal solution

Since $c_j - E_j$ is negative or zero under all columns, the solution given by table 7·61 is the optimal solution.

$$\therefore \quad Y_1 = \frac{27}{161}, \; Y_2 = \frac{62}{161} \text{ and } Y_3 = \frac{3}{161},$$

$$\frac{1}{V} = Y_1 + Y_2 + Y_3 = \frac{27}{161} + \frac{3}{161} + \frac{62}{161} = \frac{92}{161} = \frac{4}{7}.$$

$$\therefore \quad \text{Value of the game, } V = \frac{7}{4}.$$

$$\because \quad \frac{y_j}{V} = Y_j \quad \therefore \quad y_1 = Y_1 . V = \frac{27}{161} \times \frac{7}{4} = \frac{27}{92},$$

$$y_2 = Y_2.V = \frac{62}{161} \times \frac{7}{4} = \frac{31}{46} = \frac{62}{92},$$

$$y_3 = \frac{3}{161} \times \frac{7}{4} = \frac{3}{92}.$$

A's best strategies appear in the $c_j - E_j$ row under S_1, S_2 and S_3 respectively (in table 7·61) with negative signs.

$$\therefore \quad X_1 = \frac{23}{161},\ X_2 = \frac{46}{161} \text{ and } X_3 = \frac{23}{161}.$$

$$\therefore \quad x_1 = X_1 \,.\, V = \frac{23}{161} \times \frac{7}{4} = \frac{23}{92} = \frac{1}{4},$$

$$x_2 = X_2 \cdot V = \frac{46}{161} \times \frac{7}{4} = \frac{1}{2},$$

$$x_3 = X_3 \,.\, V = \frac{23}{161} \times \frac{7}{4} = \frac{1}{4}.$$

$\therefore$ Best strategies for A (Indian Oil Co.) are (1/4, 1/2, 1/4),

best strategies for B (Caltex) are (27/92, 62/92, 3/92),

value of the game (for A), $V = 7/4$.

EXAMPLE 7.10-2-2

Solve the game shown in table 7·62.

Table 7·62

		B 1	2	3
	1	3	−4	2
A	2	1	−3	−7
	3	−2	4	7

Solution

Let x_1, x_2, x_3 and y_1, y_2, y_3 be the probabilities by which A and B respectively, select their pure strategies. The first step is to look for saddle point. This step is represented in table 7·63.

Table 7·63

		B 1	2	3	
	1	3	−4	2	−4
A	2	1	−3	−7	−7
	3	−2	4	7	(−2)
		(3)	4	7	

Evidently, the game has no saddle point. The value of the game lies between −2 and +3. The next step is to see if the given matrix can be reduced by dominance. We find that it cannot be reduced. Solving it by the method of matrices we get table 7·64.

Table 7·64

			B y_1 1	y_2 2	y_3 3			
	x_1	1	3	−4	2	7	−6	12
A	x_2	2	1	−3	−7	4	4	57
	x_3	3	−2	4	7	−6	−3	52
			2	−1	9			
			3	−7	−14			
			77	55	11			143/121

Since the sum of the oddments of A and B are not equal, the problem cannot be solved by the method of matrices. The method of linear programming will be used to solve it. Since value of the game lies between −2 and +3, it is possible that the value of the game (V) may be negative or zero, because $-2 \leqslant V \leqslant 3$.

Thus a constant K is added to all the elements of the matrix which is *at least* equal to the negative of the maximin value *i.e.*, K

must be $\geqslant 2$. Let $K=3$. The given matrix is thus modified to the one shown in table 7·65.

Table 7·65

		B 1	2	3
	1	6	-1	5
A	2	4	0	-4
	3	1	7	10

The value of the game (to A) is V. Consider the game from B's point of view. B is trying to minimize V.

Then, against A_1, $6y_1 - y_2 + 5y_3 \leqslant V$,

against A_2, $4y_1 + 0y_2 - 4y_3 \leqslant V$,

against A_3, $y_1 + 7y_2 + 10y_3 \leqslant V$,

$y_1 + y_2 + y_3 = 1$, (sum of the probabilities must be equal to 1)

where $y_1, y_2, y_3 \geqslant 0$.

Dividing each of the above relations by V,

$$6\frac{y_1}{V} - \frac{y_2}{V} + 5\frac{y_3}{V} \leqslant 1,$$

$$4\frac{y_1}{V} - 0\frac{y_2}{V} - 4\frac{y_3}{V} \leqslant 1,$$

$$\frac{y_1}{V} + 7\frac{y_2}{V} + 10\frac{y_3}{V} \leqslant 1,$$

$$\frac{y_1}{V} + \frac{y_2}{V} + \frac{y_3}{V} = \frac{1}{V},$$

where $y_1, y_2, y_3 \geqslant 0$.

Putting $\frac{y_j}{V} = Y_j$, $j=1, 2, 3$, we get

$$\left.\begin{array}{l} 6Y_1 - Y_2 + 5Y_3 \leqslant 1, \\ 4Y_1 \quad\;\; - 4Y_3 \leqslant 1, \\ Y_1 + 7Y_2 + 10Y_3 \leqslant 1, \end{array}\right] \qquad ...(7\cdot21)$$

$$Y_1 + Y_2 + Y_3 = \frac{1}{V}, \qquad ...(7\cdot22)$$

where $Y_1, Y_2, Y_3 \geqslant 0$.

Since B is trying to minimize V, he must maximize $\frac{1}{V}$. Thus the problem is to maximize objective function (equation 7·22) subject to constraints (relations 7·21), which can be done by simplex method under the following steps :

Step 1. Make the Problem as N+S Co-ordinate Problem

In order to solve the problem by simplex method, the set of inequalities (7·21) are converted into equalities by introducing new non-negative variables (slack variables) S_1, S_2 and S_3 in them. The slacks contribute zero to the objective function and the following equations result :

$$6Y_1 - Y_2 + 5Y_3 + S_1 = 1,$$
$$4Y_1 \quad - 4Y_3 + S_2 = 1,$$
$$Y_1 + 7Y_2 + 10Y_3 + S_3 = 1,$$

and it is desired to maximize $1/V = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$,

where $\quad Y_1, Y_2, Y_3, S_1, S_2, S_3 \geqslant 0.$

Step 2. Make N Co-ordinates Assume Zero Value

The initial feasible solution is obtained by putting decision variables Y_1, Y_2, Y_3, each equal to zero and we get $S_1=1$, $S_2=1$ and $S_3=1$ as the first feasible solution. The above information is expressed in the form of a simple matrix or table 7·66.

Table 7·66

Objective function	c_j	1	1	1	0	0	0	
e_i	variables in current solution	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
0	S_1	6	−1	5	1	0	0	1
0	S_2	4	0	−4	0	1	0	1
0	S_3	1	7	10	0	0	1	1

Step 3. Perform Optimality Test

Find $c_j - E_j$, where $E_j = \Sigma e_i a_{ij}$. If $c_j - E_j$ is positive under any column, at least one better solution is possible. Inserting the values of E_j and $c_j - E_j$ in table 7·66 we get table 7·67.

Table 7·67

c_j		1	1	1	0	0	0		
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b	θ
0	S_1	(6)	-1	5	1	0	0	1	1/6←key row
0	S_2	4	0	-4	0	1	0	1	1/4
0	S_3	1	7	10	0	0	1	1	1
$E_j=\Sigma e_i a_{ij}$		0	0	0	0	0	0		
c_j-E_j		1	1	1	0	0	0		
		↑ K							*First feasible solution*

Since there *is tie* in the c_j-E_j row, column Y_1 is selected arbitrarily as the key column.

Step 4. Iterate Towards an Optimal Solution

'Y_1' column is the key column. Mark it 'K'. Thus Y_1 is the incoming variable. Find the 'θ' column. The minimum positive ratio is 1/6 and the row containing it is marked as key row. The key element is (6). Make this key element as unity. This is shown in table 7·68.

Table 7·68

c_j		1	1	1	0	0	0	
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
0	S_1	(1)	$-1/6$	5/6	1/6	0	0	1/6
0	S_2	4	0	-4	0	1	0	1/4
0	S_3	1	7	10	0	0	1	1

Make all the elements in key column zero except the key element which is unity. This is shown in table 7·68. The new table will have S_1 replaced by Y_1. This completes the first stage. The resulting table 7·69 is shown below.

Table 7·69

c_j		1	1	1	0	0	0		
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b	θ
1	Y_1	1	$-1/6$	5/6	1/6	0	0	1/6	-1
0	S_2	0	2/3	$-22/3$	$-2/3$	1	0	$\frac{-5}{12}$	$\frac{-15}{8}$
0	S_3	0	(43/6)	55/6	$-1/6$	0	1	5/6	5/43←K
$E_j=\Sigma e_i a_{ij}$		1	$-1/6$	5/6	1/6	0	0		
c_j-E_j		0	7/6	1/6	$-1/6$	0	0		
			↑ K						*2nd feasible solution*

Step 5

Repeat step 3 for table 7·69. On finding the value of $c_j - E_j$ for various columns, we find that it is positive for some of them. Hence at least one better solution exists. The key column and key row have also been marked in table 7·69. Key element is (43/6). In table 7·70, the key element has been made unity.

Table 7·70

c_j		1	1	1	0	0	0	
e_i	current solution variables	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
1	Y_1	1	$-1/6$	$5/6$	$1/6$	0	0	$1/6$
0	S_2	0	$2/3$	$-22/3$	$-2/3$	1	0	$-5/12$
0	S_3	0	(1)	$55/43$	$-1/43$	0	$6/43$	$5/43$

Step 6

Make all the elements in key column zero except the key element which is unity. This is shown in table 7·71. The new table will have S_3 replaced by Y_2. The resulting table 7·71 is shown below.

Table 7·71

c_j		1	1	1	0	0	0	
e_i	*c.s.v.*	Y_1	Y_2	Y_3	S_1	S_2	S_3	b
1	Y_1	1	0	$45/43$	$7/43$	0	$1/43$	$8/43$
0	S_2	0	0	$-\frac{352}{43}$	$-\frac{28}{43}$	1	$-\frac{4}{43}$	$-\frac{85}{172}$
1	Y_2	0	1	$\frac{55}{43}$	$-\frac{1}{43}$	0	$\frac{6}{43}$	$\frac{5}{43}$
$E_j = \Sigma e_i a_{ij}$		1	1	$\frac{100}{43}$	$\frac{6}{43}$	0	$\frac{7}{43}$	
$c_j - E_j$		0	0	$-\frac{57}{43}$	$-\frac{6}{43}$	0	$-\frac{7}{43}$	

Optimal solution

Since $c_j - E_j$ is not positive under any column, the solution given by table 7·71 is the optimal solution.

$$\therefore \quad Y_1 = \frac{8}{43}, \; Y_2 = \frac{5}{43}, \; Y_3 = 0,$$

$$\frac{1}{V} = Y_1 + Y_2 + Y_3 = \frac{8}{43} + 0 + \frac{5}{43} = \frac{13}{43}.$$

$\therefore$ Value of the game for the modified matrix, $V = \frac{43}{13}$.

$$\because \quad \frac{y_j}{V}=Y_j \quad \therefore \quad y_j=Y_j \,.\, V$$

$$\therefore \quad y_1=Y_1.\quad V=\frac{8}{43}\times\frac{43}{13}=\frac{8}{13},$$

$$y_2=Y_2.\quad V=\frac{5}{43}\times\frac{43}{13}=\frac{5}{13},$$

$$y_3=Y_3.\quad V=0.$$

A's best strategies appear in the c_j-E_j row under S_1, S_2, S_3 respectively (table 7·71) with negative signs.

$\therefore \quad X_1=6/43$, $X_2=0$ and $X_3=7/43$.

$\therefore \quad x_1=X_1.\quad V=6/43\times 43/13=6/13,$

$x_2=X_2.\quad V=0,$

$x_3=X_3\quad V=7/43\times 43/13=7/13.$

$\therefore$ Best strategies for A are (6/13, 0, 7/13),

best strategies for B are (8/13, 5/13, 0),

value of game for the given matrix$=43/13-3=4/13$.

EXAMPLE 7.10-2.3

Write both primal and dual L.P. problems corresponding to the following rectangular game :

Table 7.72

	Y			
	5	7	4	10
X	4	3	7	9
	1	2	5	6

Solution

Let x_1, x_2, x_3 and y_1, y_2, y_3, y_4 be the probabilities by which, A and B respectively, select their pure strategies. Then we can write

Table 7.73

		Y			
		y_1	y_2	y_3	y_4
	x_1	5	7	4	10
X	x_2	4	3	7	9
	x_3	1	2	5	6

Let the value of the game be V. Consider the game from Y's point of view. He is interested in determining strategies y_1, y_2, y_3 and y_4 that will minimize his maximum expected loss V. Thus

$$\begin{aligned}
&\text{against } X_1, \quad 5y_1+7y_2+4y_3+10y_4 \leqslant V,\\
&\text{against } X_2, \quad 4y_1+3y_2+7y_3+9y_4 \leqslant V,\\
&\text{against } X_3, \quad y_1+2y_2+5y_3+6y_4 \leqslant V,\\
\text{where} \quad & y_1+y_2+y_3+y_4=1, \quad \text{(Since sum of probabilities must be unity)}\\
& y_1, y_2, y_3, y_4, \text{ all } \geqslant 0.
\end{aligned}$$

Since maximin value of the game is greater than zero, dividing by V we get the following relations :

$$5\frac{y_1}{V}+7\frac{y_2}{V}+4\frac{y_3}{V}+10\frac{y_4}{V} \leqslant 1,$$

$$4\frac{y_1}{V}+3\frac{y_2}{V}+7\frac{y_3}{V}+9\frac{y_4}{V} \leqslant 1,$$

$$\frac{y_1}{V}+2\frac{y_2}{V}+5\frac{y_3}{V}+6\frac{y_4}{V} \leqslant 1,$$

$$\frac{y_1}{V}+\frac{y_2}{V}+\frac{y_3}{V}+\frac{y_4}{V}=\frac{1}{V},$$

where y_1, y_2, y_3, y_4, all $\geqslant 0$.

Putting $\frac{y_j}{V}=Y_j$, $j=1, 2, 3, 4$, we get

$$\left.\begin{aligned}
5Y_1+7Y_2+4Y_3+10Y_4 &\leqslant 1,\\
4Y_1+3Y_2+7Y_3+9Y_4 &\leqslant 1,\\
Y_1+2Y_2+5Y_3+6Y_4 &\leqslant 1,
\end{aligned}\right\} \qquad \text{...(7.23)}$$

$$Y_1+Y_2+Y_3+Y_4=\frac{1}{V}, \qquad \text{...(7.24)}$$

where Y_1, Y_2, Y_3, Y_4, all $\geqslant 0$. ...(7.25)

Since Y is trying to minimize V, it must maximize $\frac{1}{V}$. Thus the problem is to maximize objective function (equation 7.24) subject to constraints (7.23) and the non-negativity restriction (7.25).

Similarly, an L.P. problem can be developed for X's strategies. X is interested in determining strategies x_1, x_2, x_3 that will maximize his minimum expected gain V. Thus

$$\begin{aligned}
&\text{against } Y_1, \quad 5x_1+4x_2+x_3 \geqslant V,\\
&\text{against } Y_2, \quad 7x_1+3x_2+2x_3 \geqslant V,
\end{aligned}$$

$$\text{against } Y_3, \quad 4x_1+7x_2+5x_3 \geqslant V,$$

$$\text{against } Y_4, \quad 10x_1+9x_2+6x_3 \geqslant V,$$

$$x_1+x_2+x_3=1,$$

$$\text{where} \quad x_1, x_2, x_3, \text{ all} \geqslant 0.$$

Dividing by V and substituting $\frac{x_i}{V}=X_i$, $i=1, 2, 3$, we get

$$\left.\begin{array}{l} 5X_1+4X_2+X_3 \geqslant 1, \\ 7X_1+3X_2+2X_3 \geqslant 1, \\ 4X_1+7X_2+5X_3 \geqslant 1, \\ 10X_1+9X_2+6X_3 \geqslant 1, \end{array}\right\} \qquad ...(7.26)$$

$$X_1+X_2+X_3 \geqslant \frac{1}{V}, \qquad ...(7.27)$$

$$\text{where } X_1, X_2, X_3 \text{ all } \geqslant 0. \qquad ...(7{\cdot}28)$$

Since X is tying to maximize V, he must minimize $\frac{1}{V}$. Thus the problem is minimize objective function equation (7.27) subject to constraints (7.26) and the non-negativity restriction (7.28). If programme for X is regarded as primal, that for Y is dual and vice-versa.

7.10-3. Iterative Method of Approximate Solution

Iterative method can be applied to solve 3×3 or higher games which cannot be solved by the method of matrices and are extremely tedius to be solved by method of linear programming. This method gives an approximate solution for the value of the game and the true value can be determined to any degree of accuracy. Optimum strategies can also be determined but not so satisfactorily. The method assumes that each player acts under the assumption that past is the best guide for the future and will play in such a manner so as to maximize the expected gain or to minimize the expected loss.

EXAMPLE 7.10-3.1

Find the value and optimum strategies of the rectangular game where payoff matrix is given in table 7.74. Use iterative method of approximate solution.

Table 7.74

		B 1	2	3
	1	2	0	0
A	2	0	0	4
	3	0	3	0

Solution. In this method, player A selects any row arbitrarily and places it below the matrix. Let he select row 1. Player B examines this row and chooses a column corresponding to the *smallest* number in the row. Let it be column 3. Column 3 is then placed to the *right* of the matrix.

Player A examines this column and chooses a row corresponding to the *largest* number in this column. This is row 2. Row 2 is then added to the row last chosen and the sum of the two rows is placed beneath the row last chosen.

Player B chooses a column corresponding to the *smallest*

Table 7.75

		B 1	2	3	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	
	1	2	0	0	0	0	0	2	4	(6)	(8)	(10)	(10)	10	$\frac{4}{10}$
A	2	0	0	4	(4)	(4)	4	4	4	4	4	4	8	(12)	$\frac{3}{10}$
	3	0	3	0	0	3	(6)	(6)	(6)	6	6	6	6	6	$\frac{3}{10}$
		2	0	(0)											
		2	(0)	4											
		2	(0)	8											
		(2)	3	8											
		(2)	6	8											
		(2)	9	8											
		(4)	9	8											
		(6)	9	8											
		8	9	(8)											
		10	9	(8)											
		$\frac{5}{10}$	$\frac{2}{10}$	$\frac{3}{10}$											

number in the new row and adds this column to the column last chosen. In case of a tie the player should choose the row or column different from his last choice. The procedure is repeated for a number of iterations. Ten iterations have been shown for the present example. The smallest number in each succeeding row and the largest number in each succeeding column have been ringed. The approximate strategies are determined by dividing the *number* of ringed numbers in each row or column by the total number of ringed numbers. Thus, A's approximate strategies are (4/10, 3/10, 3/10) and B's approximate strategies are (5/10, 2/10, 3/10).

The upper bound for the game value is determined by dividing the highest number in the last column (12), by the total number of iterations *i.e.*, 10 in this case. Similarly, the lower bound is determined by dividing the lowest number in the last row (8), by the total number of iterations, 10. Thus

$$\frac{8}{10} \leqslant V \leqslant \frac{12}{10}.$$

The approximate solution gets better with further iterations.

7·11. Summary of Systematic Methods for Solving Two-Person Zero-Sum Games

1. Look for a saddle point or points. If one is found, the game is readily solved.
2. Look for dominance. If dominance is found, delete the dominated row(s) and/or column(s). Each matrix so formed must be checked for dominance.
3. If the size of the reduced matrix becomes (2×2) with no saddle point, it can be solved by arithmetic and algebraic methods described in section 7·8.
4. If the size of the reduced matrix becomes $(2 \times n)$ or $(m \times 2)$, use graphic method to reduce it to (2×2) size matrix and then solve it by arithmetic or algebraic method.

   If graphic method is not to be applied, the game can still be solved by algebraic method and method of subgames. All these methods are described in section 7·9.
5. If the reduced size of the matrix becomes (3×3) or higher, algebraic method, method of matrices, simplex method of linear programming and iterative method of approximate solution can be used for solving it. These methods are described in section 7·10.

7·12. n-Person Zero-Sum Games

These games are usually treated as if the two coalitions are formed by the n-persons involved. The characteristics of such a game are values of the various games between every possible pair of coalitions. For example, for a player A, B, C and D the following coalitions can be formed :

A against B, C, D ;
B against A, C, D ;
C against A, B, D ;
D against A, B, C ;
A, B against C, D ;
A, C against B, D ;
A, D against B, C.

If the value of the game for B, C, D coalition is V, then the value of the game for A is $-V$, since it is zero-sum game. Thus in a four-person zero-sum game there will be seven values or characteristics for the game, which are obtained from the seven different coalitions.

EXAMPLE 7·12-1

Find the values of the three-person zero-sum game in which player A has two choices X_1, X_2 ; player B has two choices Y_1, Y_2 and player C also has two choices Z_1 and Z_2. The payoff matrix is shown in table 7·76.

Table 7·76

Choice			*Payoff*		
A	B	C	A	B	C
X_1	Y_1	Z_1	3	2	−2
X_1	Y_1	Z_2	0	2	1
X_1	Y_2	Z_1	0	−1	4
X_1	Y_2	Z_2	1	3	−1
X_2	Y_1	Z_1	4	−1	0
X_2	Y_1	Z_2	−1	1	3
X_2	Y_2	Z_1	1	0	2
X_2	Y_2	Z_2	0	2	1

Solution. There are three possible coalitions :

1. A against B, C ;
2. B against A, C ;
3. C against A, B.

We shall solve each of the resulting game.

1. **A against B, C.** The payoff matrix in A's terms is shown in table 7·77.

Table 7·77

		Y_1, Z_1	Y_1, Z_2	$Y_2\ Z_1$	Y_2, Z_2	
A	X_1	3	0	0	1	(0)
	X_2	4	−1	1	0	−1
		4	(0)	1	1	

The first step is to look for a saddle point. The game has a saddle point. Thus we have the following solution for A against B, C :

A's best strategy is X_1,

B's and C's best combination of strategies is Y_1, Z_2,

value of the game for A is zero,

value of the game for B, C is zero.

2. **B against A, C :** The payoff matrix in B's terms is shown in table 7.78.

Table 7.78

A, C

		X_1, Z_1	X_1, Z_2	X_2, Z_1	$X_2\ Z_2$
B	Y_1	2	2	−1	1
	Y_2	−1	3	0	2

The first step is to look for saddle point. In this game there is none. The next step is to reduce the game by the rules of dominance. Columns X_1, Z_2 and X_2, Z_2 will be dominated and should therefore be deleted. The resulting reduced matrix is shown in table 7.79. It has no saddle point.

Table 7.79

A, C

		X_1, Z_1	X_2, Z_1		
B	Y_1	2	−1	1	1/4
	Y_2	−1	0	3	3/4
		1	3		
		1/4	3/4		

Solving this 2×2 game by the arithmetic method we get the following result :

B's best strategy is to play choice Y_1 with a frequency of 1/4 and choice Y_2 with a frequency of 3/4. A's and C's best strategy is for C to play Z_1 and for A to play X_1 with a frequency of 1/4 and X_2 with a frequency of 3/4.

$$\text{Value of the game for B}=\frac{\frac{2}{4}-\frac{3}{4}}{\frac{1}{4}+\frac{3}{4}}=-\frac{1}{4},$$

value of the game for A, C=1/4.

3. **C against A, B.** The payoff matrix in C's terms is shown in table 7.80.

Table 7.80

		A, B			
		X_1, Y_1	X_1, Y_2	X_2, Y_1	X_2, Y_2
C	Z_1	–2	4	0	2
	Z_2	1	–1	3	1

The first step is to look for saddle point. In this case there is none. The next step is to reduce the game by the rules of dominance. Columns X_2, Y_1 and X_2, Y_2 are dominated by column X_1, Y_1 and the resulting reduced matrix is shown in table 7.81.

Table 7.81

		X_1, Y_1	X_1, Y_2		
C	Z_1	–2	4	2	2/8
	Z_2	1	–1	6	6/8
		5	3		
		5/8	3/8		

This 2×2 game has no saddle point. Solving it by the arithmetic method we get the following results :
C's best strategy is to play choice Z_1 with a frequency of 2/8 and choice Z_2 with a frequency of 6/8 ;
A's and B's best strategy is for A to play X_1 and for B to play Y_1 with a frequency of 5/8 and Y_2 with a frequency of 3/8,

$$\text{value of the game for C is} = \frac{-\frac{10}{8}+\frac{12}{8}}{5/8+3/8} = \frac{2/8}{1} = \tfrac{1}{4},$$

value of the game for A, B $= -\frac{1}{4}$.

Therefore, the characteristics of the game are

$$\begin{array}{ll} V(A)=0, & V(B, C)=\ \ 0, \\ V(B)=-1/4, & V(A, C)=\ \ 1/4, \\ V(C)=\ \ 1/4, & V(A, B)=-1/4. \end{array}$$

7.13. Limitations of Game Theory and Concluding Remarks

The basic limitation of game theory is that it is difficult for the players to find the values of the payoff matrix accurately. Incorrect figures in the matrix will result in misleading output. It is not difficult to establish that one outcome is better than the other, but it is quite another thing to state exactly how much more.

Secondly, most of the decisions made by the managements cannot be categorised as two-person games as the Govt. or Society is invariably present as third or fourth party.

Further, the two parties involved in the game may not have equal intelligence or equal knowledge.

Game theory has not still reached its full potential. It is very likely that game theory will be used more and more for solving OR marketing problems as more firms employ computers to simulate their operations. The wedlock of game theory with simulation to solve marketing management problems will give game theory the required thrust to become an important tool for quantitative decision making.

7.14. Bibliographic Notes

The theory of games has quite a wealth of literature. The literature varies, ranging from purely mathematical studies to text books dealing with fundamentals and applications of this theory.

A fine mathematical treatment of the general theory is made by Mckinsay [4] in his "Introduction to the Theory of Games". The book deals mainly with two-person games. Relationship between game theory and linear programming is also discussed.

Some of the most important basic research developments in the field of game theory are presented in the two volumes edited by Kuhn and Tucker [2]. The work is intended for mathematicians.

A very elementary introduction to the theory of games is given in the book by J. Williams [13].

The book entitled,"Theory of Games and Economic Behaviour", written jointly by J. Von Neumann and O. Morgenstern [12], deals with the theory of games and its applications in economics. The book needs quite a good mathematical knowledge on the part of reader.

Undoubtedly, the best book on the subject at present is by R. Luce and H. Raiffa [3]. This book can be understood by the reader without much mathematical training. It deals with game theory and its applications.

Theory of games and its links with theory of linear programming are also dealt in a small book by S. Vajda [11].

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EXERCISES

Section 7.5

1. The following games have saddle point solutions. Determine the saddle point and optimum strategies for each player.

(a)

	Y		
X	4	6	4
	2	10	0

(b)

	Y		
X	2	−1	3
	4	5	6

(c)

	B	
A	1	4
	−1	−4
	−3	−2

(d)

	B	
A	4	4
	4	4

(*Ans.* (*a*) S.P.$=4$, X_1, Y_1 or Y_3.
(*b*) S.P.$=4$, X_2, Y_1.
(*c*) S.P.$=1$, A_1, B_1.
(*d*) S.P.$=4$, A_1 or A_2 ; B_1 or B_2.)

2. Determine the optimum strategies and values of the following games :

(a)

	B			
A	−3	4	2	9
	7	8	6	10
	6	2	4	−1

[*Delhi B. Sc. (Math.) 1972*]

(b)

		Y		
	1	7	3	4
X	5	6	4	5
	7	2	0	3

[*Delhi B. Sc. (Math.) 1972*]

(c)

		Y		
	−1	9	6	8
	−2	10	4	6
X	5	3	0	7
	7	−2	8	4

(d)

			B		
	10	4	2	9	1
A	7	6	5	7	8
	3	5	4	4	9
	6	7	3	3	2

(*Ans.* (*a*) A_2, B_3 ; $V=6$.
(*b*) X_2, Y_3 ; $V=4$.
(*c*) $0 \leqslant V \leqslant 7$.
(*d*) A_2, B_3 ; $V=5$)

3. Find the value of the games shown below. Also indicate whether they are fair or strictly determinable.

(a)

	B			
A	1	9	6	0
	2	3	8	−1
	−5	−2	10	−3
	7	4	−2	−5

(b)

	B			
A	6	−2	−3	8
	−1	−2	−7	0
	8	9	−6	−7
	9	5	−7	7

(*Ans.* (*a*) $V=0$; A_1, B_4 ; fair.
(*b*) $V=-3$; A_1, B_3; Strictly determinable)

4. Find the range of values of p and q so that the entry (2, 2) is a saddle in the following games :

(a)

	B		
A	2	q	4
	p	6	11
	7	3	4

(b)

	0	2	3
	8	5	q
	2	p	4

(*Ans.* (*a*) $p \geqslant 6$, $q \leqslant 6$.
(*b*) $p \geqslant 5$, $q \leqslant 5$)

Section 7·8-7·9.

5. Two players A and B match coins. If the coins match, then A wins one unit of value ; if the coins do not match, then B wins one unit of value. Determine optimum strategies for the players and the value of the game. [*Delhi M.B.A. 1972*]

(*Ans.* A $(\frac{1}{2}, \frac{1}{2})$, B $(\frac{1}{2}\ \frac{1}{2})$; $V=0$)

6. Solve the game whose payoff matrix is

	B	
A	5	2
	3	4

by arithmetic method and verify the results by algebraic method. Calculate the game value.

(*Ans.* A (1/4, 3/4), B (1/2, 1/2 ; V=3·5)

7. In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails, and loses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A.

[*Kuru. M.Sc. (Math.) 1977 ; Delhi M.B.A. 1971*]

(*Ans.* A (1/4, 3/4), B (1/4, 3/4) ; V=−1/8)

8. Use dominance property to solve the following game between two players A and B :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}$$

[*Meerut M.Sc. (Maths.) 1977*]

(*Ans.* A_1, B_3; V=6)

9. Solve, by using dominance property, the following game :

		B I	B II	B III
	I	1	7	2
A	II	6	2	7
	III	6	1	6

[*Meerut M.Sc. (Math.) 1974*]

(*Ans.* A (2/5, 3/5, 0), B (1/2, 1/2, 0) ; V=4)

10. Reduce the following game to 2×2 game. Solve it by arithmetic method and verify by algebraic method.

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} -2 & -4 & 3 & 4 \\ -6 & -5 & 2 & 1 \end{bmatrix} \end{array}$$

(*Ans.* X (1/3, 2/3), Y (1/5, 4/5, 0, 0) ; V=−18/5)

11. (*a*) Explain the following :

Competitive games ; zero-sum games ; strategy ; two-persons zero-sum game.

(*b*) Find the solution of the following game :

$$\begin{array}{c|ccc|} & & B & \\ \hline A & 1 & 3 & 11 \\ & 8 & 5 & 2 \\ \hline \end{array}$$

[*Sambalpur Univ. May, 1977*]

$$\left(\text{Ans. } A\left(\frac{6}{16}, \frac{10}{16}\right), B\left(\frac{9}{16}, 0, \frac{7}{16}\right); V=\frac{43}{8}\right)$$

12. (*a*) Explain clearly the following terms :

(1) Strategy.

(2) Pay off matrix.

(3) Saddle point.

(*b*) Find the best strategy and the value of the following game :

		B I	B II	B III
A	I	−1	−2	8
A	II	7	5	−1
A	III	6	0	12

[*Baroda Univ. April, 1973*]

$$\left(\text{Ans. } A\left(0, \frac{12}{18}, \frac{6}{18}\right), B\left(0, \frac{13}{18}, \frac{5}{18}\right); V=\frac{10}{3}\right)$$

13. (*a*) What is game theory ? Include in your answer various approaches in solving for strategies and game values.

[*Baroda Univ. B.E. May, 1975*]

(*b*) Describe the role of 'Theory of games' for scientific decision making.

(*c*) Describe how a 'Two-person zero-sum game' can be solved by linear programming.

[*Punjab Univ. B. Sc. (Prod.) Engg. 1977*]

14. Consider the game

		B 1	2	3
	1	5	50	50
A	2	1	1	0·1
	3	10	1	10

Verify that the strategies $\left(\frac{1}{6}, 0, \frac{5}{6}\right)$ for player A and $\left(\frac{49}{54}, \frac{5}{54}, 0\right)$ for B are optimal and find the value of the game.

$\left(Ans.\ V = \frac{55}{6}\right)$

15. Find the optimum strategies for X and Y and the value of the game :

	Y		
	–6	10	11
X	–1	–2	–3
	–1	–2	–4

$\left(Ans.\quad X\left(\frac{2}{19}, \frac{17}{19}, 0\right), Y\left(\frac{14}{19}, 0, \frac{5}{19}\right); V = -\frac{29}{19}\right)$

16. Find the optimum strategies for Y and the value of the game.

	Y				
	4	–1	4	–1	2
X	2	2	3	–4	2
	1	–3	1	0	–4

$$\left(Ans.\ \ Y\left(0, 0, 0, \frac{6}{7}, \frac{1}{7}\right);\ V=-\frac{4}{7}\right)$$

17. A and B play a game in which each has three coins: 5 paise, 10 paise, and 20 paise. Each selects a coin with the knowledge of other's choice. If the sum of the coins is an odd amount, A wins B's coin, if the sum is even, B wins A's coin. Find the best strategy for each player and the value of the game.

[*Meerut M. Sc. (Maths) 1976*]

(*Ans.* A $(\frac{1}{2}, \frac{1}{2}, 0)$, B $(\frac{2}{3}, \frac{1}{3}, 0)$, V=0)

[**Hint.** The payoff matrix for A is

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & -20 & -20 \end{bmatrix} \end{array}$$

This matrix can be solved for optimal strategies].

18. Consider the following payoff matrix for two firms. What is the best mixed strategy for both the firms and also find out the value of the game.

		Firm II	
	No advertising	*Medium advertising*	*Large advertising*
No advertising	60	50	40
Firm I Medium advertising	70	70	50
Large advertising	80	60	75

[*Delhi M.B.A. 1973*]

$$\left(Ans.\ \ I\left(0, \frac{3}{7}, \frac{4}{7}\right),\ II\left(0, \frac{5}{7}, \frac{2}{7}\right);\ V=\frac{450}{7}\right)$$

19. Two separate firms, A and B, for years have been selling a competing product which forms a part of both firms' total sales. The marketing executive of firm A raised the question, "What should be the firms' strategies in terms of advertising for the product in question?" The market research team of the firm A developed the following data for varying degrees of advertising :

(*a*) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(*b*) Firm A with no advertising : 40% of the market with medium advertising by firm B and 28% of the market with large advertising by firm B.

(*c*) Firm A using medium advertising : 70% of the market with no advertising by firm B and 45% of the market with large advertising by firm B.

(*d*) Firm A using large advertising : 75% of the market with no advertising by firm B and 47·5% of the market with medium advertising by firm B.

Based upon the foregoing information, answer the marketing executive's question.

20. In an election for M.L.A., two political parties A and B are thinking of nominating a candidate in a closed session, whose results are to be announced simultaneously. The following odds are offered for the various possible combinations of candidates:

Party A	*Odds*	*Party B*
Sharma	3:1	Singh
Sharma	4:1	Gill
Sharma	1:3	Bajwa
Goel	3:7	Singh
Goel	3:2	Gill
Goel	1:4	Bajwa
Kapoor	4:1	Singh
Kapoor	1:4	Gill
Kapoor	2:1	Bajwa

The parties want to select candidate in accordance with standard minimax criterion. What are the optimal strategies for party A and B ?

(*Ans.* A (Sharma), B (Bajwa); V = —0·50)

[**Hint** : If the payoff for a party are taken as its probability of winning, the two payoff matrices in the 'game' would be

		Party B		
		Singh	Gill	Bajwa
	Sharma	0·75	0·80	0·25
Party A	Goel	0·30	0·60	0·20
	Kapoor	0·80	0·20	0·25

and

Payoff matrix for party A

		Party B		
		Singh	Gill	Bajwa
	Sharma	0·25	0·20	0·75
Party A	Goel	0·70	0·40	0·80
	Kapoor	0·20	0·80	0·75

Payoff matrix for party B

With these payoff matrices, the game is not zero-sum. The game can be converted to zero-sum by taking the payoffs to be *differences* between the corresponding win probabilities.

The payoff matrix for party A becomes

		Party B		
		Singh	Gill	Bajwa
	Sharma	0·50	0·60	−0·50
Party A	Goel	−0·40	0·20	−0·60
	Kapoor	0·60	−0·60	−0·50

[This matrix can be solved for optimal strategies.]

21. A steel company is negotiating with its union for revision of wages to its employees. The management, with the help of a mediator, has prepared a payoff matrix shown below. Plus sign

represents wage increase, while negative sign stands for wage decrease. Union has also constructed a table which is comparable to that developed by management. The management does not have the specific knowledge of game theory to select the best strategy (or (strategies) for the firm. You have been called to assist the management on the problem. What game value and strategies you suggest for the opposing groups ?

Additional costs to Steel Co. (Rs.)

Union strategies

		U_1	U_2	U_3	U_4
	C_1	+2·50	+2·70	+3·50	−0·20
	C_2	+2·00	+1 60	+0·80	+0·80
Steel Co. strategies	C_3	+1·40	+1·20	+1·50	+1·30
	C_4	+3·00	+1·40	+1·90	0

[**Hint**. Solve the matrix by the arithmetic method by reducing it to 3×2 matrix by using the principle of dominance].

$\left(\textit{Ans.}\quad C_1=\frac{1}{30},\ C_2=0,\ C_3=\frac{29}{30},\ C_4=0;\ U_1=0,\ U=\frac{1}{2},\ U_3=0,\ U=\frac{1}{2};\ V=1{\cdot}25\right)$

22. In a well known children's game each player says 'stone' or 'scissors' or paper'. If one says 'stone' and the other 'scissors', then the former wins a rupee. Similarly 'scissors' beats 'paper' and 'paper' beats 'stones', *i.e.*, the player calling the former word wins a rupee. If the two players name the same item, then there is a tie *i.e.*, there is no payoff. Write down the payoff matrix, find the value of the game and hence write down the optimal strategies for both the players.

[*Bombay B.Sc.* (*Stat.*) *1977*]

Section 7.9-3

23. Solve the following games by reducing them to 2×2 games by graphical method :

(*a*)

		B				
A	3	0	6	−1	7
	−1	5	−2	2	1

$\left(Ans.\ \ A\left(\frac{3}{7}, \frac{4}{7}\right), B\left(\frac{3}{7}, 0, 0, \frac{4}{7}, 0\right), V=\frac{5}{7}\right)$

(*b*)

		B				
		1	2	3	4	5
A	1	0	4	−8	−5	1
	2	1	5	8	−4	0

(*Ans.* A_2, B_4; $V=-4$)

24. Solve the following 5×2 games graphically :

(*a*)

	B	
A	2	3
	6	7
	−6	10
	−3	−2
	3	2

(*b*)

	B	
A	−4	3
	−7	1
	−2	−4
	−5	−2
	−1	−6

(*Ans.* (*a*) A_2, B_1; $V=6$.

(*b*) $A\left(\frac{5}{12}, 0, 0, 0, \frac{7}{12}\right), B\left(\frac{9}{12}, \frac{3}{12}\right);\ V=-\frac{9}{4}\right)$

Section 7.10-1

25. Solve the following problem by the method of matrices :

$$\begin{array}{c|ccc|} & & B & \\ \hline & 1 & 0 & 2 \\ A & 3 & 0 & 0 \\ & 0 & 2 & 1 \\ \hline \end{array}$$

(*Ans.* A$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, B$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; $V=1$)

26. Solve the following 3×3 game by the method of matrices :

$$\begin{array}{c|c|c|c|} & & B & \\ \hline & 1 & -1 & -1 \\ \hline A & -1 & -1 & 3 \\ \hline & -1 & 2 & -1 \\ \hline \end{array}$$

$\left(\textit{Ans.} \ A\left(\frac{6}{13}, \frac{3}{13}, \frac{4}{13}\right), B\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}\right); \ V=-\frac{1}{13} \right)$

27. Solve the following rectangular game by the method of matrices :

$$\begin{array}{c|cccc|} & & B & & \\ \hline & 3 & -1 & 1 & 2 \\ A & -2 & 3 & 2 & 3 \\ & 2 & -2 & -1 & 1 \\ \hline \end{array}$$

[*Delhi B.Sc. (Math.) 1974*]

$\left(\textit{Ans.} \ A\left(-\frac{1}{9}, \frac{4}{9}, \frac{6}{9}\right), B\left(\frac{7}{9}, \frac{11}{9}, -1\right); \ V=\frac{1}{9} \right)$

Section 7.10-2

28. Solve the following game by linear programming :

		B	
	0	2	2
A	3	−1	3
	4	4	−2

$\left(\textit{Ans.}\ \ A\left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}\right), B\left(\frac{5}{22}, \frac{8}{22}, \frac{9}{22}\right);\ V=\frac{17}{11}\right)$

29. Write both the primal and the dual L.P. programs for the following payoff matrix :

2	1	0	−2
1	0	3	2

[*Delhi M.Sc.* (*Math.*) *1970*]

$\Big($*Ans.* Primal : Minimize $\frac{1}{V}=X_1+X_2$,

$$\text{s.t. } 2X_1+X_2 \geqslant 1,\ X_1 \geqslant 1,\ X_2 \geqslant 3,$$
$$-2X_1+2X_2 \geqslant 1;\ X_1,\ X_2 \geqslant 0.$$

Dual : Maximize $\frac{1}{V}=Y_1+Y_2+Y_3+Y_4$,

$$\text{s.t. } 2Y_1+Y_2-2Y_3 \leqslant 1,\ Y_1+3Y_3+2Y_4 \geqslant 1;$$
$$Y_1,\ Y_2,\ Y_3,\ Y_4 \geqslant 0)$$

30. Solve the following 3×3 game by linear programming :

		Paper B	
	1	−1	−1
Player A	−1	−1	3
	−1	2	−1

[*Kuruk. M.Sc.* (*Math.*) *1975*]

$\left(\textit{Ans.}\ \ A\left(\frac{6}{13}, \frac{3}{13}, \frac{4}{13}\right), B\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}\right);\ V=-\frac{1}{13}\right)$

31. Solve exercise no. 27 by the method of linear programming.

$$\left(Ans.\ \ A\left(-\frac{1}{9}, \frac{4}{9}, \frac{4}{-9}\right), B\left(\frac{7}{9}, \frac{11}{9}, \frac{-1}{-9}\right); V=\frac{1}{9}\right)$$

32. Write both the primal and dual L.P. programmes for the payoff matrix shown below.

$$\begin{array}{c|cccc|} & & & Y & \\ \hline & 3 & 5 & 2 & 8 \\ X & 2 & 6 & 5 & 1 \\ & 4 & 7 & 3 & 9 \\ \hline \end{array}$$

$$\left(Ans.\ \text{Primal : Minimize } \frac{1}{V}=X_1+X_2+X_3, \right.$$

$$\begin{aligned} \text{subject to}\quad & 3X_1+2X_2+4X_3 \geqslant 1, \\ & 5X_1+6X_2+7X_3 \geqslant 1, \\ & 2X_1+5X_2+3X_3 \geqslant 1, \\ & 8X_1+X_2+9X_3 \geqslant 1, \\ & X_1, X_2, X_3 \geqslant 0. \end{aligned}$$

$$\text{Dual : Maximize } \frac{1}{V}=Y_1+Y_2+Y_3,$$

$$\left. \begin{aligned} \text{subject to}\quad & 3Y_1+5Y_2+2Y_3+8Y_4 \leqslant 1, \\ & 2Y_1+6Y_2+5Y_3+\ Y_4 \leqslant 1, \\ & 4Y_1+7Y_2+3Y_3+9Y_4 \leqslant 1; \\ & Y_1, Y_2, Y_3, Y_4 \geqslant 0 \end{aligned} \right).$$

33. Write both the primal and dual L.P. programmes for the following rectangular game. Solve the game by simplex method.

$$\begin{array}{|ccc|} \hline 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \\ \hline \end{array}$$

34. There are two competing departmental stores R and C in a city. Both stores have equal reputation and the total number of customers is equally divided between the two. Both the stores plan to run annual discount sales in the last week of December. For this they want to attract more number of customers by using advertisement through newspaper, radio and television. By seeing the market trend, the store R constructed the following payoff matrix where the

numbers in the matrix indicate a gain or a loss of customers. Find optimal strategies for stores R and C by L.P. method.

	Store C		
	40	50	−70
Store R	10	25	−10
	100	30	60

[*Punjab Univ. B.Sc. (Mech.) Engg. 1978*]

$$\left(Ans.\ \ R\left(\frac{1}{5}, 0, \frac{4}{5}\right), C\left(0, \frac{13}{15}, \frac{2}{15}\right); V=34.\right)$$

Section 7.10-3

35. The matrix of a certain 3×3 game is given below.

	Y		
	2	0	1
X	0	2	1
	3	0	0

Obtain an approximate solution by iteration.

$$\left(Ans.\ \ X\left(\frac{3}{10}, \frac{6}{10}, \frac{1}{10}\right), Y\left(\frac{3}{10}, \frac{5}{10}, \frac{2}{10}\right); 1 \leqslant V \leqslant 1 \cdot 2.\right)$$

36. Solve exercise no. 25 by iterative method of approximate solution.

(*Ans.* A $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, B $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; V = 1.)

37. Use iterative method to find approximate solution for the following game :

	Y				
	3	4	5	−2	3
X	2	−2	1	0	−2
	1	−1	2	1	6

8

Dynamic Programming

While considering the situations of allocation, transportation, assignment, scheduling and planning, it was assumed that the values of decision variables do not change over the planning horizon. Thus these problems were of static nature and were solved as specific situations occurring at a certain moment. However, we come across a number of situations where the decision variables vary with time, and these situations are considered to be dynamic in nature. The technique dealing with these types of problems is called *dynamic programming*. It will be shown in the body of the chapter that time element is not an essential variable, rather any multistage situation in which a series of decisions are to be made is considered a dynamic programming problem.

8·1. Introduction

Dynamic programming is a mathematical technique dealing with the optimization of multistage decision problems. The technique was originated in 1952 by Richard Bellman and G.B. Dantzig, and was initially referred to as the *stochastic linear programming* To-day dynamic programming has been developed as a mathematical technique to solve a wide range of decision problems, and the technique forms an important part of every operation researcher's tool kit. By this technique decisions regarding a certain problem are typically optimized in stages rather than simultaneously. The original problem is broken into sub-problems (stages), which can then be handled more efficiently from the computational viewpoint. Alternatively, the situation may be such as to require a series of decisions with the outcome of each depending upon the results of the previous decision in the series. For example, a production manager may not

neglect plant maintenance to obtain greater output in present month, instead, he may like to sacrifice the present production to obtain greater production next month. Thus, individually, each decision in the series may not be optimal. A sacrifice at one stage may result in greater gains at some other stage. The technique of dynamic programming aims at optimizing the decision for the situation as a whole, and the decisions for the stage may be sub-optimal.

Though the originator of the technique, Richard Bellman, himself, has said, "we have coined the term 'dynamic programming' to emphasize that there are problems in which time plays an essential role", yet, in many dynamic programming problems time is not a relevant variable. For example, a decision regarding allocation of a fixed quantity of resources to a number of alternative users constitutes one decision to be taken at one time, but the situation can be handled as a dynamic programming problem. For instance, suppose a company has marked capital C to be spent on advertising its products through three different medias *i.e.*, of newspaper, radio and television. In each media the advertisement can appear a number of times per week. Each appearance has associated with it certain costs and returns. How many times the product should be advertised in each media so that the returns are maximum and the total cost is within the prescribed limit ? In this situation time is not a variable, but the problem can be divided into stages and solved by dynamic programming.

8·2. Examples on the Applications of Dynamic Programming

Before we discuss the technique of dynamic programming, it will be worthwhile to be familiar with the type of situations in which this quantification technique may be applicable. To meet the objective, a few examples illustrating the nature of situations are given below.

EXAMPLE 8·2·1 (Capital Budgeting Problem)

A manufacturing company has three sections producing automobile parts, b-cycle parts and sewing machine parts respectively. The management has allocated Rs. 20,000 for expanding the production facilities. In the auto-parts and bicycle parts sections, the production can be increased either by adding new machines or by replacing some old inefficient machines by automatic machines. The sewing machine parts section was started only a few years back and thus the additional amount can be invested only by adding new machines to the section. The cost of adding and replacing the machines, along with the associated expected returns in the different sections is given in table 8·1. Select a set of expansion plans which may yield the maximum return.

Table 8 1

Alternatives	*Auto-parts section*		*Bicycle parts section*		*Sewing machine parts section*	
	Cost (Rs.)	Return (Rs)	Cost (Rs.)	Return (Rs.)	Cost (Rs.)	Return (Rs.)
1. *No expansion*	0	0	0	0	0	0
2. *Add new machines*	4,000	8,000	8,000	12,000	2,000	8,000
3. *Replace old m/cs*	6,000	10,000	12,000	18,000	—	—

EXAMPLE 8·2·2 (Selection of Advertising Media)

A cosmetics manufacturing company is interested in selecting the advertising media for its product and the frequency of advertising in each media. The data collected over the past two years regarding the frequency of advertising in three medias of newspaper, radio and television and the related sales of the product gives the following results :

Table 8·2

Expected sales in thousands of rupees

Frequency/week	*Television*	*Radio*	*Newspaper*
1	220	150	100
2	275	250	175
3	325	300	225
4	350	320	250

The cost of advertising in newspaper is Rs. 500 per appearance, in radio and in television, Rs. 1,000. and Rs. 2,000 respectively per appearance. The budget provides Rs 4,500 per week for advertisement. The problem is of determining the optimal combination of advertising media and advertising frequency.

EXAMPLE 8·2·3 (Production Scheduling Problem)

The manufacturing capacity of a factory is 3,500 units/month in regular time and 1,500 units/ month in overtime. Requirements for

the first, second and third months are 3,000, 5,000 and 4,500 units. It is estimated that the cost of production is Rs. 10 per unit in regular time and Rs. 12 per unit in overtime in the first month, Rs. 14 and Rs. 16 respectively in the second month and Rs. 19 and Rs. 22 respectively in the third month. Cost of storage is Rs. 3 per unit per month. Production Manager is faced with the problem of deciding the production schedule for the three months so that the total cost of production and storage is the minimum.

EXAMPLE 8·2·4 (Employment Smoothening Problem)

A firm has divided its marketing area into three zones. The amount of sales depends upon the number of salesmen in each area. The firm has been collecting the data regarding sales and salesmen in each area over a number of past years. The information is summarized in table 8·3. For the next year firm has only 9 salesmen and the problem is to allocate these salesmen to three different zones so that the total sales are maximum.

Table 8·3

Profits in thousands of rupees

No. of Salesmen	*Zone* 1	*Zone* 2	*Zone* 3
0	30	35	42
1	45	45	54
2	60	52	60
3	70	64	70
4	79	72	82
5	90	82	95
6	98	93	102
7	105	98	110
8	100	109	110
9	90	100	110

8·3. Need of Dynamic Programming

In the situations cited above we have noticed that the decision making process consists of selecting a combination of (decision) plans from a large number of alternative combinations. The number of alternative combinations is usually very large. For example, in the case of selecting the advertising media and the frequency of advertising in each media, a very large number of combinations is possible. The problem can be solved by first making an exhaustive list of the different combinations and then selecting the one which gives the largest

return and also satisfies the budget constraint. Thus before making a decision, it is required that

(*i*) all the decisions of a combination are specified before a decision is evaluated.

(*ii*) the optimal policy (combination of decisions) can be selected only after all the combinations are evaluated.

This enumerative method being too much time and effort consuming is inefficient because

(*a*) all combinations may not satisfy the limitations (such as the limitation on capital allocated) and thus may be infeasible.

(*b*) the number of combinations may be too large to allow exhaustive listing.

(*c*) in large problems due to the number of combinations being large, with each combination encompassing the entire decision problem, the decision process may become infeasible from computation viewpoint.

The dynamic programming approach deals with such situations by breaking the total problem into sub-problems or stages. Only one stage is considered at a time and the various infeasible combinations are eliminated. The solution is obtained by moving from one stage to the next and is completed when the final stage is reached.

8·4. Distinguishing Characteristics of Dynamic Programming

The important features of dynamic programming which distinguish it from other quantitative techniques of decision making can be summarized as follows :

1. It involves a multistage process of decision making. The stages may be certain time intervals or certain sub-divisions of the problems, for which independent feasible decisions are possible.

2. In dynamic programming the outcome of decisions depends upon a small number of variables; that is, at any stage only a few variables should define the problem. For example, in the production smoothening problem, all that one needs to know at any stage is the production capacity, cost of production in regular and overtime, storage costs and the time remaining to the last decision.

3. A stage decision does not alter the number of variables on which the outcome depends, but only changes the numerical value of these variables. For the production smoothening problem, the number of variables which describe the problem *i.e.*, production capacity, production costs, storage costs and time to the last decision, remain

the same at all stages. No variable is added or dropped. The effect of decision at any stage will be to alter the used production capacity, storage cost, production cost and time remaining to the last decision.

4. *Principle of Optimality.* Dynamic programming is based on Bellman's Principle of Optimality, which states, "*An optimal policy (a sequence of decisions) has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*". This principle implies that a wrong decision taken at one stage does not prevent from taking of optimum decisions for the remaining stages.

8·5. Dynamic Programming Approach

Before discussing the solutions to numerical problems, it will be worthwhile to know a little more about some fundamental concepts of dynamic progamming. The first concept is *stage*. As already discussed, the problem is broken down into sub-problems and each sub-problem is referred to as a stage. A stage signifies a portion of decision problem for which a separate decision can be made. At each stage there are a number of alternatives and the decision making process involves the selection of one feasible alternative which may be called as *stage decision*. The stage decision may not be optimal for the considered stage, but contributes to make an overall optimal decision for the entire problem.

The other important concept is *state*. The variables which specify the condition of decision process and summarize the current '*status*' of the system are called *state variables*. For example, in the capital budgeting problem, the capital is the state variable. The amount of capital allocated to the present stage and the preceding stages (or the capital remaining) defines the status of the problem. The number of state variables should be as small as possible. With the increase in number of state variables, increases the difficulty of problem solving.

In a decision making process, at each stage, a decision is made to change the state of the problem, with the aim of maximizing the return. At the next stage, decisions are made using the values of the state variables that result from the preceding stage decisions. The procedure is best illustrated with the help of an example.

EXAMPLE 8·5·1

Let us consider the employment smoothening problem (example 8·2-4). The problem is of allocating 9 salesmen to three marketing

zones 1, 2 and 3 in such a way that the total profits of the company are maximized. In this example the stages are three zones and the state variables are the number of salesmen. We start with zone 1. The return corresponding to the different number of salesmen allocated to zone 1 as given in table 8·3 are reproduced in table 8·4.

Table 8·4

Zone 1

No. of salesmen	0	1	2	3	4	5	6	7	8	9
Profits (in thousands of rupees)	30	45	60	70	79	90	98	105	100	90

Now we consider the first two zones, zone 1 and 2. Nine salesmen can be divided into two zones in 10 different ways ; as 9 in zone 1 and none in zone 2, 8 in zone 1 and 1 in zone 2, 7 in zone 1 and 2 in zone 2, ... Each combination will have associated with it certain returns. This is given in table 8·5.

Table 8·5

Zones 1 *and* 2

No. of salesmen	*Zone* 1	x_1	9	8	7	6	5	4	3	2	1	0
	Zone 2	x_2	0	1	2	3	4	5	6	7	8	9
Profits (in thousands of rupees)	*Zone* 1	$f_1(x_1)$	90	100	105	98	90	79	70	60	45	30
	Zone 2	$f_1(x_2)$	35	45	52	64	72	82	93	98	100	100
		Total	125	145	157	162	162	161	163	158	145	130

In the above table x_1 and x_2 denote the number of salesmen in zones 1 and 2 and f_1 (x_1) and f_2 (x_2) denote the profits from zone 1 and zone 2 respectively. If S denotes the total profits from each combination, then the various alternatives are

$$S=f_1\,(9)+f_2\,(0),$$

$$S=f_1\,(8)+f_2\,(1),$$

$$\vdots \quad \vdots \quad \vdots$$

$$S=f_1(0)+f_2\,(9).$$

In the general form, $S = f_1(x) + f_2(9 - x)$,

or $$S = f_1(x) + f_2(A - x),$$

where A is the total number of salesmen. The objective is to maximize the profits.

i.e., maximize $S = F(A) = [f_1(x) + f_2(A - x)]$. ...(8·1)

Here, F (A) is the maximum profit that can be achieved by allocating A salesmen to the two marketing zones, 1 and 2. Equation (8·1) can be applied to determine the optimum distribution of any number of salesmen *i.e.*, A = 9, 8, 7, 6, *etc.* This can be best done in the tabular form as shown in table 8·6.

Table 8·6

Zone 2 \ Zone 1		0	1	2	3	4	5	6	7	8	9	*No. of salesmen*
		30	45	60	70	79	90	98	105	100	90	*Profits*
0	35	65*	80*	95*	105*	114	125*	133	140	135	125	
1	45	75	90	105*	115*	124	135*	143*	150	145		
2	52	82	97	112	122	131	142	150	157			
3	64	94	109	124	134	143*	154*	162				
4	72	102	117	132	142	151	162					
5	82	112	127	142	152	161						
6	93	123	138	153	163*							
7	98	128	143	158								
8	100	130	145									
9	100	130										
No. of salesmen	*Profits*											

The numbers marked* are the maximum along each diagonal.

Expected profits are computed for all the possible combinations. For a particular number of salesmen, the values of profits for different

combinations can be read along the diagonal. For instance, if four salesmen are to be distributed between two zones, the values of sales for the combination $4+0$, $3+1$, $2+2$, $1+3$ and $0+4$ can be read along diagonal $4-4$. The maximum profit of Rs. 1,15,000 for four salesmen results from the combination of three salesmen in zone 1 and one in zone 2. Similarly, for five salesmen, a maximum profit of Rs. 1,25,000 can be obtained if all the five are assigned to zone 1 and none to zone 2. The optimum returns for the various combinations are tabulated in table 8.7.

Table 8.7

	Maximum returns from optimum allocation of salesmen in Zones 1 and 2									
No. of salesmen A	0	1	2	3	4	5	6	7	8	9
Total profit $f_1(x_1)+f_2(x_2)$	65	80	95	105	115	125	135	143	154	163
(x_2+x_1)	0+0	0+1	0+2	0+3 1+2	1+3	0+5	1+5	3+4 1+6	3+5	6+3

Now we move to the next stage and consider the distribution of 9 salesmen into three zones, zones 1, 2 and 3. The decision at this stage will result in allocating certain number of salesmen to zone 3 and the remaining to zones 1 and 2 combined, and then by following the backward process they will be distributed in zones 1 and 2. For instance, if we assign 4 sallesmen to zones 1 and 2, and the remaining 5 to zone 3, then the best profits would be

$$S=F(4)+f_3(5),$$

where F (4) is the optimum sales given by four salesmen in zones 1 and 2, and $f_3(5)$ are the profits corresponding to 5 salesmen in zone 3.

In the general form we can write

$$S=F(x)+f_3(A-x),$$

where x is the number of salesmen allocated to zones 1 and 2 and $(A-x)$ to zone 3. The objective is to find a combination of x and $(A-x)$ which maximizes the value of S.

i.e., maximize $S=F(A)=[F(x)+f_3(A-x)]$,

$$0 \leqslant x \leqslant A.$$

If we use subscript 2 for the first two zones, such that

$$F_2(A_2)=\text{Maximum}\,[f_1(x)+f_2(A_2-x)],$$

$$0 \leqslant x \leqslant A_2\,;$$

with A as the number of salesmen assigned to first two zones,

then, $F_3(A_3) = \text{Maximum}\ [F_2(A_2) + f_3(A_3 - A_2)]$,

$$0 \leqslant x \leqslant A_3.$$

The computations for selecting the optimum combination of A_2 and $(A_3 - A_2)$ with $A_3 = 9$, can be carried in the same way as for the first two zones. This is given in table 8.8.

Table 8.8

No. of salesmen in Zone 1+Zone 2

No. of salesmen in zone 3		0	1	2	3	4	5	6	7	8	9
		65	80	95	105	115	125	135	143	151	163
0	42	107*	122*	137*	147	157	167	177	185	196	205
1	54	119	134	149*	159*	169*	179*	189	199	208	
2	60	125	140	155	165	175	185	195	203		
3	70	135	150	165	175	185	195	205			
4	82	147	162	177	187	197	207				
5	95	160	175	190*	200*	210*					
6	102	167	182	197	207						
7	110	175	190	205							
8	110	175	190								
9	110	175									

The numbers marked * are the maximum along each diagonal

The optimum combination as we see along the diagonal 9.9 is 5 salesmen in zones 3 and 4 in zones 1 and 2 combined. This gives a maximum profit of Rs. 2,10,000. Now proceeding backwards, from table 8.6 we find that the optimum division of four salesmen in the first two zones is 3 in zone 1 and one in zone 2. Similarly, we can determine the optimum allocation of any other number of salesmen in the three zones. For example, 5 salesmen will give the optimum profit of Rs. 1,69,000 when distributed as one in zone 3, three in zone 1 and one in zone 2.

The problem can be extended to four marketing zones.

$$F_4(A_4)=\text{Max.}\,[F_3(A_3)+f_4(A_4-A_3)],$$
$$0\leqslant x\leqslant A_4.$$

For n marketing zones the equation can be written as

$$F_n(A_n)=\text{Max.}\,[F_{n-1}(A_{n-1})+f_n(A_n-A_{n-1})],$$
$$0\leqslant x\leqslant A_n,$$
$$n=2, 3, 4, \ldots.$$

If $x_1, x_2, x_3, \ldots, x_n$ are the number of salesmen assigned to zones 1, 2, 3, ..., n, then the equation for determining the maximum profit can be expressed as

$$F_n(A_n)=\text{Max.}\,[f_1(x_1)+f_2(x_2)+\cdots+f_n(x_n)],$$
$$0\leqslant x_n\leqslant A_n.$$

Thus the solution is obtained by a *step by step* analysis. The algorithm used utilized the result obtained at a particular stage to the solution at the subsequent stage. The process is of recursive nature and the equation formulated for the maximum profits may be referred to as the *recursive equation*. The real life problems are usually too cumbersome for manual computations and can be handled only with electronic computers.

EXAMPLE 8.5-2 : Solve Example 8.2-1.

Solution : In this example each department of the firm can be regarded as a stage. At each stage a certain number of alternatives are possible for expansion of the department. The effect of each stage decision would be to alter the capital allocated, or we may say that capital is the state variable in this problem. Let us consider the first stage *i.e.*, expansion of auto-parts section. There are three alternatives, no expansion, add new machines, and replace old machines. The amount which may be allocated to stage 1 may vary from 0 to Rs. 20,000 ; of course, it will be overspending if it is more than Rs. 6,000. The values of the various alternatives can best be evaluated in the form of a table (table 8.9).

When the capital allocated is zero or Rs. 2,000, only first alternative (no expansion) is possible. Return is, of course, zero. When the amount allocated is, say, Rs. 6,000, all the three alternatives are possible, giving returns of zero, Rs. 8,000 and Rs. 10,000. So we will select the alternative 3 with a return of Rs. 10,000. To express the decision process mathematically :

Let the capital allocated to stage 1 be x_1 with $0\leqslant x_1\leqslant C$, where C is the total capital available.

Table 8.9

Stage 1

State x_1	*Evaluation of alternatives (Values in thousands of rupees.)* 1	2	3	*Optimal solution*	
	Cost $C_{11}=0$ *Return* $R_{11}=0$	*Cost* $C_{12}=4$ *Return* $R_{12}=8$	*Cost* $C_{13}=6$ *Return* $R_{13}=10$	*Optimal return*	*Decision*
0	0	—	—	0	1
2	0	—	—	0	1
4	0	8	—	8	2
6	0	8	10	10	3
8	0	8	10	10	3
10	0	8	10	10	3
12	0	8	10	10	3
14	0	8	10	10	3
16	0	8	10	10	3
18	0	8	10	10	3
20	0	8	10	10	3

Let us express the cost of alternative j $(j=1, 2, 3)$ at stage 1 as C_{1j} and the return which is a function of C_{1j} as R_{1j} (C_{ij}).

Now, if $f_1(x_1)$ is the return from optimal policy at stage 1, it is required to select an alternative which has maximum value of R_{1j} (C_{1j}) with $j=1, 2, 3$.

$$i.e., f_1(x_1)=\max. R_{1j}(C_{1j}),$$

$$j=1, 2, 3.$$

Now we move to the second stage. Here, again, three alternatives are available. The computations are again carried in the tabular form (Table 8.10).

Table 8.10

Stage 2

State x_2	Evaluation of alternatives (Values in thousands of rupees) 1	2	3	Optimal solution	
	Cost $C_{21}=0$ Return $R_{21}=0$	Cost $C_{22}=8$ Return $R_{22}=12$	Cost $C_{23}=12$ Return $R_{23}=18$	Optimal return	Decision
0	$0+0=0$	—	—	0	1
2	$0+0=0$	—	—	0	1
4	$0+8=8$	—	—	8	1
6	$0+10=10$	—	—	10	1
8	$0+10=10$	$12+0=12$	—	12	2
10	$0+10=10$	$12+0=12$	—	12	2
12	$0+10=10$	$12+8=20$	$18+0=18$	20	2
14	$0+10=10$	$12+10=22$	$18+0=18$	22	2
16	$0+10=10$	$12+10=22$	$18+8=26$	26	3
18	$0+10=10$	$12+10=22$	$18+10=28$	28	3
20	$0+10=10$	$12+10=22$	$18+10=28$	28	3

Here, the state x_2 signifies the total capital allocated to the current stage (stage 2) and the preceding stage (stage 1). Similarly, the return corresponds to the sum of the return of current stage and the preceding stage (Principle of Optimality). Thus when $x_2 <$ Rs. 6,000, only the first alternative (no expansion) is possible. But with $x_2 =$ Rs. 6,000, a return of Rs. 10,000 is possible by selecting the third alternative at stage 1. With $x_2 =$ Rs. 12,000, three alternatives are possible with the maximum return of Rs. 20,000 from alternative 2. The optimal policy consists of a set of two decisions, namely adopt alternative 2 at second stage and again alternative 2 at the first stage.

If $f_2(x_2)$ is the return from the optimal policy, then at $x_2=12$ (hereafter, the values of $x=12, 20, \ldots$, etc., stand for Rs. 12, 000, Rs. 20,000...),

$$f_2(x_2)=20=12+8$$
$$=R_{22}(C_{22})+R_{12}(C_{12})$$
$$=R_{22}(C_{22})+R_{12}(x_2-C_{22}).$$

In general form we can write

$$f_2(x_2)=\text{Max.}\,[R_{2j}(C_{2j})+f_1(x_2-C_{2j})],$$
$$j=1, 2, 3,$$
$$0 \leqslant C_2 \leqslant x_2.$$

Similarly, the recursive equation for the third stage is written as

$$f_3(x_3)=\text{Max.}\,[R_{3j}(C_{3j})+f_2(x_3-C_{3j})],$$
$$j=1, 2,$$
$$0 \leqslant C_3 \leqslant x_{3j}.$$

The computations for the third stage are given in table 8.11.

Table 8.11

Stage 3

State x_3	*Evaluation of alternatives (values in thousands of rupees)*		*Optimal solution*	
	Cost $C_{31}=0$ *Return* $R_{31}=0$	*Cost* $C_{32}=2$ *Return* $R_{32}=8$	*Optimal return*	*Decision*
0	$0+0=0$	—	0	1
2	$0+0=0$	$8+0=8$	8	2
4	$0+8=8$	$8+0=8$	8	1, 2
6	$0+10=10$	$8+8=16$	16	2
8	$0+12=12$	$8+10=18$	18	2
10	$0+12=12$	$8+12=20$	20	2
12	$0+20=20$	$8+12=20$	20	1, 2
14	$0+22=22$	$8+20=28$	28	2
16	$0+26=26$	$8+22=30$	30	2
18	$0+28=28$	$8+26=34$	34	2
20	$0+28=28$	$8+28=36$	63	2

For $x_3 = C = 20$, the optimal decision for stage 3 is second alternative, which gives a total return of Rs. 36,000. Now proceeding back to stage 2, $x_2 = x_3 - C_{32} = 20 - 2 = 18$, for which the optimal solution of alternatives at stage 2 is alternative 3. Further, going back to stage 1, $x_1 = x_2 - C_{23} = 18 - 12 = 6$, corresponding to which the best stage decision at stage 1 is third alternative.

Thus when the capital allocated is Rs. 20,000, the optimal policy of expanding production facilities is 3—3—2, which can be elaborated as

replace old machines with automatics in auto parts section,

replace old machines with automatics in bicycle parts section,

add new machines to the sewing machines parts section. This policy gives the optimal return of Rs. 36,000.

EXAMPLE 8.5-3. Solve example 8·2·2.

Solution. The situation can be tackled in a number of ways. We will use the same computational procedure as per the capital budgeting problem. The problem can be decomposed into three stages corresponding to the three medias of advertising. In each media four alternatives (frequencies) are possible. Each alternative, has associated with it certain cost and return (expected sales). Here again, the capital marked for allocation to different medias is the state variable. A combination of media and frequency is to be selected in such a way as to maximize the total sales with expenditure not exceeding the specified limit of Rs. 4,500.

Let us consider the advertising media of television as the first stage. If x_1 is the capital allocated to stage 1, and $R_{1j}(C_{1j})$ is the return (expected sales) corresponding to cost C_{1j}, then, the optimal return is

$$f_1(x_1) = \text{Max.}\ [R_{1j}(C_{1j})],$$

$$j = 0, 1, 2, 3, 4,$$

with $0 \leqslant x_1 \leqslant C$.

By applying this equation at various levels of expenditure, the various alternatives are evaluated and the one giving the largest expected sales is selected. The selected frequencies and the optimal return for different values of x_1 are given in table 8·12.

Table 8·12

Stage 1

State x_1	*Return*	*Cost = Rs. 2,000* *Frequency*
0	—	0
500	—	0
1,000	—	0
1,500	—	0
2,000	220	1
2,500	220	1
3,000	220	1
3,500	220	1
4,000	275	2
4,500	275	2

For $x_1=0$, Rs. 500, Rs. 1,000 and Rs. 1,500; it is not possible to advertise in this media, since the cost of one appearance per week is Rs. 2,000. For $x_1=$Rs. 2,000, Rs. 2,500, Rs. 3,000 and Rs. 3,500, the product can be advertised only once, giving a return of Rs. 2,20,000.

With $x_1=$Rs. 4,000, Rs. 4,500, two appearances can occur giving a return of Rs. 2,75,000.

Table 8·13

Stage 2

Cost per appearance = Rs. 1,000

State x_2	0	1	2	3	4	*Return*	*Freq.*
500	0	0	—	—	—	0	0
1,000	0	150	—	—	—	150	1
1,500	0	150	—	—	—	150	1
2,000	220	150+0	250	—	—	250	2
2,500	220	150+0	250+0	—	—	250	2
3,000	220	150+220	250+0	300	—	370	1
3,500	220	150+220	250+0	300+0	—	370	1
4,000	275	150+220	250+220	300+0	320	470	2
4,500	275	150+220	250+220	300+0	320+0	470	2

Now let us move to the second stage. Again for advertising in radio, four alternatives (frequencies) are possible. Here, the state x_2 will signify the expenditure incurred at the first stage and at the current stage.

At any value of state x_2 $(0 \leqslant x_2 \leqslant C)$,

$$\text{optimal return } f_2(x_2) = \text{Max. } [R_{2j}(C_{2j}) + f_1(x_1)], \text{ for } j = 0 \text{ to } 4$$

$$= \text{Max. } [R_{2j}(C_{2j}) + f_1(x_2 - C_{2j})], \text{ for } j = 0 \text{ to } 4.$$

The evaluation of alternatives is carried in the tabular form shown in table 8·13. To illustrate, at $x_2 = 3{,}000$, four alternatives are possible, *i.e.*, do not advertise, advertise once, twice or thrice. It is not possible to advertise four times because that needs a sum of Rs. 4,000. If we do not purchase any advertisement (frequency = 0), the amount of Rs. 3,000 can purchase one advertisement in media T, giving expected sales of Rs. 2,20,000. If one advertisement is purchased in media R, this will cost Rs. 1,000, and with amount of Rs. 2,000 left one advertisement can be purchased in media T, giving total return of Rs. $(150+220) \times 1{,}000$ = Rs. 3,70,000. If two advertisements are purchased in R, costing Rs. 2,000, the balance amount of Rs. 1,000 will be of no use in media T, and thus will give total sales as Rs. $(300+0) \times 1{,}000$ = Rs. 3,00,000. The maximum return comes when we purchase one advertisement in media R. This is the optimal decision for x_2 = Rs. 3,000.

Now, we move to the third stage.

$$f_3(x_3) = \text{Max. } [R_{3j}(C_{3j}) + f_2(x_2)], \text{ for } j = 0 \text{ to } 4$$

$$= \text{Max. } [R_{3j}(C_{3j}) + f_2(x_3 - C_{3j})], \text{ for } j = 0 \text{ to } 4.$$

Table 8·14

Stage 3 State x_3	*0*	*1*	*2*	*3*	*4*	*Optimal Decision Total sales*	*Frequency*
500	0	100	—	—	—	100	1
1,000	150	100+0	**175**	—	—	175	2
1,500	150	**100+150**	175+0	225	—	250	1
2,000	250	100+150	**175+150**	225+0	250	325	2
2,500	250	100+250	175+150	**225+150**	250+0	375	3
3,000	370	100+250	175+250	**225+150**	250+150	375	3
3,500	370	100+370	175+250	**225+250**	250+150	475	3
4,000	470	100+370	**175+370**	225+250	250+250	545	2
4,500	470	100+370	175+370	**225+370**	250+250	595	3

The computations are given in table 8·14. For the allocated capital of Rs. 4,500, the maximum sales that can be expected are of Rs. 5,95,000. From table 8·14 the optimal decision is purchase three advertisements in newspaper. This will cost Rs. 1,500. The amount left is Rs. 3,000, and corresponding to that at stage 2, the optimal decision is purchase one advertisement in radio. This costs Rs. 1,000 which leaves behind an amount of Rs. 2,000 which can purchase one advertisement in television (stage 1).

Similarly, if the firm wants to spend only Rs. 4,000 per week, the optimal policy will be : purchase two advertisements in newspaper costing Rs. 1,000, one in radio costing Rs. 1,000, and one in television costing Rs. 2,000. This will give an optimal expected sale worth Rs. 5,45,000.

EXAMPLE 8·5·4 (An Optimal Routing Problem)

Figure 8·1 shows a network of cities spread over a state. A company has to transport some goods from city A to city J. The cost of transportation between the different cities is given along the lines connecting the nodes. A nóde represents a city. It is required to determine the optimal route connecting cities A and J.

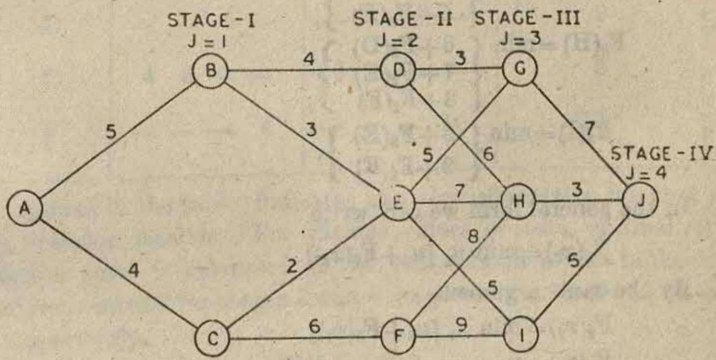

Fig. 8·1

This is a shortest route problem and can very conveniently be solved by the graphical methods discussed earlier in Section 5·11. Here the situation will be analysed by using the dynamic programming approach. The problem can be divided into four stages. Each stage decision will comprise of selecting a path out of a number of possible alternatives. At stage 1, one of the two paths AB and AC is to be selected, and at stage 3 one of the alternatives DG, DH, EG, EH, EI, FH and FI is to be selected. The optimal policy will consist of a set of paths connecting A with J. The node reached will r pre

sent the state of the system. If x_j represents the state, then at $j=0$, $x_0=A$ and at $j=2$, x_2 will have two values B and C. Possible alternative paths from one stage to next will be referred to as decision variables. Let u_j represent the decision variable which takes us from state $j-1$ to state j and with this the state changes from x_{j-1} to x_j. The minimum transportation cost from x_0 to x_j will be denoted by $F_j(x_j)$. The return from a decision u_j, being the function of the decision, will be represented by $f_j(u_j)$, which in this case is the cost of transporting along a path, or is the cost of decision variable, and thus we can take $f_j(u_j)=u_j$.

Let us start with node J, that is, with stage 4 and follow the backward process.

At $j=4$, u_4 has three values 7, 3 and 6, leading back to state x_3, which has three values G, H and I. The least of the values of $f_4(x_4)$ is to be selected.

$$f_j(x_j)=f_4(x_4)=\min \begin{Bmatrix} 7+F_3(G) \\ 3+F_3(H) \\ 6+F_3(I) \end{Bmatrix},$$

$$=\min u_4\ [u_4+f_3(x_3)].$$

Similarly at stage 3, $(j=3)$

$$F_3(G)=\min \begin{Bmatrix} 3+F_2(D) \\ 5+F_2(E) \end{Bmatrix},$$

$$F_3(H)=\min \begin{Bmatrix} 6+F_2(D) \\ 7+F_2(E) \\ 8+F_2(F) \end{Bmatrix},$$

$$F_3(I)=\min \begin{Bmatrix} 5+F_2(E) \\ 9+F_2(F) \end{Bmatrix}.$$

In the general form we can write

$$F_3(x_3)=\min u_3\ [u_3+F_2(x_2)].$$

By the same argument

$$F_2(x_2)=\min u_2\ [u_2+F_1(x_1)],$$

$$F_1(x_1)=u_1.$$

The whole process can thus be expressed by the recursion equation

$$F_j(x_j)=\min u_j\ [u_j+F_{j-1}\ (x_{j-1})],\ j=4, 3, 2;$$

$$F_1(x_1)=u_1.$$

This equation will help us in determining the optimal policy consisting of a set of decisions such that the cost $F_4\ (x_4)$ is minimum.

Now let us start with stage 1.

State x_1 has two values B and C. The decision variables u_1 are two with returns (costs) of 5 and 4. The information is tabulated in table 8.15.

Table 8·15

Stage 1

State x_1	*Decision variable* u_1	F_1 (x_1)
B	5	5
	4	4

At stage 2, state x_2 has three values D, E and F and there are four values for the decision variable u_2. The values of F_1 (x_1) are transferred from stage 1 to stage 2, to which the values of u_2 are added. (See table 8.16.)

Table 8·16

Stage 2

x_2 \ u_2	F_1 (x_1) 2	3	4	6	$u_2+F_1(x_1)$ 2	3	4	6	*Optimal* $F_2(x_2)$
D	—	—	5	—	—	—	9	—	9
E	4	5	—	—	6	8	—	—	6
F	—	—	—	4	—	—	—	10	10

A dash in the table indicates that transformation for that pair of x_j, u_j is not feasible. For various values of state, optimal return (minimum cost) is tabulated in the cost column of the table. The procedure is carried for stages 3 and 4, as given in table 8·17 and table 8·18 respectively.

Table 8·17

Stage 3

x_3 \ u_3	$F_2(x_2)$ 3	5	6	7	8	9	$u_3+F_2(x_2)$ 3	5	6	7	8	9	*Optimal* $F_3(x_3)$
G	9	6	—	—	—	—	12	11	—	—	—	—	11
H	—	10	9	6	—	—	—	15	15	13	—	—	13
I	—	6	—	—	—	10	—	11	—	—	—	19	11

Table 8·18

Stage 4

u_4 / x_4	$F_3(x_3)$ 3	6	7	$u_4+F_3(x_3)$ 3	6	7	*Optimal* $F_4(x_4)$
J	13	11	11	16	17	18	16

The least cost of the route connecting A and J is 16. Now to find the route we follow the backward pass.

$$F_4(x_4)=16=u_4+F_3(x_3)$$
$$=3+13 \text{ (from table 8·17).}$$
$$\therefore \quad u_4=3, \text{ path HJ.}$$
$$F_3(x_3)=13=u_3+F_2(x_2)$$
$$=7+6 \text{ (from table 8·16).}$$
$$\therefore \quad u_3=7, \text{ path EH.}$$

Similarly $u_2=2$, path CE,

$u_1=4$, path AC

Thus the optimal route connecting A and J is A-C-E-H-J.

EXAMPLE 8·5·5. (Dealing under Uncertainty)

A dealer has to dispose of certain goods within five weeks' time. The market prices are fluctuating from week to week. It is estimated that the chances of getting Rs. 2,000 for the whole stock is 45%, chances of getting Rs. 2,500 are 35% and there are 20% chances that the goods may sell at Rs. 3,000. If the goods are not sold in the first four weeks, then they will have to be disposed of in the fifth week at the prevailing market price in that week. When should the stocks be sold ?

Solution. The decision process can be displayed conveniently by the so called 'decision tree' as shown in figure 8·2. The five weekly periods can be treated as five stages. At each stage the dealer has to make a choice among the two alternatives, "Act and Wait". If the market price is more than what he expects in the following weeks he should 'Act' and if it is less, he should 'Wait'.

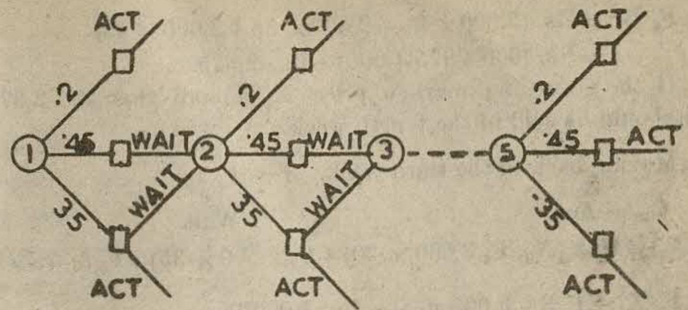

Fig. 8·2

These types of problems can conveniently be handled by starting from the last stage and solving by the ***backward induction*** or ***backward process***. As it is essential to dispose of the stock by the fifth week, decision can be expressed as

$$F_5(X_5)=X_5,$$

where, F_5 (X_5) is the highest expected return if the prevailing market price in the fifth week is X_5.

$$F_5(X_5)=\text{Rs. } 3{,}000 \text{ if } X_5=\text{Rs. } 3{,}000,$$
$$F_5(X_5)=\text{Rs. } 2{,}500 \text{ if } X_5=\text{Rs. } 2{,}500,$$
$$F_5(X_5)=\text{Rs. } 2{,}000 \text{ if } X_5=\text{Rs. } 2{,}000.$$

In the general form $F_n(X_n)=X_n$ is the highest expected return in the nth week if the prevailing price in nth week is X_n.

Here n refers to stages, and in this case $n=1$, 2, 3, 4 or 5 and X is the stage variable.

Now moving back to fourth stage, the price may be at Rs. 2,000, Rs. 2,500 or Rs. 3,000. If it is Rs. 3,000, then naturally the stock should be sold (Act), but if it is less than Rs. 3,000, then decision may be Act or Wait. The decision equation can be written as

$$F_4(X_4)=\max\left[\overbrace{X_3,}^{\text{Act}}\ \overbrace{(3{,}000\times\cdot 20)(+2{,}500\times .35)+(2{,}000\times\cdot 45)}^{\text{Wait}}\right].$$

The decision between Wait and Act depends upon the expected return in the fifth week *i.e.*, F_5 (X_5). If X_4, the prevailing price in 4th week is more than F_5 (X_5), then it is to Act, otherwise to wait..

$$F_4(X_4)=\max\left[\begin{array}{ll}\text{Act} & \text{Wait}\\ X_4, & F_5\ (3000\times\cdot 20)+F_5(2{,}500\times\cdot 35)+F_5(2000\times .45)\end{array}\right].$$

The break even point between the Act and Wait alternatives

can be calculated as

$$F_4\,BE = \text{Rs. } (3{,}000 \times \cdot 20 + 2{,}500 \times \cdot 35 + 2{,}000 \times \cdot 45)$$
$$= \text{Rs. } (600 + 875 + 900) = \text{Rs. } 2{,}375.$$

If the prevailing market price X_4 is more than Rs. 2,375 the goods should be sold in the fourth week.

Moving back to the third week.

$$F_3(X_3) = \text{Max } [\underset{\text{Act}}{X_3},\ \underset{\text{Wait}}{F_4(3{,}000 \times \cdot 20) + F_4(2{,}500 \times \cdot 35) + F_4(2{,}375 \times \cdot 45)}].$$

Here $F_4(X_4) = \begin{bmatrix} \text{Rs. } 3{,}000 \text{ if } X_4 = \text{Rs. } 3{,}000 \\ \text{Rs. } 2{,}500 \text{ if } X_4 = \text{Rs. } 2{,}500 \\ \text{Rs. } 2{,}375 \text{ break even} \end{bmatrix}.$

The break even point between the Act and Wait alternatives for the third week can be calculated as

$$F_3BE = \text{Rs. } (3{,}000 \times \cdot 20 + 2{,}500 \times \cdot 35 + 2{,}375 \times \cdot 45) = \text{Rs. } 2{,}544.$$

If the market price in third week is more than Rs. 2,544, goods should be diposed of, otherwise Wait.

Similarly for the second week, the break even point is

$$F_2BE = \text{Rs. } (3{,}000 \times \cdot 20 + 2{,}544 \times \cdot 35 + 2{,}544 \times \cdot 45)$$
$$= \text{Rs. } 2{,}735{\cdot}20 \approx \text{Rs. } 2{,}735.$$

And the break even price for the first week,

$$F_1BE = \text{Rs. } (3{,}000 \times \cdot 20 + 2{,}735 \times \cdot 35 + 2{,}735 \times \cdot 45)$$

$$= \text{Rs. } 2{,}738.$$

Thus we reach at the following optimal decision policy : The dealer should sell the goods if the market price in the first, second and third weeks is Rs. 3000; otherwise wait. In the fourth week, if he can get Rs. 2,500 or Rs. 3000, he should dispose of the goods; if it is less than Rs. 2,500, he should wait. If the goods remain unsold upto the fifth week he should dispose of the stock at what so ever value he can get in that week.

8.6. Additional Examples

EXAMPLE 8.6-1

A food processing firm has compiled the following data for future monthly production requirements and production costs in regular and overtime.

Table 8·19

Month	*Quantity (units)*	*Cost per unit*	
		Regular (Rs.)	*Overtime (Rs.)*
September	4,000	20	30
October	5,200	25	35
November	5,000	24	34
December	3,700	26	36
January	4,200	20	30
February	3,000	20	30

The production capacity of the firm is 6,000 units in regular time and 3,000 units in overtime. The cost of carrying storage is Rs. 7·50 per unit per month. If at the end of August, there are 3,500 units in stock at a cost of Rs. 25 each, what is optimal production schedule and the total associated cost ? Note that no inventory is required at the end of six months.

EXAMPLE 8·6.2.

A drug manufacturing concern has ten medical representatives working in three sales areas. The profitability for each representative in three sales areas is as follows :

Table 8·20

No. of representatives		0	1	2	3	4	5	6	7	8	9	10
Profitability (thousands of rupees)	*Area 1*	15	22	30	38	45	48	54	60	65	70	70
	Area 2	26	35	40	46	55	62	70	76	83	90	95
	Area 3	30	38	44	50	60	65	72	80	85	90	85

Determine the optimum allocation of medical representatives in order to maximize the profits.

What will be the optimum allocation if the number of representatives available at present is only six ?

EXAMPLE 8·6·3

A student of OR has five days at his disposal to revise the subject before examination. The course is divided into four sections. He decides to devote a whole day to the study of some section so that he may study a section for one day, two days, three days etc., or not at all. The expected grade points he will get for different alternative arrangements are as follows:

Table 8·21

Study days	*Course sections* I	II	III	IV
0	1	1	0	0
1	2	1	0	1
2	2	2	1	2
3	3	3	2	2
4	4	3	3	3
5	4	4	3	4

How should he distribute the available days to the different sections of the course so that he maximizes his grade point average ?

EXAMPLE 8·6·4

A company has three media A, B and C available for advertising its product. The data collected over the past years about the relationship between the sales and frequency of advertisement in the different media is as follows:

Table 8·22

Frequency/month	*Estimated sale (units) per month* A	B	C
1	125	180	300
2	225	290	350
3	260	340	450
4	300	370	500

The cost of advertisement is Rs. 5,000 in medium A, Rs. 10,000 in medium B and Rs. 20,000 in medium C. The total budget allocated for advertising the product is Rs. 40,000. Determine the optimal combination of advertising media and frequency.

EXAMPLE 8·6·5

If in example 8.5-5 of dealing under uncertainty, instead of selling the goods, the problem is of making purchase, determine the optimal policy so that the goods are purchased at the lowest cost.

8·7. APPLICATIONS OF DYNAMIC PROGRAMMING

We have discussed some over-simplified examples from the various fields of applications of dynamic programming. Many more applications are found for this decision making technique. Whereas the linear programming has found its applications in large scale complex situations, dynamic programming has more applications in smaller scale systems. Following are a few of the large number of fields in which dynamic programming has been successfully applied:

1. **Production**. In the production area, this technique has been employed for *production, scheduling* and *employment smoothening*, in the face of widely fluctuating demand requirements.

2. **Inventory Control**. This technique has been used to determine the optimum inventory level and for formulating the inventory recording rules, indicating when to replenish an item and by what amount.

3. **Allocation of Resources**. It has been employed for allocating the scarce resources to different alternative uses, such as, allocating salesmen to different sales zones and *capital budgeting procedures*.

4. **Selection of an advertising media**. (See example 8·2-2).

5. **Spare part level determination** to guarantee high efficiency utilisation of expensive equipment.

6. **Equipment replacement policies.** To determine at which stage equipment is to be replaced for optimal return from the facilities.

7. Scheduling methods for routine and major overhauls on complex machinery.

8. Systematic plan or search to discover the whereabouts of a valuable resource.

These are only a few of the wide range of situations to which the dynamic programming has been successfully applied. Many real operating systems call for thousands of such decisions each week. The dynamic programming models make it possible to make all these decisions, of course with the help of computers. These decisions individually may not appear to be of much economic benefit, but in aggregate they exert a major influence on the economy of a firm.

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9

Probability Theory

9·1. Introduction

Some knowledge of the basic elements of probability is essential for understanding the remaining chapters of this book. The information which forms the basis for making decisions is usually in the form of numbers; and these numbers may be uncertain to a certain extent. Probability theory expresses the degree of uncertainty in terms of mathematical equations.

This chapter deals with the various definitions, different types of probability distributions and their applications. The material is by no means exhaustive. Complete volumes have been written on this topic which forms only one chapter of this book. However, it provides the basic background for maintaining continuity of presentation in the chapters that follow. Several good references are given at the end of the chapter.

9·2. Historical Background

The origin of probability theory dates back to the 17th century when a French nobleman Chevelier de Me' re' (a man of ability and great experience in gambling), who played games at Monte Carlo, tried to find theoretical explanation of some of his gambling experiences. However, he did not have the necessary mathematical knowledge to analyse the situations himself and mentioned his difficulties to the best mathematicians of his country—Pascal and Fermat. This brought about the famous exchange of letters between these two mathematicians regarding applications of mathematical knowledge to explain the simple gambling games. This exchange of letters formed the beginning of probability theory.

Huygens (1629—1695), a great Dutch scientist became acquainted with contents of this correspondence and published the first book on probability in 1654. This book contains many interesting and difficult problems on probabilities of games of chance.

However, it was Jacob Bernoulli, an Italian mathematician, who gave the first formal definition of probability. Bernoulli's contribution to the theory of probability is generally compared to that of Newton's to mechanics. He made a systematic and unified presentation of the theory of probability in his book which, however, was published eight years after his death, in 1713 by his nephew.

The next great successor was Abraham de Moivre (1667—1754), whose most important book on probability, "The Doctrine of Chances", was published in 1718. It describes many new and powerful methods in solving more difficult problems.

Laplace (1749—1827), well known for his contributions to astronomy, was extremely interested in the theory of probability, right from the beginning of his scientific career. His book published in 1812, contains new ideas, methods and results and is regarded as one of the most outstanding contributions to mathematical literature.

In 19th century, Gregor Mendel, a monk from Austria, demonstrated that probability theory could be applied for biological investigations and founded the science of genetics. A Russian mathematician, A.N. Kolmogorov, gave three simple rules or axioms, which probabilities are supposed to obey. Since then, there have been great developments in the theory of probability and it has been applied to a wide variety of problems in physics, chemistry and other sciences.

In recent past, John Von Neumann applied the mathematical theory of probability for the study of economics and sociological problems. Because of its application to various different fields, the theory of probability forms a fundamental topic to be studied by every student of science.

9·3. Terminology in Probability Theory

Before a formal definition of probability can be given, it is necessary to introduce a few basic terms.

Experiment. An *experiment* is an operation whose output cannot be predicted with certainty.

Outcome. Output of an experiment is called *outcome*. The number of outcomes depends upon the nature of the experiment and may be finite or infinite. For example, consider the experiment of hitting a particular target by a marksman. There are only two

outcomes, either hit or miss. However, if the experiment involves the measurement of time between successive failures of an electronic equipment, the outcomes are given by time-to-failure, which can have any positive real values.

Sample Space. The set of all possible outcomes of a given experiment is called the *sample space* or the *event space* or the *probability space*. We now consider a few examples which will explain the above definitions.

EXAMPLE 9.3-1

A single, regular die consisting of 6 faces is rolled once. Thus the experiment is the roll of the die. A sample space for this experiment could be

$$S=\{1, 2, 3, 4, 5, 6\},$$

where integers 1 to 6 represent the face which may come up then the die stops.

EXAMPLE 9.3-2

Consider an experiment of tossing a coin once. The possible outcomes of this experiment are

(*a*) the coin lies with its head H up,

(*b*) the coin lies with its tail T up.

The chances of the coin resting on its edge are extremely small and hence this outcome is neglected. The associated sample space is

$$S=\{H, T\}.$$

EXAMPLE 9.3-3

Consider the experiment of tossing of two coins. The possible outcomes are HH, HT, TH and TT. The associated sample space for the experiment is

$$S=\{HH, HT, TH, TT\}.$$

EXAMPLE 9.3-4

Consider the experiment of selection of one card from a standard deck of 52 and let it be assumed that the cards are numbered from 1 to 52. The associated sample space is

$$S=\{1, 2, 3, ..., 52\},$$

since the selected card must correspond to one of these integers.

Finite and Infinite Sample Spaces

A sample space is called *finite* if the number of possible outcomes in the sample space is finite, otherwise it is called *infinite*. The four examples cited above have finite sample spaces.

Discrete and Continuous Sample Spaces

A sample space is called *discrete* if it contains only finite or infinite but countable number of possible outcomes, otherwise it is called *continuous.*

EXAMPLE 9.3-5

In the random experiment of counting the number of persons coming per hour for tickets at the booking window of cinema-hall, the sample space $S=\{1, 2, 3, ...\}$ is discrete.

EXAMPLE 9.3-6

In the experiment of measuring the time t between successive failures of an electronic equipment, the sample space is

$$S=\{0 \leqslant t \leqslant \infty\},$$

which is continuous.

Event. An *event* or *sample point* is a subset of sample space. Every subset is an event. An event *occurs* if any one of its elements is the outcome of the experiment.

Elementary Event

An *Elementary event* or *simple event* is a subset containing a single sample point. The sample space S itself is called the *certain event* or *sure event.* An event that contains no sample point is called *impossible event* and is denoted by ϕ.

EXAMPLE 9.3-7

When a single, regular die is rolled once, the associated sample space is

$$S=\{1, 2, 3, 4, 5, 6\}.$$

Then each of the subsets

$A=\{1\}$, $B=\{2, 4, 6\}$, $C=\{1, 3, 5\}$, $D=\{1, 2, 4, 5\}$, $E=\{2, 3, 4, 5\}$, is an event. In fact these are not the only events (there are many more), since these are not the only subsets of S. Evidently A is an elementary event. Events A, B, C, D and E are different because their subsets are different. If we actually roll the die once and we get face 1 up, then events A, C and D are said to have occurred, since all these events have 1 as an element. Similarly, if we get the face 3 up, events C & E are said to have occurred as they have 3 as an element. Event $F=\phi=\{7\}$ is, obviously, an impossible event.

EXAMPLE 9.3-8

Let the experiment involve selection of one student from the total number of 1,500 students on the rolls of a college. The sample

space will be represented by

$$S=\{1, 2, 3, \ldots, 1{,}500\}.$$

Then the subsets

$$A=\{1, 100\},\ B=\{10\},\ C=\{2, 5, 16, 80, 400\},$$
$$D=\{201, 202, 203, \ldots, 1{,}000\}$$

etc., are all events. If the student numbered 80 happens to be selected, then event C is said to have occurred ; if student numbered 1,350 happens to be selected, none of the events A, B, C and D has occurred since this number is not contained by any of them.

EXAMPLE 9.3-9

If a trial consists of rolling of two dice simultaneously, then since each die can show one of the faces 1 to 6, there are 36 possible outcomes and the sample space is

$$S=\{x_1, x_2\}, \text{ where } x_1=1, 2, \ldots, 6 \,;\ x_2=1, 2, \ldots, 6.$$

And let, events A, B and C be such that

A occurs if the sum of two dice is 4,

B occurs if the sum of two dice is 9,

C occurs if the two dice have the same number.

Then these events are subsets

$$A=\{(1, 3), (2, 2), (3, 1)\},$$
$$B=\{(3, 6), (4, 5), (5, 4), (6, 3)\},$$
$$C=\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), 6, 6)\}.$$

EXAMPLE 9.3-10

Consider again the experiment of measuring time between successive failures of an electronic equipment with sample space

$$S=\{0\leqslant t\leqslant\infty\}.$$

Then event with inter-failure time of 10 is $A=\{0\leqslant t\leqslant 10\}$.

Mutually Exclusive Events

Two events A and B are said to be *mutually exclusive* or *disjoint* if they do not have simple events in common. In other words A and B are mutually exclusive events if they cannot occur simultaneously.

Independent Events

An event A is said to be *independent* of event B if the probability that A occurs is not affected by whether B has or has not occurred.

9.4 Definition of Probability

Probability of an event has been defined in the following three ways.

The a-priori Probability

If there are n *mutually exclusive, exhaustive* and *equally likely* outcomes of an experiment and if m of them are favourable to an event A, then the probability of occurrence of A, denoted by P (A) is defined as the ratio $\frac{m}{n}$.

i.e.,
$$P(A)=\frac{m}{n},$$

where $0 \leqslant m \leqslant n,\ n>0.$ ...(9.1)

EXAMPLE 9.4-1

A card is drawn from a deck of cards. What is the probability that the card drawn is a heart ? What is the probability that the card drawn is an ace ?

Solution

In drawing a card there are 52 mutually exclusive and equally likely cases and there are 13 cases favourable to the drawing of a heart; hence the probability of drawing a heart is $13/52=1/4$. For the second event, there are 4 cases favourable to the drawing of an ace, hence the probability is $4/52=1/13$.

EXAMPLE 9.4-2

An urn contains 10 black, 15 white and 5 red balls. What is the probability of drawing a black, a white or a red ball ?

Solution

There are 30 equally likely cases. Out of these 30 cases, there are 10, 15 and 5 cases favourable respectively to a black, a white or a red ball. Hence the probability of drawing a black, white or a red ball is 1/3, 1/2 and 1/6 respectively.

The probability defined as above is called *a-priori probability* or *mathematical probability*. This definition works well with games of chance—tossing of an ideal coin, rolling of a die or playing a game of cards. In fact this definition is true only for those cases where the outcomes are equally likely and this may not always be true.

EXAMPLE 9.4-3

In a class of 50 students what is the probability that a particular student X will pass an examination ?

Solution.

There are obviously two cases ; either the student will pass or fail. In finding the probability of passing, the two cases cannot be taken as equally likely to give a value of 1/2 since X may be a hardworking and intelligent student and probability of his passing may be 1.

2. The a-posteriori Probability

This is also called *statistical* or *empirical probability*. It overcomes the shortcomings of the previous definition of probability. If n represents *sufficiently large* number of trials made to see whether an event A occurs or not and m represents the number of trials in which it is observed to occur then the probability of occurrence of A is given by

$$P(A)=\lim_{n\to\infty} \frac{m}{n}, \qquad \ldots(9.2)$$

provided the circumstances from trial to trial remain the same.

Clearly, $0 \leqslant P(A) \leqslant 1$,

where $P(A)=0$ signifies that the event is impossible, while $P(A)=1$ signifies that it is certain. For example, the probability of the event that rolled die will show a "seven" is zero (impossible), while the probability that a tossed coin will turn up a head or a tail is one (certain).

3. Axiomatic Definition of Probability

With every A in a finite sample space S, we associate a real number P(A), which is called the probability of event A if it satisfies the following axioms :

(*i*) $0 \leqslant P(A) \leqslant 1$, for each subset A of S, which implies that probability of an event always varies from 0 to 1.

(*ii*) $P(S)=1$, where S is the sample space, also called *certain event*.

(*iii*) $P(\phi)=0$, where ϕ is the *impossible event*.

(*iv*) $P(A \text{ or } B)=P(A)+P(B)$, where A and B are two *mutually exclusive* events.

9.5 Laws of Probability

Now we shall study the various laws of probability. The proofs given refer to the first definition of probability (mathematical probability. However, these laws can be proved by using the second definition of probability (statistical probability) with requisite modifications.

9.5-1 Complementary Events

When an event A fails to occur, one may say that event 'non-A' has occurred. The event 'non-A' is called the *complementary event* of A and is represented by $\bar{A}$ or A$'$. For example, in flipping up of an ideal coin, the event that head appears is complementary to the event that tail appears.

1st Law :

If p is the probability of event A and q is the probability of its complementary event $\bar{A}$, then

$$p=1-q$$

or $$\mathrm{P(A)}=1-\mathrm{P(\overline{A})}. \qquad \text{...(9·3)}$$

Proof. Let n be the total number of possible outcomes, of which m are favourable to the occurrence of event A. Then, obviously, the remaining $n-m$ outcomes are favourable to 'non-A' or $\overline{\mathrm{A}}$

$$\therefore \qquad q=\mathrm{P}\ (\overline{\mathrm{A}})=\frac{n-m}{n}=1-\frac{m}{n}=1-p=1-\mathrm{P(A)}.$$

EXAMPLE 9.5-1

An illiterate servant is given 5 cards addressed to 5 different persons residing in the same city. What is the probability that the servant hands over the card to a wrong person ?

Solution.

All possible different ways to distribute 5 cards to 5 different persons are (5 !) or 120. There is only one way of handing over the cards to all the five right addressees. Let this event be A.

Then $p=\mathrm{P(A)}=\frac{1}{120}$.

The probability of handing over cards to wrong addressees,

$$q=1-p=1-\frac{1}{120}=\frac{119}{120}.$$

9.5-2. Mutually Exclusive Events

Two events A and B are said to be *mutually exclusive* if the occurrence of A precludes the occurrence of B and vice versa *i.e.*, if they cannot occur *simultaneously*. For example, in the random experiment of tossing an unbiased coin, the two events defined by

A = Head appears,

B = Tail appears,

are mutually exclusive.

2nd Law (Law of Addition) :

If A and B are two mutually exclusive events with probabilities p_1 and p_2 respectively, then the probability of either of them (A or B) is equal to the sum of their individual probabilities.

i.e., $$p=p_1+p_2$$

or $$P(A \text{ or } B)=P(A)+P(B). \qquad ...(9\cdot4)$$

Proof. Let n be the total number of possible outcomes, of which m_1 are favourable to A and m_2 are favourable to B. Then $p_1=\frac{m_1}{n}$ and $p_2=\frac{m_2}{n}$.

Since A and B are mutually exclusive, m_1 outcomes favourable to A are not favourable to B and vice versa. Hence there are exactly m_1+m_2 outcomes favourable to the event 'A or B'.

$$p=\frac{m_1+m_2}{n}=\frac{m_1}{n}+\frac{m_2}{n}=p_1+p_2,$$

or $$P(A \text{ or } B)=P(A)+P(B).$$

Generalising, if A_1, A_2, A_3, ..., A_n are the mutually exclusive events with probabilities $p_1, p_2, p_3, ..., p_n$, then

$$P(A_1 \text{ or } A_2, ..., \text{ or } A_n)=P(A_1)+P(A_2)+...+P(A_n)$$

or $$p=p_1+p_2+...+p_n. \qquad ...(9\cdot5)$$

EXAMPLE 9.5-2

An urn contains 4 white and 6 black balls. What is the probability that the ball drawn is white or black ?

Solution

Since the ball drawn can be either white or black, the probability is $4/10+6/10=1$.

9.5-3. Mutuallay Exclusive and Exhaustive Events

Events A_1, A_2, A_3, ..., A_n are called mutually exclusive and exhaustive events if

(*i*) when one of them occurs, none out of the remaining will occur,

(*ii*) one or the other event must occur in any trial.

For example, in the trial of tossing of a fair coin, the events that head appears and tail appears are two mutually exclusive and exhaustive events, as the chances of the coin resting on its edge are almost zero.

3rd Law :

If A_1, A_2, A_3, ..., A_n are n mutually exclusive and exhaustive

events with probabilities $p_1, p_2, \ldots, p_n$ respectively, then

$$p_1+p_2+p_3+\ldots+p_n=1 \qquad \ldots(9\cdot6)$$

Proof. Let n be the total number of possible outcomes, of which m_1 are favourable to event A_1, m_2 to A_2, ..., m_n to A_n, so that

$$p_1=\frac{m_1}{n},\ p_2=\frac{m_2}{n},\ \ldots,\ p_n=\frac{m_n}{n}.$$

As the events are mutually exclusive, m_1 outcomes are favourable only to A_1 and not to $A_2, A_3, \ldots, A_n$ and so on. Moreover, they are also exhaustive.

$$\therefore \qquad m_1+m_2+\ldots+m_n=n$$

or $$\frac{m_1}{n}+\frac{m_2}{n}+\ldots+\frac{m_n}{n}=1$$

or $$p_1+p_2+\ldots+p_n=1$$

or $P(A_1)+P(A_2)+\ldots+P(A_n)=1.$

9.5-4. Mutually Independent Events

Two events A and B are said to be independent events if the occurrence or non-occurrence of A does not depend upon the occurrence of B and vice versa.

4th Law :

If two events A and B are mutually independent with individual probabilities p_1 and p_2 respectively, then the probability p of their simultaneous occurrence is equal to the product of their individual probabilities.

i.e., $$P(A \text{ and } B)=P(A).P(B)$$

or $$P(AB)=P(A).P(B)$$

or $$p=p_1 \cdot p_2. \qquad \ldots(9\cdot7)$$

Proof. Let n_1 and m_1 be the total number of possible and favourable outcomes for event A and n_2 and m_2 for event B so that

$$p_1=\frac{m_1}{n_1} \quad \text{and} \quad p_2=\frac{m_2}{n_2}.$$

As the two events are independent, n_1 possible outcomes for event A can be associated with each of the n_2 possible cases for event B, so that the total number of possible cases for 'A and B' is n_1n_2. Similarly, the total number of favourable cases for 'A and B' is m_1m_2.

$\therefore$ Probability of 'A and B',

$$p=\frac{m_1m_2}{n_1n_2}=\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=p_1p_2.$$

or $P(A \text{ and } B)=P(A).P(B)$

or $\quad P(AB)=P(A).P(B).$

Similarly, if there are three mutually independent events A, B and C (this means that the probability of any one of these taking place does not depend upon whether the remaining two have taken place), then

$$P(A \ \& \ B \ \& \ C) = P(A \ \& \ B).\ P(C)$$

or $$P(ABC) = P(A).\ P(B).\ P(C). \qquad \ldots(9.8)$$

Generalising, for n mutually independent events $A_1, A_2, \ldots, A_n$ with individual probabilities of $p_1, p_2, \ldots, p_n$ respectively, the probability of event '$A_1, A_2, \ldots,$ and A_n' is the product $p_1\, p_2 \ldots p_n$.

i.e., $$P(A_1, A_2, \ldots, \text{and } A_n) = P(A_1).\ P(A_2) \ldots P(A_n). \qquad \ldots(9.9)$$

Thus the probability for joint occurrence of any number of mutually independent events is equal to the product of the probabilities of these events.

EXAMPLE 9.5-4.1

A worker is to look after three machines. In any given hour, the probability of first machine not requiring worker's attention is 0.8, for the second machine it is 0.85 and for the third it is 0.75. What is the probability that none of the machines will require the worker's attention during a given hour ?

Solution.

Assuming that the machines work independently of each other, the required probability is

$$= (0.8)\ (0.85)(0.75)$$
$$= 0.51.$$

EXAMPLE 9.5-4.2

In the previous problem, what is the probability that at least one of the three machines will not require the worker's attention during a given hour ?

Solution.

In this problem we are to deal with probability of the form P(A or B or C) and hence we first think of the addition rule. However, this rule cannot be applied directly, since any two of the three events are mutually compatible (nothing prevents any two of the machines from working normally during a given hour). Moreover, the sum of the three given probabilities also exceeds unity.

However, the probability of the first machine requiring the worker's attention is 0.2, for the second machine it is 0.15 and for the third it is 0.25. Since the three events are mutually independent,

the probability that all the three events will take place is

$$(0.2)\,(0.15)\,(0.25)=0.00075.$$

But the events "all the three machines will require attention" and "at least one of the three will work quietly" are mutually incompatible. Hence their sum must be unity and, therefore, the probability that at least one machine will not require attention is

$$1-0.00075=0.99925.$$

As this value is very near to unity, the probability that at least one machine will not require the worker's attention is almost a certainty. Thus at least one machine will operate normally during a given hour.

EXAMPLE 9.5-4.3

An urn contains 5 white and 8 black balls. Another urn contains 6 white and 10 black balls. One ball is taken out from each of the urns. What is the probability that the balls taken out are both white ?

Solution

Probability of white ball from first urn $=5/13$ and probability of white ball from second urn $=6/16$. Now the colour of the ball drawn from the second urn does not depend upon the colour of the ball drawn from the first urn. Hence the two events are independent and the required probability is

$$\left(\frac{5}{13}\right)\left(\frac{6}{16}\right)=\frac{15}{104}.$$

EXAMPLE 9.5-4.4

The probability of shooting down an enemy aircraft by one rifle shot is 0.004. Find the probability of shooting down the plane with simultaneous shots from 250 rifles.

Solution

Probability of not shooting down the aircraft with a single shot is $1-0.004=0.996$. Since the events are independent, the probability that it will not be downed by 250 shots $=(0.996)^{250}$.

$\therefore$ The probability that at least one of 250 shots will down the plane

$$=1-(0.996)^{250}=\frac{5}{8}\text{ (approx.)}$$

9.6 Modified Addition Law

The previous sections dealt with the laws governing the probability of compound events consisting of mutually exclusive or

mutually independent events. However, there may be many events which are neither mutually exclusive nor independent. The modified addition law for such events states :

The probability that at least one of the events A and B occurs is obtained by adding the probability that A occurs and the probability that B occurs and then subtracting the probability that both A and B occur.

i e., $$\text{P(A or B)} = \text{P(A)} + \text{P(B)} - \text{P(AB)}. \qquad \text{...(9.10)}$$

Proof. The various possible combinations of two events A and B are

(*i*) A occurs and B occurs (n_{11}),

(*ii*) A occurs and B does not occur (n_{12}),

(*iii*) A does not occur and B occurs (n_{21}),

(*iv*) A does not occur and B does not occur (n_{22}).

Let n be the total number of possible outcomes of the combination of A and B and let n_{11}, n_{12}, n_{21} and n_{22} represent the possible outcomes that favour occurrence of (*i*), (*ii*), (*iii*) and (*iv*) respectively. Here subscript 1 stands for the occurrence of event A or B and subscript 2 stands for the non-occurrence of event A or B. For example, n_{12} represents the number of possible outcomes in which event A occurs and event B does not occur.

Since event 'A or B' means one of the above combinations (*i*) (*ii*) and (*iii*), the number of possible outcomes favourable to 'A or B' is $n_{11}+n_{12}+n_{21}$.

$$\therefore \quad \text{P(A or B)} = \frac{n_{11}+n_{12}+n_{21}}{n} = \frac{(n_{11}+n_{12})+(n_{11}+n_{21})-n_{11}}{n}$$

$$= \frac{n_{11}+n_{12}}{n} + \frac{n_{11}+n_{21}}{n} - \frac{n_{11}}{n}$$

$$= \text{P(A)} + \text{P(B)} - \text{P(AB)}.$$

Generalising, the modified addition law for three events becomes

$$\text{P (A or B or C)} = \text{P(A)} + \text{P(B)} + \text{P(C)} - [\text{P(AB)} + \text{P(BC)} + \text{P(AC)}] + \text{P (ABC)}. \qquad \text{...(9.11)}$$

Proof. Let us consider events 'B or C' as event D. Then applying the modified addition law for two events A and D, we get

$$\text{P(A or D)} = \text{P(A)} + \text{P(D)} - \text{P(AD)},$$

$$\text{P(D)} = \text{P(B or C)} = \text{P(B)} + \text{P(C)} - \text{P(BC)}$$

and $$\text{P(AD)} = \text{P(A \& `B or C')} = \text{P(AB or AC)}$$

$$= \text{P(AB)} + \text{P(AC)} - \text{P(ABC)}.$$

$\therefore$ $$P(A \text{ or } D)=P(A)+P(B)+P(C)-P(BC)-P(AD)$$

or $$P(A \text{ or } B \text{ or } C)=P(A)+P(B)+P(C)-$$
$$-[P(AB+P(BC)+P(CA)]+P(ABC).$$

9.7 Law of Conditional Probability

Sometimes it may be given that event A has occurred and it may be required to find the probability that B also occurs. For example, it may be given that the card drawn from a deck of 52 cards is red and it may be required to find the probability that the card drawn is a king of hearts. Or, it may be found on medical research that a randomly selected person has a family history of leprosy and further it may be required to find the probability of this particular person also to suffer from leprosy. In such situations we are, obviously, given that an event A has occurred and we are to find the probability that B also occurs.

The event B is dependent on A and occurs only if A has occurred. The probability attached to such an event is called *conditional probability*. It is denoted by $P(B/A)$ *i.e.*, probability of B given that A has occurred and is expressed as

$$P(B|A)=\frac{P(AB)}{P(A)}. \qquad \ldots(9.12)$$

Proof. Let n denote the total number of possible outcomes of which m are favourable to event A. The cases favourable to A *and* B are to be found from the m cases favourable to A. Let m_1 be the number of such cases. Then, from the definition of probability,

$$P(AB)=\frac{m_1}{n}.$$

This can also be written as

$$P(AB)=\frac{m}{n}\cdot\frac{m_1}{m},$$

where $\frac{m}{n}$ is the probability of A. To understand the second factor, we observe that assuming the occurrence of A, there are only m equally likely cases left, out of which m_1 are favourable to B. Hence ratio $\frac{m_1}{m}$ represents the conditional probability $P(B/A)$ of B assuming that A has actually occurred.

Thus $\frac{m}{n}=P(A)$ and $\frac{m_1}{m}=P(B/A)$.

$\therefore$ $P(AB)=P(A)\,.\,P(B/A)$.

Similarly, $P(AB)=P(B).P(A/B)$. $\qquad \ldots(9{\cdot}13)$

Thus probability of the product AB of the two events is equal to the product of probability of event A and conditional probability of B under the condition A or is equal to the product of probability of event B and conditional probability of A under the condition B. This is called law of *conditional probability or compound probability.*

This result can be easily extended to three or more events. For example, let us consider three events A, B & C. The occurrence of *A and B and C* is evidently equivalent to the occurrence of the compound event *AB and C*. Therefore, we have

$$P(ABC)=P(AB).\ P(C/AB).$$

Also $$P(AB)=P(A).P(B/A).$$

$$\therefore \quad P(ABC)=P(A).P(B/A).P(C/AB). \qquad ...(9.14)$$

This formula means. *Probability of the product of three events is equal to the product of probability of first event, conditional probability of second event when the first event has occurred, and conditional probability of the third event when the product of the first and the second events has occurred.*

Generalising, if $A_1, A_2, \ldots, A_n$ are the random events, then

$$P(A_1 . A_2 \ldots A_n)=P(A_1).P(A_2/A_1).P(A_3/A_1A_2)\ldots P(A_n/A_1\ldots A_{n-1}) \qquad ...(9 \cdot 15)$$

If events A and B are *independent*, the conditional probability P(B/A) is the same as the probability of B, P(B) found without any reference to A. Hence the compound probability of two independent events can be written as

$$P(AB)=P(A).P(B),$$

a result already proved under section 9·5·4. This result, obviously, can be extended to three or more events.

9·8. Bayes' Theorem

If $A_1, A_2, \ldots, A_n$ are mutually exclusive events whose union is the sample space S, where $P(A_i)) \neq 0$ for $i=1, 2, \ldots, n$ and if B is any random event for which $P(B)=0$, then for all i

$$P(A_i/B)=\frac{P(A_i).P(B/A_i)}{P(A_1).P(B/A_1)+P(A_2).P(B/A_2)+\ldots+P(A_n).P(B/A_n)} \qquad ...(9 \cdot 16)$$

Proof. Evidently, Bayes' theorem concerns with finding the conditional probability $P(A_i/B)$. The probability of compound event A_iB can be written in two forms :

$$P(A_iB)=P(A_i).P(B/A_i)$$

or $$P(A_iB)=P(B).P(A_i/B).$$

Equating the right hand sides, we derive the following expres-

sion for the unknown probability $P(A_i/B)$:

$$P(A_i/B)=\frac{P(A_i).P(B/A_i)}{P(B)} \qquad ...(9\cdot17)$$

Since the event B can occur in mutually exclusive forms

$$A_1B,\ A_2B,\ldots,\ A_nB,$$

by applying the theorem of total probability, we get

$$P(B)=P(A_1).P(B/A_1)+P(A_2).P(B/A_2)+\ldots+P(A_n).P(B/A_n).$$

Substituting the value of P(B) in equation (9·17), we get

$$P(A_i/B)=\frac{P(A_i).P(B/A_i)}{P(A_1).P(B/A_1)+P(A_2).P(B/A_2)+\ldots+P(A_n).P(B/A_n)}.$$

9·9. Finite Stochastic Processes and Tree Diagrams

A finite sequence of experiments in which each expriment has a finite number of outcomes with given probabilities is called *finite stochastic process*. A convenient way of describing such a process and finding the probability of any event is by means of a *tree diagram.* Tree diagrams have been used in solving problems 9·9-29 to 9·9-32.

9·10. Additional Examples

EXAMPLE 9.10-1

A drawer contains 50 nuts and 100 washers. Half the nuts and half of the washers are rusted. If an item is randomly chosen, what is the probability that this item is a nut or it is rusted ?

Solution

Let event A denote that the item is a nut and event B denote that it is rusted. Then

$$P(A)=\frac{50}{150}=\frac{1}{3}$$

and $$P(B)=\frac{25+50}{50+100}=\frac{1}{2}.$$

Then P (A or B)=P(A)+P(B)−P(AB)

$$=\frac{1}{3}+\frac{1}{2}-\frac{25}{150}$$

$$=\frac{1}{3}+\frac{1}{2}-\frac{1}{6}=\frac{2}{3}$$

EXAMPLE 9.10-2

An integer is chosen at random from the first 100 positive integers. What is the probability that the integer chosen is divisible by 4 or by 6 ?

Solution

Let event A = the integer chosen is divisible by 4,

event B = the integer chosen is divisible by 6,

and sample space S = (1, 2, 3, ..., 100).

Then $$P(A)=\frac{25}{100}=\frac{1}{4},$$

$$P(B)=\frac{16}{100}=\frac{4}{25}$$

and $$P(AB)=\frac{8}{100}=\frac{2}{25},$$

since integers among the first 100 which are divisible both by 4 and 6 *i.e.*, by 12 are only $=\frac{100}{12}=8$.

Now P (A or B) = P(A) + P(B) − P(AB).

$$\therefore \quad P(A \text{ or } B)=\frac{1}{4}+\frac{4}{25}-\frac{2}{25}=\frac{33}{100}.$$

EXAMPLE 9.10.3

Three horses H_1, H_2 and H_3 are in a race; H_1 is twice as likely to win as H_2 and H_2 is twice as likely to win as H_3. Find

(*i*) their individual probabilities of winning,

(*ii*) the probability that H_2 or H_3 wins.

Solution.

(*i*) Let $P(H_3)=p$; then $P(H_2)=2p$ and $P(H_1)=4p$.

Now the sum of the probabilities must be unity; hence

$$p+2p+4p=1 \quad \text{or} \quad p=\frac{1}{7}.$$

$$\therefore \quad P(H_1)=\frac{4}{7},\ P(H_2)=\frac{2}{7} \text{ and } P(H_3)=\frac{1}{7}.$$

(*ii*) $$P(H_2 \text{ or } H_3)=P(H_2)+P(H_3)=\frac{2}{7}+\frac{1}{7}=\frac{3}{7}.$$

EXAMPLE 9.10.4

Two cards are drawn from a pack of 52 cards. What is the probability that both cards drawn are aces?

Solution

As there are 42 cards in the pack, there are 52 ways of drawing the first card. After the first card has been drawn, the second card can be extracted in 51 ways. Therefore, the total number of ways to draw two cards is 52×51. All these cases are equally likely.

Let us now find the number of cases favourable to drawing of aces. As there are 4 aces, there are 4 ways to draw the first ace. After it has been drawn, there are 3 ways to extract the second ace. Therefore, the total number of ways to draw two aces is 4×3.

$\therefore$ The required probability is

$$\frac{4 \times 3}{52 \times 51} = \frac{1}{13 \times 17} = \frac{1}{221}.$$

EXAMPLE 9.10-5

Two cards are drawn from 8 cards numbered 1 to 8. Find the probability that the sum is odd if

(*i*) the two cards are drawn one after the other without replacement.

(*ii*) the two cards are drawn one after the other with replacement.

(*iii*) the two cards are drawn together.

Solution

(*i*) There are $8 \times 7 = 56$ different possible ways to draw two cards one after the other without replacement. The sum will be odd if one number is even and the other is odd. There are $4 \times 4 = 16$ ways to draw an even number and then an odd number; also there are $4 \times 4 = 16$ ways to draw an odd number followed by an even number.

$\therefore$ Probability $p = \dfrac{16+16}{56} = \dfrac{32}{56} = \dfrac{4}{7}$.

(*ii*) There are $8 \times 8 = 64$ ways to draw two cards one after the other with replacement. As before, there are $4 \times 4 = 16$ ways to draw an even number and then an odd number; also there are $4 \times 4 = 16$ ways to draw an odd number followed by an even number.

$\therefore$ Probability $p = \dfrac{16+16}{64} = \dfrac{32}{64} = \dfrac{1}{2}$.

(*iii*) There are ${}^8C_2 = \dfrac{8\,!}{2\,!\,6\,!} = \dfrac{8 \times 7}{2} = 28$ ways to draw two cards together out of 8. There are $4 \times 4 = 16$ ways to draw an even and an odd number.

$\therefore$ Probability $p = \dfrac{16}{28} = \dfrac{4}{7}$.

EXAMPLE 9.10-6

A card is drawn at random from a set of 20 cards, numbered 1, 2, 3,..., 20. Find the probability that its number is divisible by 3 or 7.

Solution

Let A be the event that the number extracted is divisible by 3 and B the event that the number extracted is divisible by 7.

Then $P(A)=\frac{6}{20}$, $P(B)=\frac{2}{20}$ and $P(AB)=0$.

$$\therefore \quad P(A \text{ or } B)=P(A)+P(B)-P(AB)$$
$$=\frac{6}{20}+\frac{2}{20}=\frac{8}{20}=\frac{2}{5}.$$

EXAMPLE 9.10-7

A gambler draws a card from an ordinary pack of 52 cards and bets that it is a heart or a king. What are the odds in favour of his winning this bet ?

Solution

Let A and B be the events that the card drawn is a heart and a king respectively, then $P(A)=\frac{13}{52}=\frac{1}{4}$, $P(B)=\frac{4}{52}=\frac{1}{13}$. Since there is only one king of heart, $P(AB)=\frac{1}{52}$.

$\therefore$ The required probability is

$$P(A \text{ or } B)=P(A)+P(B)-P(AB)$$

or
$$p=\frac{1}{4}+\frac{1}{13}-\frac{1}{52}$$
$$=\frac{13+4-1}{52}=\frac{16}{52}=\frac{4}{13}.$$

If q represents the probability of losing the bet then

$$q=1-p=1-\frac{4}{13}=\frac{9}{13}.$$

$\therefore$ Odds in favour are $\frac{4}{13}:\frac{9}{13}$ or **4 : 9**

EXAMPLE 9.10-8

(*a*) A coin is tossed n times ; what is the probability of head occurring m times ?

(*b*) Find the probability of head occurring thrice in four throws.

(*c*) What is the probability of getting tail at least once in part (*b*) ?

Solution

(*a*) If the coin is tossed twice, there are four (2^2) different possible arrangements HH, HT, TH, TT.

(*b*) If the coin is tossed thrice, there are eight (2^3) different possible arrangements HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

Thus if the coin is tossed n times, there will be 2^n different possible arrangements. The number of possible arrangements for the head to occur m times is same as the number of combinations of n elements taken m at a time *i.e.*,

$$^nC_m = \frac{n\,!}{m\,!\,(n-m)\,!},$$

and the required probability is

$$= \frac{^nC_m}{2^n} = \frac{n\,!}{2^n\, m\,!\,(n-m)\,!}.$$

(*b*) The required event occurs if the head comes thrice or four times in four throws.

Now probability of occurrence of head thrice

$$= \frac{4\,!}{2^4\,.\,3\,!\,.\,1\,!} = \frac{4}{2^4} = \frac{1}{4},$$

and probability of head occurring four times

$$= \frac{4}{2^4\,.\,4\,!\,0\,!} = \frac{1}{16}.$$

As the events are mutually exclusive, the required probability is $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$.

(*c*) Total number of possible cases for four throws is 2^4. Out of these, there is only one case (H, H, H, H) when tail does not turn up. Therefore, the number of cases favourable to obtaining tail at least once are $2^4 - 1 = 15$, so that the required probability is $\frac{15}{16}$.

EXAMPLE 9.10-9

During random tossing of a coin, a player scores one point for every head 'H' and two points for every tail 'T'. He plays until his score reaches n. What is probability of attaining exactly n points ?

Solution.

The player can attain exactly n points by the following two mutually exclusive ways:

(*a*) if he throws head 'H' when the score already obtained is $n-1$.

(*b*) if he throws tail 'T' when the score already obtained is $n-2$.

Let p_n denote the probabilities of exact n points. Since $P(H)=\frac{1}{2}$, $P(T)=\frac{1}{2}$; the probabilities in (*a*) and (*b*) are $\frac{p_{n-1}}{2}$ and $\frac{p_{n-2}}{2}$ respectively.

$$\therefore \qquad p_n=\frac{p_{n-1}}{2}+\frac{p_{n-2}}{2}=\frac{1}{2}(p_{n-1}+p_{n-2}).$$

Similarly, $p_{n-1}=\frac{1}{2}(p_{n-2}+p_{n-3})$,

$$\begin{aligned} p_{n-2}&=\tfrac{1}{2}(p_{n-3}+p_{n-4}),\\ &\vdots\\ p_4&=\tfrac{1}{2}(p_3+p_2),\\ p_3&=\tfrac{1}{2}(p_2+p_1),\\ p_2&=\tfrac{1}{2}(p_1+p_0). \end{aligned}$$

Adding, $p_n+\frac{1}{2}p_{n-1}=p_1+\frac{1}{2}p_0$

or $$p_n=-\frac{p_{n-1}}{2}+p_1+\frac{p_0}{2}$$

Now, $p_0=1, p_1=\frac{1}{2}$ $\therefore$ $p_n=-\frac{p_{n-1}}{2}+\frac{1}{2}+\frac{1}{2}=-\frac{p_{n-1}}{2}+1.$

EXAMPLE 9.10-10

A pair of dice is rolled once. Find the probability that the sum of two numbers is

(*i*) three,

(*ii*) eight,

(*iii*) eleven.

Solution

Sample space S is $=\{(x_1, x_2)$, where $x_1=1, 2, \ldots, 6$; $x_2=(1, 2, \ldots, 6)\}$. Since the first die can have any number from 1 to 6 on it and the second die, quite independently, can also have any number from 1 to 6 on it, there are $6\times 6=36$ elements in the set S. Let A be the event that the sum is 3, B the event that the sum is 8 and C the event that the sum is 11. Then

$$A=\{(1, 2), (2, 1)\},$$
$$B=\{(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1)\},$$
$$C=\{(5, 6), (6, 5)\}.$$

$$\therefore \quad P(A)=\frac{2}{36}=\frac{1}{18},\ P(B)=\frac{7}{36} \text{ and } P(C)=\frac{2}{36}=\frac{1}{18}$$

EXAMPLE 9.10-11

A die is thrown *n* times. What is the probability of obtaining

'four' at least once ? Also find the number of throws sufficient to assure that there are more chances to obtain 'four' at least once than there are chances to fail.

Solution

There are six different cases in the first throw and each case of the first throw can combine with each of the six cases of the second throw and so on, the total number of cases in n throws will be 6^n. Instead of finding the cases favourable to 'four', it will be easier to find the number of cases unfavourable to getting 'four'. In one throw there are 5 such cases and in n throws there will be 5^n such cases.

$\therefore$ Number of cases favourable to 'four' $= 6^n - 5^n$,

and the required probability p is $= \dfrac{6^n - 5^n}{6^n} = 1 - \left(\dfrac{5}{6}\right)^n$.

Now, the statement that there are more chances to obtain 'four' at least once than there are chances to fail means that the probability p should be $> \frac{1}{2}$.

$\therefore$ We put $\quad 1-(5/6)^n > \frac{1}{2}$

or $\quad (5/6)^n - 1 < -\frac{1}{2}$.

or $\quad (5/6)^n < \frac{1}{2}$ or $(6/5)^n > \dfrac{2}{1}$

or $\quad n > \dfrac{\log 2}{\log 6 - \log 5}$ or $n > 3.8 \quad \therefore \quad n = 4$.

Hence if a single die is thrown 4 times, there are more chances to obtain 'four' at least once than there are chances to fail.

EXAMPLE 9.10-12

Two dice are thrown one after the other for n times. What is the probability of obtaining 'double four' at least once ? How many times the two dice be thrown to ensure a probability $> \frac{1}{2}$ of obtaining 'double four' at least once ?

Solution

This problem is similar to the previous problem. There are 36 cases in every throw and total number of cases for n throws will be 36^n. In one throw there are 35 cases unfavourable to 'double four' and in n throws there will be 35^n such cases.

$\therefore$ Number of favourable cases $= 36^n - 35^n$,

and the required probability $p = \dfrac{36^n - 35^n}{36^n} = 1 - \left(\dfrac{35}{36}\right)^n$.

For the remaining part, $p > \frac{1}{2} \quad \therefore \quad 1 - \left(\dfrac{35}{36}\right)^n > \frac{1}{2}$

or $$\left(\frac{35}{36}\right)^n < \tfrac{1}{2}$$

or $$n > \frac{\log 2}{\log 36 - \log 35} \text{ or } n > 24\ 6.$$

$\therefore$ $$n = 25.$$

$\therefore$ In 25 throws there are more chances of obtaining 'double four' at least once than not to obtain at all.

EXAMPLE 9.10-13

A die is so weighted that when it is tossed, the probability of a number appearing on it is proportional to the given number (*e.g.*, 4 has twice the probability of appearing than 2). If event A represents a prime number ; B, an event number and C, an odd number, find

(*i*) the probability space,

(*ii*) P(A), P(B) and P(C),

(*iii*) (*a*) P(prime or even number),

(*b*) P (prime and odd number),

(*c*) P (A but not B).

Solution

(*i*) Let $P(1)=p$. Then $P(2)=2p$, $P(3)=3p$, ..., $P(6)=6p$. Since the sum of probabilities must be unity, $p+2p+3p+4p+5p+6p=1$ or $p=1/21$.

$$\therefore \quad P(1)=\frac{1}{21},\ P(2)=\frac{2}{21},\ P(3)=\frac{3}{21}=\frac{1}{7},\ P(4)=\frac{4}{21},$$

$$P(5)=\frac{5}{21},\ P(6)=\frac{6}{21}=\frac{2}{7}.$$

(*ii*) $A=\{2, 3, 5\}$ $\therefore$ $P(A)=\dfrac{2+3+5}{21}=\dfrac{10}{21}$.

$B=\{2, 4, 6\}$ $\therefore$ $P(B)=\dfrac{2+4+6}{21}=\dfrac{4}{7}$

$C=\{1, 3, 5\}$ $\therefore$ $P(C)=\dfrac{1+3+5}{21}=\dfrac{3}{7}$.

(*iii*) (*a*) $P(\text{A or B})=P(2, 3, 4, 5, 6)=\dfrac{20}{21}$.

(*b*) $P(\text{A or C})=P(3, 5)=\dfrac{3+5}{21}=\dfrac{8}{21}$.

(*c*) $P(\text{A but not B})=P(3, 5)=\dfrac{8}{21}$.

EXAMPLE 9·10-14

An urn contains 7 pieces of paper of equal sized marked 1 to 7. Find the probability of drawing a strip

(*i*) with an even number,

(*ii*) with an odd number.

Solution. (*i*) Probability of drawing any strip of paper is 1/7. There are 3 strips with even number *e.g.* 2, 4, 6.

$\therefore$ Probability of drawing a strip with even number $=$ **3/7**.

(*ii*) Probability of drawing a strip with odd number $=1-3/7$ $=$ **4/7**.

EXAMPLE 9.10-15

An urn contains 6 white and 3 black balls. One ball is drawn, its colour unnoted and it is laid aside. Another ball is then drawn. Find the probability that it is white or black.

Solution

As there are 9 balls in all, there are 9 ways to draw the first ball and whatever its colour, there remain 8 ways to draw the second ball. Therefore, the total number of equally likely cases is $9 \times 8 = 72$.

Now let us find the number of cases favourable to drawing a second white or black ball. Suppose we are interested in white ball in the second drawing. If the first ball were white it could be drawn in 6 ways and the second white ball obviously could be drawn in 5 ways, so that the number of favourable cases is $6 \times 5 = 30$. Again, supposing the first ball were black, it could be drawn in 3 ways and the second white ball could be drawn in 6 ways, so that the number of favourable cases is $3 \times 6 = 18$. The total number of favourable cases is $30 + 18 = 48$.

$\therefore$ The required probability of white ball $= \dfrac{48}{72} = \dfrac{2}{3}$.

Similarly, it can be found that probability of black ball $= \frac{1}{3}$.

EXAMPLE 9.10-16

An urn contains 6 white and 3 black balls. A white ball is drawn first and then a second ball is drawn. What is the probability of second ball to be white ?

Solution

The first write ball can be drawn in 6 ways and the second ball in 8 ways.

$\therefore$ Number of equally likely cases $= 6 \times 8 = 48$.

Number of cases favourable to getting the second white ball

$$=6\times5=30.$$

$\therefore$ The required probability is $=\frac{30}{48}=\frac{5}{8}$.

EXAMPLE 9.10-17

An urn contains m white balls and n black balls. $a+b$ balls are drawn from this urn; find the probability that among them there are exactly a white and b black balls.

Solution

Total number of possible and equally likely ways to draw $a+b$ balls out of the total of $m+n$ balls is ${}^{a+b}C_{m+n}$. Now a white ball can be drawn from a total of m white balls in ${}^{a}C_{m}$ ways; and likewise b black balls can be drawn from a total of n black balls in ${}^{b}C_{n}$ ways.

$\therefore$ Total number of ways to draw exactly a white and b black balls

$$={}^{a}C_{m}\,.\,{}^{b}C_{n}.$$

$\therefore$ The required probability is

$$=\frac{{}^{a}C_{m}\,.\,{}^{b}C_{n}}{{}^{a+b}C_{m+n}}.$$

EXAMPLE 9·10-18

A box contains 20 pieces of which 8 are defective. Two pieces are drawn at random from the box. Find the probability that

(*i*) both pieces are good,

(*ii*) both are defective,

(*iii*) one is good and the other is defective.

Solution

(*i*) Let event A = {first piece is good},

and event B = {second piece is good};

then event AB = {both pieces are good}.

$\therefore$
$$P(AB)=P(A).P(B/A)$$
$$=\frac{12}{20}\cdot\frac{11}{19}=\frac{33}{95}$$

(*ii*) Similarly, probability that both pieces are defective

$$=\frac{8}{20}\cdot\frac{7}{19}=\frac{14}{95}.$$

(*iii*) *First Method.* Let event C

= {One piece is good and one is defective}

Then P (A)+P(B)+P(C)=1, since they are mutually exclusive events.

$$\therefore \quad P(C) = 1 - \frac{33}{95} - \frac{14}{95} = \frac{48}{95}.$$

Second Method. Let event D

= {first piece is good and second is defective}

and event E = {first piece is defective and second is good},

$\therefore$ P(C)=P(D)+P(E), since the two events are mutually exclusive.

$$= \frac{12}{20} \cdot \frac{8}{19} + \frac{8}{20} \cdot \frac{12}{19} = \frac{24}{95} + \frac{24}{95} = \frac{48}{95}.$$

EXAMPLE 9.10.19.

Ten digits 0, 1, 2, ..., 9 are in random order. What is the probability that the digits 5 and 6 come together ?

Solution

Digits 5 and 6 can come together in the following 18 ways :

Event A_1: 56— — — — — — — —,

Event A_2: —56— — — — — — —,

⋮

Event A_9: — — — — — — — — —56,

Event B_1: 65— — — — — — — —,

Event B_2: —65— — — — — — —,

⋮

Event B_9: — — — — — — — — —65.

$\therefore$ Probability of digits 5 and 6 to come together

$$= P(A_1 + A_2 + \ldots + A_9 + B_1 + B_2 + \ldots + B_9)$$

$$= P(A_1) + P(A_2) + \ldots + P(A_9) + P(B_1) + P(B_2) + \ldots + P(B_9)$$

$= 18.P(A_1)$ (since the events are mutually exclusive and equally likely)

$$= 18.\left(\frac{1}{10} \cdot \frac{1}{9}\right)$$

$$= \frac{1}{5}.$$

EXAMPLE 9·10·20.

If two integers A and B (B>A) are selected at random, what is the probability that they have no common divisor ?

Solution

The probability that A has a prime p as a factor is $\frac{1}{p}$. The probability that both A and B contain p as a factor is $\frac{1}{p^2}$. The probability that both A and B do not contain p is $1-\frac{1}{p^2}$.

Now the probability that no prime number is a common factor of A and B is given by

$P=p_2 \cdot p_3 \cdot p_4 \cdots p_n \cdots$, where p_2 is the probability that 2 is not a common factor of A and B, p_3 is the probability that 3 is not a common factor of A and B and so on.

∴ The probability that A and B do not have a common divisor

$$=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots=\frac{6}{\pi^2}.$$

EXAMPLE 9.10-21

A man aged 50 years has odds 4:7 in favour of his living upto 75 years, while his wife aged 45 years has odds 3:4 in favour of her living upto 70 years. Find the probability of

(*i*) the couple living alive 25 years hence,

(*ii*) at least one of them living alive 25 years hence.

Solution

(*i*) Probability of the man living alive 25 years hence,

$$P(A)=\frac{4}{11}.$$

Probability of his wife living alive 25 years hence,

$$P(B)=\frac{3}{7}.$$

∴ Probability of the couple living alive 25 years hence,

$$P(AB)=P(A).P(B)=\frac{4}{11}\cdot\frac{3}{7}=\frac{12}{77}.$$

(*ii*) Probability of at least one of them living 25 years hence,

$$P(A \text{ or } B)=P(A)+P(B)-P(AB)$$

$$=\frac{4}{11}+\frac{3}{7}-\frac{12}{77}=\frac{7}{11}$$

EXAMPLE 9.10-22

Three men M_1, M_2, M_3, and two women W_1 and W_2 play in a cthessournament. Persons of the same sex have equal probability

of winning, but the probability of winning for a man is twice as great as that of a woman. Find the probability that

(*i*) a man wins the tournament,

(*ii*) one of M_1 and W_1 wins the tournament.

Solution

Let $P(W_1)=p$; then $P(W_2)=p$ and $P(M_1)=P(M_2)=P(M_3)=2p$. As the sum of probabilities must be unity,

$$p+p+2p+2p+2p=1 \therefore p=\frac{1}{8}.$$

(*i*) Probability that a man wins $=P(M_1 \text{ or } M_2 \text{ or } M_3)$

$$=P(M_1)+P(M_2)+P(M_3)$$

$$=\frac{6}{8}=\frac{3}{4}.$$

(*ii*) Probability that one of M_1 and W_1 wins $=P(M_1 \text{ or } W_1)$

$$=P(M_1)+P(W_1)$$

$$=\frac{2+1}{8}=\frac{3}{8}.$$

EXAMPLE 9·10-23

Three light bulbs are drawn at random from 12 bulbs of which 4 are defective. Find the probability that

(*i*) none is defective,

(*ii*) exactly one is defective,

(*iii*) at least one is defective.

Solution

There are ${}^{12}C_3$

$$=\frac{12\,!}{3\,!\,9\,!}=\frac{12\times11\times10}{3\times2\times1}=220$$ ways to draw 3 bulbs from the 12 bulbs.

(*i*) Since there are $12-4=8$ non-defective bulbs, there are ${}^{8}C_3=\frac{8\,!}{3\,!\,5\,!}=\frac{8\times7\times6}{3\times2\times1}=56$ ways to draw 3 non-defective bulbs.

$\therefore$ The associated probability is $=\frac{56}{220}=\frac{14}{55}$.

(*ii*) There are 4 defective bulbs and ${}^{8}C_2=\frac{8\,!}{2\,!\,6\,!}=\frac{8\times7}{2\times1}=28$ different pairs of non-defective bulbs ; hence there are $4\times28=112$ ways to choose three bulbs so that exactly one

of them is defective.

$\therefore$ The associated probability is $=\frac{112}{220}=\frac{28}{55}$.

(iii) The event that at least one bulb is defective is complementary to the event that none of the bulbs is defective.

$\therefore$ The associated probability $=1-\frac{14}{55}=\frac{41}{55}$.

EXAMPLE 9.10-24 :

Two guns G_1 and G_2 are used to fire a target. In a given interval G_1 can fire 8 shots while G_2 can fire 10. On an average 8 shots out of 10 fired by G_1 and 7 shots out of 10 fired by G_2 hit the target. Both guns fire and the target is hit. What is the probability that the shot fired by G_2 hits the target ?

Solution

Let events A_1 and A_2 denote a shot fired from guns G_1 and G_2 respectively. As G_1 fires 8 shots in the time in which G_2 fires 10, $P(A_1)=0.8\ P(A_2)$. Further, let B denote the event the target is hit. Then $P(B/A_1)=0.8$ and $P\ (B/A_2)=0.7$.

Then by Bayes' formula,

$$P(A_2/B)=\frac{P(A_2).\ P(B/A_2)}{P(A_1).\ P(B/A_1)+P(A_2).\ P\ (B/A_2)}$$

$$=\frac{P(A_2).\ (0.7)}{0.8\ P(A_2).\ (0.8)+P\ (A_2).\ (0.7)}$$

$$=\frac{0.7}{0{\cdot}64+0.7}=\frac{0.7}{1.34}=0.52.$$

EXAMPLE 9.10-25

Given the information that a family contains two children and that at least one of these two children is a girl, find the probability that both are girls.

Solution

The original sample space is {GG, GB, BG, BB}, in which each outcome has a probability of $\frac{1}{4}$. The reduced sample space with outcomes ensuring that there is at least one girl is GG, GB, BG and out of this, the probability of the outcome that both are girls is $\frac{1}{3}$.

EXAMPLE 9.10-26

A ball is transferred from an urn containing two white and

three black balls to another containing four white and five black balls. A white ball is then taken out from the second urn. What is the probability that the transferred ball was white ?

Solution

Let A_1 and A_2 be the two events of transferring a white and a black ball respectively and B be the event of drawing a white ball from the second urn.

Then, $$P(A_1)=\frac{2}{2+3}=\frac{2}{5},\ P\ (B/A_1)=\frac{5}{5+5}$$
$$=\frac{1}{2} \text{ and } P\ (A_2)=\frac{3}{2+3}=\frac{3}{5},$$
$$P(B/A_2)=\frac{4}{4+6}=\frac{2}{5}.$$

$\therefore$ $$P(A_1/B)=\frac{P\ (A_1).\ P(B/A_1)}{P(A_1).\ P(B/A_1)+P(A_2).\ P(B/A_2)}$$
$$=\frac{\frac{2}{5}\times\frac{1}{2}}{\frac{2}{5}\times\frac{1}{2}+\frac{3}{5}\times\frac{2}{5}}=\frac{\frac{1}{5}}{\frac{1}{5}+\frac{6}{25}}=\frac{\frac{1}{5}}{\frac{11}{25}}$$
$$=\frac{5}{11}.$$

EXAMPLE 9.10-27

An urn contains a white balls and b black balls ; another contains c white and d black balls. One ball is transferred from the first urn to the second and then a ball is drawn from the latter. What is the probability that it will be a white ball ?

Solution

Let A_1 and A_2 be the two events of transferring a white and a black ball respectively and B be the event of drawing a white ball from the second urn. Then,

$$P(A_1)=\frac{a}{a+b},\ P\ (B/A_1)=\frac{c+1}{c+d+1};$$
$$P(A_2)=\frac{b}{a+b},\ \ P\ (B/A_2)=\frac{c}{c+d+1}$$

Now $$P\ (B)=P\ (A_1).\ P\ (B/A_1)+P\ (A_2).\ P\ (B/A_2)$$
$$=\frac{a}{a+b}\cdot\frac{c+1}{c+d+1}+\frac{b}{a+b}\cdot\frac{c}{c+d+1}$$
$$=\frac{a+ac+bc}{(a+b)\ (c+d+1)}.$$

EXAMPLE 9.10-28

Two players play the game of tossing a die with faces marked 1 to 6. He who first gets face 1, wins the game. What is the probability that the player who starts the game, wins it ? What is the probability of the other player winning it ?

Solution

Let A and B be the two players and let A start the game. The game can be won by A, first, if he gets face 1 in the first throw ; second, if A and B do not get face 1 in the first throw and A gets it in the second throw ; third, if A and B do not get face 1 in the first and second throws and A gets it in the third throw ; and so on.

Now probability of A winning in the first throw

$$=\frac{1}{6},$$

probability of A winning in the second throw

$$=\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6},$$

probability of A winning the 3rd throw

$$=\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}, \text{ and so on.}$$

$\therefore$ Total probability of A winning the game

$$=\frac{1}{6}+\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}+\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}+\cdots$$

$$=\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\cdots\right]$$

$$=\frac{1}{6}\cdot\frac{1}{1-(5/6)^2}$$

$$=\frac{6}{11}.$$

$\therefore$ Probability of B winning the game $=1-\frac{6}{11}=\frac{5}{11}$.

EXAMPLE 9.10-29

Three boxes B_1, B_2 and B_3 contain light bulbs. B_1 contains 9 bulbs of which 4 are defective, B_2 5 bulbs of which 2 are defective and B_3 7 bulbs of which 3 are defective. A bulb is drawn at random from any one box. What is the probability that the bulb is defective ?

Solution

Two operations are performed in this experiment :

(*a*) one out of the three boxes is selected,

(*b*) a bulb which is either defective (D) or non-defective (N) is drawn.

The following tree diagram describes this process and gives the probability of each branch of the tree.

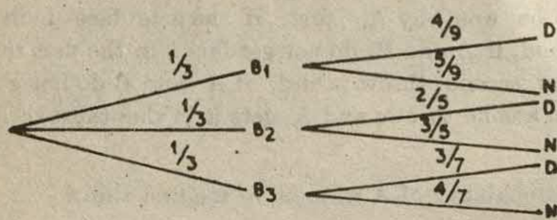


Fig. 9·1

From the diagram, the probabilities of selecting the defective bulb from boxes B_1, B_2 and B_3 are $\frac{1}{3}\cdot\frac{4}{9}$, $\frac{1}{3}\cdot\frac{2}{5}$ and $\frac{1}{3}\cdot\frac{3}{7}$ respectively.

$\therefore$ Probability of getting a defective bulb from B_1 or B_2 or B_3

$$=\frac{1}{3}\cdot\frac{4}{9}+\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{3}{7}$$

$$=\frac{4}{27}+\frac{2}{15}+\frac{1}{7}=\frac{401}{945}.$$

EXAMPLE 9·10-30

A weighted coin having $P(H)=\frac{3}{5}$ and $P(T)=\frac{2}{5}$ is tossed. If head comes up, a number is selected at random from 1 to 11 ; if tail comes up, then a number is selected at random from 1 to 7. Find the probability that an even number is selected.

Solution

The tree diagram with respective probabilities is

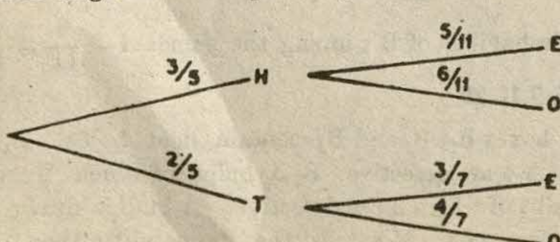


Fig. 9·2

Probability of selecting an even number from numbers 1 to 11 is $\frac{5}{11}$, since there are 5 even numbers out of 11. Similarly, probability of selecting an even number from numbers 1 to 7 is 3/7. Now two paths lead to an even number and the required probability is

$$\frac{3}{5}\cdot\frac{5}{11}+\frac{2}{5}\cdot\frac{3}{7}$$

$$=\frac{3}{11}+\frac{6}{35}=\frac{171}{385}.$$

EXAMPLE 9·10·31

In a factory, three machines A, B and C produce 30%, 25% and 45% of the items. Of their output, 1, 1·5 and 2 per cent are defective. If an item is drawn at random, what is the probability that it is defective ? What is the probability that it was produced by A, B or C ?

Solution

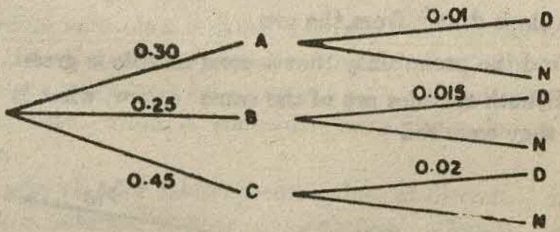


Fig. 9·3

Probability that the item is defective

$$=0{\cdot}30\times0{\cdot}01+0{\cdot}25\times0{\cdot}015+0{\cdot}45\times0{\cdot}02$$
$$=0{\cdot}0030+0{\cdot}00375+0{\cdot}0090$$
$$=0{\cdot}01575$$
$$\simeq0{\cdot}016.$$

Probability that the item was produced by

$$A=\frac{0{\cdot}003}{0{\cdot}016}=0{\cdot}19.$$

Probability that the item was produced by

$$B=\frac{0{\cdot}00375}{0{\cdot}016}\simeq\frac{0{\cdot}004}{0{\cdot}016}=0{\cdot}25$$

Probability that the item was produced by

$$C=\frac{0{\cdot}009}{0{\cdot}016}=0{\cdot}56.$$

Alternatively, let E be the event that the item is defective.

Then $P(E)=P(A).\ P(E/A)+P(B).P(E/B)+P(C).\ P(E/C)$

$$=0{\cdot}30\times0{\cdot}01+0{\cdot}25\times0{\cdot}015+0{\cdot}45\times0{\cdot}02$$

$$=0{\cdot}016.$$

Using Bayes' theorem,

$$P(A/E)=\frac{P(A).P(E/A)}{P(A).P(E/A)+P(B).P(E/B)+P(C).P(E/C)}$$

$$=\frac{0{\cdot}30\times0{\cdot}01}{0{\cdot}30\times0{\cdot}01+0{\cdot}25\times0{\cdot}015+0{\cdot}45\times0{\cdot}02}$$

$$=\frac{0{\cdot}003}{0{\cdot}016}$$

$$=0{\cdot}19.$$

Similarly, $P(B/E)=0{\cdot}25$ and $P(C/E)=0{\cdot}56$.

EXAMPLE 9·10·32

An urn contains 4 green and 6 red marbles. A marble is drawn from the urn and a marble of other colour is put into it. Then a second marble is drawn from the urn.

(*i*) Find the probability that second marble is green.

(*ii*) If both marbles are of the same colour, what is the probability that they were red ?

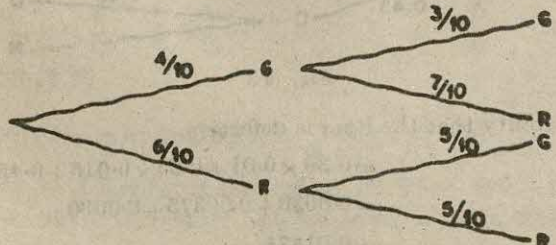


Fig. 9·4

Solution

(*i*) Probability that the second marble is green

$$=\frac{4}{10}\cdot\frac{3}{10}+\frac{6}{10}\cdot\frac{5}{10}$$

$$=\frac{12+30}{100}=0{\cdot}42.$$

(*ii*) Probability that both marbles are of the same colour

$$=\frac{4}{10}\cdot\frac{3}{10}+\frac{6}{10}\cdot\frac{5}{10}=0{\cdot}42,$$

probability that both marbles are red

$$=\frac{6}{10}\cdot\frac{5}{10}=0\cdot30.$$

$\therefore$ The conditional probability

$$=\frac{0\cdot30}{0\cdot42}=\frac{6}{7}.$$

9·11. Random Variables

A random variable is also called *variate, chance variable or stochastic variable*. It is a real valued function defined over the sample space of an experiment. Thus random variable is a variable whose value is a number determined by the outcome of an experiment with which is associated a sample space.

EXAMPLE 9·11·1

In the experiment of rolling of a die, the random variable is represented by the set of outcomes {1, 2, 3, 4, 5, 6}. In the experiment of tossing of a coin, the outcomes, head (H) and tail (T) can be represented as a random variable by assigning 0 to H and 1 to T.

9·12. Discrete and Continuous Random Variables

A random variable x is called *discrete* if the number of possible values of x (*i.e.* the range space) is finite or countably infinite *i.e.*, possible values of x may be $x_1, x_2, \ldots, x_n, \ldots$. The list terminates in the finite case, while it continues indefinitely in the countably infinite case.

A discrete variable takes specific values at discrete points on the real line. For instance, number of members of a family, number of students in a class, number of passengers in a bus, tossing of a coin and rolling of a die, all are examples of discrete variables.

A random variable is called *continuous* if its range space is an interval or a collection of intervals. A continuous variable can assume any value over a continuous range of the real line. For instance, heights of school children, temperatures and barometric pressures of different cities are examples of continuous variables.

9·13. Probability Distribution of a Discrete Random Variable

Let x be a discrete random variable on a sample space S of at most a countably infinite number of values $x_1, x_2, \ldots, x_n, \ldots$. With each possible outcome x_i, we associate a number $P(x_i)$ and call it the probability of x_i. The number $P(x_i)$ must satisfy the following two conditions :

(*i*) $P(x_i)\geqslant 0$, for all values of i,

(*ii*) $$\sum_{i=1}^{n} P(x_i)=1. \quad ...(9\cdot18)$$

The function P is called *probability function* of the random variable x.

EXAMPLE 9·13-1

A pair of fair dice is rolled once. Let x be the random variable whose value for any outcome is the sum of the two numbers on the dice.

(*i*) Find the probability function of x. Construct the probability table and a probability chart.

(*ii*) Find the probability that x is an odd number.

(*iii*) Find $P(3 \leqslant x_i \leqslant 9)$ and $P(0 \leqslant x_i \leqslant 4)$.

Solution

(*i*) As x is the random variable whose value for any outcome is the sum of the two numbers on the dice, the range for x is

(2, 3, 4, ..., 11, 12).

Event for which $x=2$ is (1, 1) *i.e.*, one,

Events for which $x=3$ are (1, 2), (2, 1) *i.e.*, two,

$x=4$ are (1, 3), (2 2), (3, 1) *i.e.*, three,

$x=5$ are (1, 4), (2, 3), (3, 2), (4, 1) *i.e.*, four,

$x=6$ are (1, 5), (2, 4), (3, 3), (4, 2), (5, 1) *i.e.*, five,

$x=7$ are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) *i.e.*, six,

$x=8$ are (2, 6), (3, 5), (4, 4), 5, 3), (6, 2) *i.e.*, five,

$x=9$ are (3, 6), (4, 5), (5, 4), (6, 3) *i.e.*, four,

$x=10$ are (4, 6), (5, 5), (6, 4) *i.e.*, three,

$x=11$ are (5, 6), (6, 5) *i.e.*, two,

event for which $x=12$ is (6, 6) *i.e.*, one.

Thus the probability distribution is as shown in table 9·1 below.

Table 9·1

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The probability distribution is represented on the chart as shown in figure 9·5.

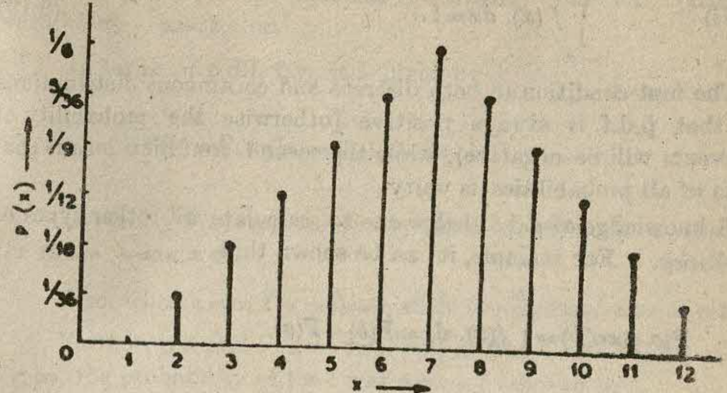


Fig. 9·5

(*ii*) Probability that x is an odd number is

$$P(x=3)+P(x=5)+P(x=7)+P(x=9)+P(x=11)$$
$$=\frac{2}{36}+\frac{4}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36}=\frac{1}{2}.$$

(*iii*) $$P(3\leqslant x_i\leqslant 9)=\sum_{i=3}^{9}P(x_i)=[P(x=3)+P(x=4)+P(x=5)$$
$$+P(x=6)+P(x=7)+P(x=8)+P(x=9)]$$
$$=\frac{2}{36}+\frac{3}{36}+\frac{4}{36}+\frac{5}{36}+\frac{6}{36}+\frac{5}{36}+\frac{4}{36}=\frac{29}{36}.$$

$$P(0\leqslant x_i\leqslant 4)=\sum_{i=2}^{4}P(x_i)=[P(x=2)+P(x=3)+P(x=4)]$$
$$=\frac{1}{36}+\frac{2}{36}+\frac{8}{36}=\frac{1}{6}.$$

9·14. Probability Distribution of a Continuous Random Variable

If x is a continuous random variable, it will have infinite number of values in any interval howsoever small. The probability that this variable lies in the infinitesimal interval $(x, x+dx)$ is expressed as $f(x).dx$, where the function $f(x)$ is called *probability density function* (p.d.f.). If x is a continuous random variable on the range $(-\infty, \infty)$ its p.d.f. $f(x)$ must satisfy the following two conditions :

(i) $f(x) \geqslant 0, \quad -\infty < x < \infty,$

(ii) $$\int_{-\infty}^{+\infty} f(x).\, dx = 1. \qquad \ldots(9 \cdot 19)$$

The first condition in both discrete and continuous distributions means that p.d.f. is always positive (otherwise the probability of some events will be negative), while the second condition means that the sum of all probabilities is unity.

A knowledge of p.d.f. helps one to calculate all other types of probabilities. For example, it can be shown that

$$P(a \leqslant x \leqslant b) = \int_a^b f(x).\, dx = F(b) - F(a).$$

EXAMPLE 9·14-1

Verify that

$$F_x(t) = 0,\ t < -1;$$
$$= \frac{t+1}{2},\ -1 \leqslant t \leqslant 1;$$
$$= 1,\ t > 1;$$

is a distribution function and specify the probability density function for x. Use it to compute $P\left(-\frac{1}{2} < x < \frac{1}{2}\right)$.

Solution

$$\frac{d}{dt} F_x(t) = 0, \text{ for } t < -1 \text{ and } t > 1;$$
$$= \frac{1}{2}, \text{ for } -1 < t < 1.$$

The derivative does not exist at $t = -1$ and $t = 1$, but we can easily define

$$f(x) = \frac{1}{2}, \text{ for } -1 < x < 1;$$
$$= 0, \text{ otherwise};$$

which is the required p.d.f.

$$\text{Now } P\left(-\frac{1}{2} < x < \frac{1}{2}\right) = \int_{-1/2}^{1/2} \frac{1}{2}.dx = \frac{1}{2}\Big[\, x \,\Big]_{-1/2}^{1/2}$$
$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}.$$

9.15 Cumulative Density Function (or Cumulative Distribution Function or Simply Distribution Function)

(*a*) **Continuous Case.** Let x be the continuous random variable, and $F(x)$ the cumulative density function (**CDF**) for variable x, $-\infty < x < \infty$.

In terms of p.d.f. $f(x)$, it is given by

$$F(a) = P(x \leqslant a) = \int_{-\infty}^{a} f(x) . dx. \qquad \text{...(9.20)}$$

Evidently $F(a)$ represents the area enclosed under $f(x)$ within the range $-\infty < x < a$.

Also, when $x = a$, $P(x = a) = 0$, since the enclosed area is zero.

Further, if a and b are two real numbers such that $-\infty < a < b < \infty$, the probability of the event $a < x < b$ is given by

$$P(a < x < b) = \int_{a}^{b} f(x) . dx = \int_{-\infty}^{b} f(x) . dx - \int_{-\infty}^{a} f(x) . dx$$

$$= F(b) - F(a). \qquad \text{...(9.21)}$$

The cumulative distribution function $F(a)$ has the following properties :

$$\left.\begin{array}{ll} (i) & \lim\limits_{a \to \infty} F(a) = \lim\limits_{a \to \infty} \displaystyle\int_{-\infty}^{a} f(x) . dx = 1, \\ (ii) & \lim\limits_{a \to -\infty} F(a) = \lim\limits_{a \to -\infty} \displaystyle\int_{-\infty}^{a} f(x) . dx = 0. \end{array}\right\} \qquad \text{...(9.22)}$$

These properties indicate that $F(a)$ is a ***monotone, non-decreasing*** function of a.

Lastly, from the relationship between $f(x)$ and $F(x)$, it can be concluded that

$$f(x) = \frac{d}{dx} [F(x)]. \qquad \text{...(9.23)}$$

It follows that the probability law of a random variable x is defined completely by either $f(x)$ or $F(x)$.

(*b*) **Discrete Case.** Let x be a random variable, a be a real number and $F(a)$ the probability that x takes values less than or

equal to *a i.e.*,

$$F(a)=P(x\leqslant a). \qquad ...(9.24)$$

Then the function F(a) as defined above is called *cumulative distribution function* of x.

Thus various results for CDF in the discrete case can be obtained by simply substituting P(x) for $f(x)$ in all the above properties. Obviously, integration will be replaced by summation and differentiation by differences. The CDF will be step function since the p.d.f. is defined at discrete points only.

EXAMPLE 9.15-1

If x is a continuous random variable with the following distribution :

$$f(x=\begin{cases}\frac{x}{2}, & 0\leqslant x\leqslant 2\\ 0, & \text{otherwise,}\end{cases}$$

find CDF and represent it graphically.

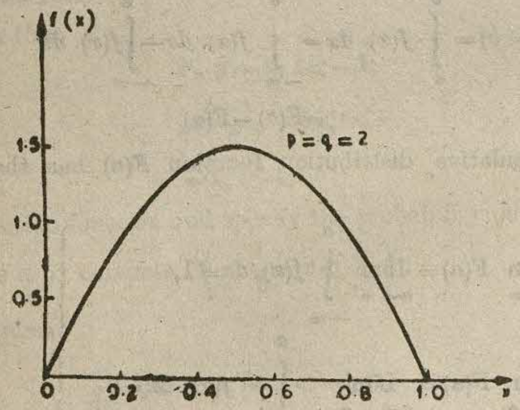


Fig. 9.6

Solution

The CDF for the range $0\leqslant x\leqslant 2$ is defined as

$$F(x)=\int_0^x f(t).\,dt=\int_0^x \frac{t}{2}.dt=\frac{x^2}{4}.$$

$$\therefore \quad F(x)=\begin{cases}0 & , x<0\\ x^2/4 & , 0\leqslant x\leqslant 2\\ 1 & , x>2.\end{cases}$$

It is shown graphically in Fig. 9.6.

EXAMPLE 9.15.2

A fair die is rolled once. The faces are numbered 1, 2,..., 6 and the probability of any face coming up is same *i.e.*, 1/6. Find the CDF and represent it graphically.

Solution

The various events with their associated probabilities are

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Then $F(x)=\sum_{t=1}^{t=x} 1/6=\frac{x}{6}$, $x=1, 2, 3, \ldots, 6$.

The complete CDF is shown in Fig. 9.7.

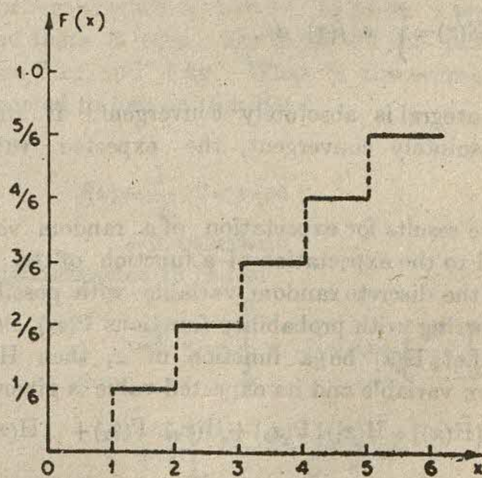


Fig. 9.7

The CDF curve is a step curve consisting of horizontal line segments only.

9.16 Mathematical Expectation of a Random Variable

It has been shown in the previous sections that if x is a discrete random variable, either distribution function $P(x)$ or probability function p can be used to evaluate probability statements about x. If, however, x is a continuous random variable, either distribution function $F(x)$ or density function $f(x)$ can be used to evaluate probability statements about x. However, very often the *average* or *expected value* of x and not merely the probability statement of x, in a certain nterval is asked for.

If x is a *discrete random variable* which takes mutually exclusive values $x_1, x_2, \ldots, x_n$ with associated probability functions $P(x_1)$, $P(x_2), \ldots, P(x_n)$, then the mathematical expectation of x, denoted by $E(x)$ is given by

$$E(x)=x_1\,P(x_1)+x_2\,P(x_2)+\ldots\ldots+x_n\,P(x_n)$$

$$=\sum_{i=1}^{n} x\,P(x), \qquad \ldots(9{\cdot}25)$$

so long as the sum is absolutely convergent.

This number is also called the *expected value or mean value of x.*

Similarly, if x is the *continuous random variable* with probability density function $f(x)$, the expected value of x is defined as

$$E(x)=\int_{-\infty}^{\infty} x.\,f(x).\,dx, \qquad \ldots(9.26)$$

so long as the integral is absolutely convergent. If integral or the sum is not absolutely convergent, the expected value does not exist.

The above results for expectation of a random variable can be easily extended to the expectation of a function of the random variable. Let x be the discrete random variable with possible values x_1, $x_2, \ldots, x_n$ occurring with probability functions $P(x_1), P(x_2), \ldots, P(x_n)$ respectively. Let $H(x)$ be a function of x, then $H(x)$ is also a discrete random variable and its expected value is given by

$$E\,[H(x)]=H(x_1).\,P(x_1)+H(x_2).\,P(x_2)+\ldots H(x_n).\,P(x_n)$$

$$=\sum_{i=1}^{n} H(x_i).\,P(x_i), \qquad \ldots(9.27)$$

provided the series is absolutely convergent.

If x is a continuous random variable with probability density function $f(x)$, then integration rather than summation will be used to get the expected value of $H(x)$.

$$\text{Thus } E[H(x)] = \int_{-\infty}^{\infty} H(x).\; f(x).\,dx, \qquad \ldots(9.28)$$

provided the integral is absolutely convergent.

Laws of Mathematical Expectation

1. $E(k)=k$ *i.e.*, expectation of a constant = constant itself.

2. $E(kx)=k\ E(x)$ *i.e.*, expectation of a product of a constant and chance variable = product of constant and expectation of chance variable.
3. $E(x\pm y\pm z\pm\ldots)=E(x)\pm E(y)\pm E(z)\pm\ldots$

   i.e., expectation of sum (or difference) = sum (or difference) of expectations.
4. $E(xyz\ldots)=E(x).\ E(y).\ E(z)\ \ldots$

   i.e., expectation of a product of independent chance variables = product of their expectations.
5. $E(k\pm lx)=k\pm l\ E(x)$, where k and l are constants.

EXAMPLE 9.16-1

A doctor recommends a patient to go on a particular diet for two weeks and there is equal likelihood for the patient to lose his weight between 2 kg. and 4 kg. What is the average amount the patient is expected to lose on this diet ?

Solution

$$f(x)=\frac{1}{2},\ 2<x<4\ ;$$
$$=0, \text{ otherwise.}$$

The weight expected to be lost

$$=E(x)=\int_2^4 x.\ \frac{1}{2}\ dx=\left[\frac{x^2}{4}\right]_2^4$$

$$=\frac{1}{4}\left[\ 4^2-2^2\ \right]=3 \text{ kg.}$$

EXAMPLE 9.16-2

In the game of rolling a fair die with faces numbered 1 to 6, a person gets as many rupees as the number of the face that turns up. What is the mathematical expectation of his earnings ?

Solution

The possible earnings together with their associated probabilities are given below.

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\therefore\quad E(x)=\sum_{i=1}^{6} x_i\ P(x_i)=1.\ \frac{1}{6}+2.\ \frac{1}{6}+3.\ \frac{1}{6}+4.\ \frac{1}{6}$$

$$+5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

$$= \text{Rs. } 3{\cdot}50.$$

EXAMPLE 9.16-3

In the game of rolling two dice simultaneously, a person is to get as many rupees as the sum of the numbers on the faces of the two dice. What is the mathematical expectation of his earnings ?

Solution

As discussed in example 9.13-1, the various possible earnings together with their associated probabilites are

x_i	2	3	4	5	6	7	8	9	10	11	12
$P(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\therefore \quad E(x) = \sum_{i=2}^{12} x_i \, P(x_i)$$

$$= 2.\frac{1}{36} + 3.\frac{2}{36} + 4.\frac{3}{36} + 5.\frac{4}{36} + 6.\frac{5}{36} + 7.\frac{6}{36}$$

$$+ 8.\frac{5}{36} + 9.\frac{4}{36} + 10.\frac{3}{36} + 11.\frac{2}{36} + 12.\frac{1}{36}$$

$$= \text{Rs. } 7.$$

EXAMPLE 9·16-4

Two persons A and B, play the game of tossing a die with faces marked 1 to 6. He who first gets face 1, wins the game. If A begins the game and each player wins an amount of money equal to the amount of tosses required to win, find their respective mathematical expectations.

Solution

Game can be won A, first, if he gets face 1 in the first throw; second, if A and B do not get face 1 in the first throw and A gets it in the second throw; third, if A and B do not get face 1 in the first and second throws and A gets it in the third throw, and so on.

Now probability of Awinning in the first throw $= \frac{1}{6}$

probability of A winning in the second throw

$$=\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}=\left(\frac{5}{6}\right)^2\cdot\frac{1}{6},$$

probability of A winning in the third throw

$$=\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}=\left(\frac{5}{6}\right)^4\cdot\frac{1}{6},$$

probability of A winning in the fourth throw

$$=\left(\frac{5}{6}\right)^6\cdot\frac{1}{6} \text{ and so on.}$$

$\therefore$ A's mathematical expectation is

$$E(x)=1.\ \frac{1}{6}+3.\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+5.\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}$$
$$+7\left(\frac{5}{6}\right)^6\cdot\frac{1}{6}+\ldots \qquad \ldots(9\cdot29)$$

This is an arithmetico-geometric progression with $\left(\frac{5}{6}\right)^2$ as the common ratio for the geometric progression.

$$\therefore\ \left(\frac{5}{6}\right)^2.E(x)=\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+3.\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}+5.\left(\frac{5}{6}\right)^6\cdot\frac{1}{6}+\ldots \qquad \ldots(9\cdot30)$$

Subtracting equation (9·30) from (9·29),

$$\frac{11}{36}\ E(x)=1.\ \frac{1}{6}+2.\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+2\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}$$
$$+2.\left(\frac{5}{6}\right)^6\cdot\frac{1}{6}+\ldots$$
$$=\frac{1}{6}+2.\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\ldots\right]$$
$$=\frac{1}{6}+\frac{50}{216}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right]$$
$$=\frac{1}{6}+\frac{50}{216}\left[\left(\frac{36}{11}\right)\right]$$
$$=\frac{1}{6}+\frac{50}{66}$$
$$=\frac{61}{66}$$

$$\therefore\ E(x)=\frac{61}{66}\cdot\frac{36}{11}=\frac{366}{121}$$

Similarly B's mathematical expectation is

$$E(y)=2.\left(\frac{5}{6}\right)\cdot\frac{1}{6}+4.\left(\frac{5}{6}\right)^3\cdot\frac{1}{6}+6.\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}+\cdots$$

$$\therefore\ \left(\frac{5}{6}\right)^2E(y)=2.\left(\frac{5}{6}\right)^3\cdot\frac{1}{6}+4\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}+\cdots$$

Subtracting,

$$\frac{11}{36}E(y)=2.\left(\frac{5}{6}\right)\cdot\frac{1}{6}+2.\left(\frac{5}{6}\right)^3\cdot\frac{1}{6}+2.\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}+\cdots$$

$$=2.\left(\frac{5}{6}\right)\cdot\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\cdots\right]$$

$$=\frac{10}{36}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right]$$

$$=\frac{10}{36}\cdot\frac{36}{11}$$

$$=\frac{10}{11}$$

$$\therefore E(y)=\frac{10}{11}\cdot\frac{36}{11}=\frac{360}{121}.$$

9·17. Central Tendency

Having collected the data for a random variable x, and analysed it in the form of frequency distribution, the next step is to find the *nature of distribution*. *Central tendency*—a property for values of x to tend towards the centre is quite important in this context. The three most important measures of central tendency are *mean*, *mode* and *median*.

Mean. If x is the random variable, then its expected value $E(x)$ itself is called the *mean* or *average* value of x and denoted by $\bar{x}$. Mean value of the random variable locates the middle of its probability function.

Mode. The mode of a random variable x is that value of the variable which occurs with the greatest frequency and is denoted by $\hat{x}$. It is possible that a particular distribution may not have a mode, or if it has a mode, it may not be unique. A distribution is called unimodal, bimodal, trimodal, ... depending upon whether it has one, two, three, ... modes.

For a discrete distribution, mode $\hat{x}$ is determined by the follow-

ing inequations :

$$\left.\begin{array}{ll} & P(x=x_i)\leqslant P(x=\hat{x}),\ x_i\leqslant\hat{x} \\ \text{and} & P(x=x_j)\leqslant P(x=\hat{x}),\ x_j\leqslant\hat{x} \end{array}\right\} \quad \ldots(9\cdot31)$$

For a continuous distribution it is determined by the following equations/inequations :

$$\left.\begin{array}{ll} & \dfrac{d}{dx}[f(x)]=0, \\ \text{and} & \dfrac{d^2}{dx^2}[f(x)]=0. \end{array}\right\} \quad \ldots(9\cdot32)$$

Median. For a discrete or continuous distribution of a random variable x, the median is defined as the variate—value X which satisfies the following inequations :

$$\left.\begin{array}{ll} & P(x\leqslant X)=\frac{1}{2}, \\ \text{and} & P(x\geqslant X)=\frac{1}{2}. \end{array}\right\} \quad \ldots(9\cdot33)$$

It is denoted by $\tilde{x}$. Thus if a continuous distribution function has a p.d.f. $f(x)$ in the range (a, b), then $\tilde{x}$ is given by

$$\int_a^{\tilde{x}} f(x)dx=\frac{1}{2}=\int_b^{\tilde{x}} f(x).dx. \quad \ldots(9\cdot34)$$

Note. If mean and median are known, the mode can be calculated from the empirical formula

Mean$-$Mode$=$3(Mean$-$Median).

9·18. Dispersion or Variability

The probability function of a random variable x indicates the possible values that x can have together with their associated probabilities. The mean or E(x) indicates where the centre of the probability function mass of the probability function is located. It gives a quick picture of the long run average result when an experiment is repeated a large number of times. However, it gives no idea as to how results of one performance vary from the other.

The property which indicates the degree of variability of data about the central value is *called dispersion.* Two important measures of dispersion are *mean deviation* and *standard deviation* (*or Variance*).

(*i*) **Mean Deviation.** Mean deviation (M.D.), '$\delta(a)$' of a sample or population from a value 'a' is defined as

(*a*) $\delta(a)=E(\,|\,x-a\,|\,)$

$$=\frac{1}{N}\Sigma f_i \mid x_i-a \mid . \qquad \ldots(9\cdot35)$$

The above equation (9·35) is a general expression applicable to all statistics.

For example, mean deviation from mean is given by

$$\delta(\bar{x})=E(\mid x-\bar{x} \mid),$$

and mean deviation from median is given by

$$\delta(\tilde{x})=E(\mid x-\tilde{x} \mid).$$

Unless otherwise mentioned, $\delta(a)$ stands for mean deviation from the mean *i.e.*, $\delta(\bar{x})$.

Though mean deviation is a good measure of dispersion, it is difficult to be treated mathematically.

(*ii*) **Variance**. If x is a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $f(x_1), f(x_2), \ldots, f(x_n)$, then the variance of x is denoted by Var(x) or σ_x^2 or S^2 and is defined by

$$\left.\begin{aligned}\text{Var}(x)&=\sigma_x^2=E(x-\mu)^2\\&=\frac{1}{N}\sum_{i=1}^{n} f(x_i)\,(x_i-\mu)^2,\end{aligned}\right\} \qquad \ldots(9\cdot36)$$

(For a population with mean $E(x)=\mu$).

and

$$\left.\begin{aligned}\text{Var}(x)&=S^2=E(x-\bar{x})^2\\&=\frac{1}{N}\sum_{i=1}^{n} f(x_i)\cdot(x_i-\bar{x})^2.\end{aligned}\right\} \qquad \ldots(9\cdot37)$$

(For a large sample with mean $E(x)=\bar{x}$).

Calculations of Variance

For a population,

$$\begin{aligned}\text{Var}(x)&=\sigma_x^2=E(x-\mu)^2\\&=E(x^2-2\mu x+\mu^2)\\&=E(x^2)-2\mu.E(x)+\mu^2\\&=E(x^2)-2\mu^2+\mu^2\\&=E(x^2)-\mu^2=E(x^2)-[E(x)]^2. \qquad \ldots(9\cdot38)\end{aligned}$$

$\therefore$ Variance can be calculated as average of squares of x minus square of the average of x.

For a large sample,

$$\text{Var}(x)=s^2=\frac{1}{N}\sum_{i=1}^{n} f(x_i)\,(x_i-\bar{x})^2$$

$$=\frac{1}{N}\sum_{i=1}^{n}(x_i^2-2x_i.\,\bar{x}+\bar{x}^2).\,f(x_i)$$

$$=\frac{1}{N}\left[\sum_{i=1}^{n}x_i^2\,f(x_i)-2\bar{x}\sum_{i=1}^{n}x_i\,f(x_i)+\bar{x}^2\sum_{i=1}^{n}f(x_i)\right]$$

$$=\frac{1}{N}\left[\sum_{i=1}^{n}x_i^2\,.\,f(x_i)-2\bar{x}.\,\bar{x}+\bar{x}^2\,.\,1\right]$$

$$=\frac{1}{N}\left[\sum_{i=1}^{n}x_i^2\,f(x_i)-\bar{x}^2\right]. \qquad ...(9.39)$$

Variance is also called *second moment of dispersion.*

(*iii*) **Standard Deviation.** The positive square root o variance is called *standard deviation* (S.D.) and is denoted by σ_x or S.

Thus $\sigma_x=+\sqrt{\text{Var }(x)}$

$$=[\text{E }(x-\mu)^2]^{\frac{1}{2}} \quad \text{(for a population)} \qquad ...(9.40)$$

$$=[\text{E }(x-\bar{x})^2]^{\frac{1}{2}} \quad \text{(for a large sample).} \qquad ...(9.41)$$

If random variable x is expressed in some units, units of variance will be squares of the units of x. However, units of standard deviation are the same as the units of x and hence S.D. is of more interest to calculate dispersion.

EXAMPLE 9.18-1

Calculate variance and standard deviation of the random variable x defined in example 9.16-1.

Solution

$$\text{Variance } \sigma_x^2=\int_2^4(x-3)^2.\,\frac{1}{2}.\,dx$$

$$=\frac{1}{2}\left[\frac{(x-3)^3}{3}\right]_2^4=\frac{1}{6}\left[(1)^3-(-1)^3\right]=\frac{1}{3},$$

and $$\sigma_x=\frac{1}{\sqrt{3}}.$$

EXAMPLE 9.18-2

Calculate variance and standard deviation of the random variable x defined in example 9.16-3.

Solution

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

As already calculated in example 9.16-3, mean $E(x)=7$.

$$E(x^2)=2^2.\frac{1}{36}+3^2.\frac{2}{36}+4^2.\frac{3}{36}+5^2.\frac{4}{36}+6^2.\frac{5}{36}+7^2.\frac{6}{36}$$

$$+8^2.\frac{5}{36}+9^2.\frac{4}{36}+10^2.\frac{3}{36}+11^2.\frac{2}{36}+12^2.\frac{1}{36}$$

$$=54.8.$$

$$\therefore \quad \text{Var}(x)=E(x^2)+[E(x)]^2$$

$$=55.8-49$$

$$=5.8,$$

and $\quad S.D.=\sqrt{5.8}=2{\cdot}4.$

EXAMPLE 9.18-3

A box contains electric bulbs, proportion p of which are defective. A bulb is drawn at random; if x has value 1 when the bulb is defective and zero otherwise, obtain the variance of x.

Solution

x	:	1	0
$P(x)$	:	p	$1-p(=q)$

$$\therefore \quad E(x)=1.p+0.q=p,$$

$$E(x^2)=1^2.p+0^2.q=p.$$

$$\therefore \quad \text{Var}(x)=E(x^2)-[E(x)]^2=p-p^2=p(1-p)=pq.$$

EXAMPLE 9.18-4

Mean and variance of x are 25 and 2 respectively. Find

(ii $E(x^2)$,

(ii) Var $(3x-2)$,

(iii) σ_{3x-2},

(iv) Var $(-x)$,

(v) σ_{-x}.

Solution

(i) $\text{Var}(x)=E(x^2)-[E(x)]^2$

$\therefore \qquad 2 = E(x^2) - (25)^2$

$\therefore \qquad E(x)^2 = 627.$

(*ii*) $Var\ (3x-2) = Var\ (3x) = 9\ Var\ (x) = 9 \times 2 = 18.$

(*iii*) $\sigma_{3x-2} = +\sqrt{18} = 4{\cdot}24.$

(*iv*) $Var\ (-x) = Var\ (x) = 2.$

(*v*) $\sigma_{-x} = +\sqrt{2} = 1.414.$

9.19 DISCRETE PROBABILITY DISTRIBUTIONS

In this section a brief account is given of a few discrete probability distributions which have a wider use in practice.

9.19-1 Bernoulli Trials and Binomial Distribution

Bernoulli Trial. It is an experiment which has only two possible outcomes, success (S) and failure (F). Various examples of Bernoulli trials are : tossing of a coin (head or tail), firing a target (hit or miss), fighting an election (win or not win), playing a game (win or lose), etc. In fact, any chance mechanism whose outcomes can be grouped into two classes may be regarded as a Bernoulli trial.

Binomial Distribution. We make n trials. The result of each trial is random and can either be success or failure. Let p be the probability of success and $q (= 1 - p)$ be the probability of failure. The results of n trials are independent *i.e.*, the outcome of any particular trial depends neither on the outcomes of the previous trials nor the trials that follow. Since the trials are independent, the probabilities are multiplied. Suppose we are interested only in the total number of successes in n Bernoulli trials. The number of successes can be 0, 1, 2, ..., n. We want to find the probabilities of 0, 1, 2, ..., n successes. Probability of any simple event to occur with k successes and $n-k$ failures in n trials is

$$p^k \cdot q^{n-k},\ k = 0, 1, 2, \ldots, n.$$

Further, since k successes can be chosen among n trials in ${}^nC_k$ ways and the corresponding sample events all have the same probability $(p^k \cdot q^{n-k})$; probability of k successes in n repeated trials will be

$$b(k\ ;\ n, p) = {}^nC_k\ p^k\ q^{n-k},\ k = 0, 1, 2, \ldots, n.$$

Thus we have proved that

If a series of n independent trials is performed such that for each trial, probability of success is p and probability of failure is q ; then the probability of an event occurring with k successes and $n-k$ failures is

$$b(k\ ;\ n, p) = {}^nC_k\ p^k \cdot q^{n-k},\ k = 0, 1, 2, \ldots, n. \qquad \ldots(9.42)$$

If n and p are regarded as constants, then the above function $P(k) = b(k; n, p)$ is a *discrete probability distribution* :

k	0	1	2	...	n
$P(x=k)$	q^n	${}^nC_1\, pq^{n-1}$	${}^nC_2\, p^2 q^{n-2}$	...	p^n

It is called *binomial distribution*, since for $k=0, 1, 2, \ldots, n$ it corresponds to the successive terms of the binomial expansion :

$$(q+p)^n = q^n + {}^nC_1\, pq^{n-1} + {}^nC_2\, p^2 q^{n-2} + \ldots + p^n.$$

Binomial distribution satisfies the condition for *p.d.f.* since

$$P(x=k) \geqslant 0, \; k=0, 1, 2, \ldots, n;$$

and $$\sum_{k=0}^{n} P(x=k) = \sum_{k=0}^{n} {}^nC_k\, p^k q^{n-k} = (q+p)^n = 1.$$

$\therefore$ $P(x=k) = b(k; n, p)$ is a *probability function.*

Properties of binomial distribution are

mean $\mu = np$,

variance $\sigma_x^2 = npq$,

standard deviation $\sigma_x = \sqrt{npq}$.

9·19·2. Negative Binomial Distribution (Pascal Distribution)

In binomial distribution n, the number of trials is fixed and the random variable is the number of successes (or failures) to occur. In *negative binomial distribution*, the random variable is given by the number of independent trials to be carried out until a given number of successes (or failures) occur. If j denotes the number of trials necessary for c, a fixed number of successes, then probability of j trials until c successes occur = probability of $(c-1)$ successes in $(j-1)$ trials $\times$ probability of a success in the jth trial.

$$\therefore \quad P(x=j) = [{}^{j-1}C_{c-1} \cdot p^{c-1} \cdot q^{j-c}] \cdot p.$$

$$= {}^{j-1}C_{c-1}\, p^c \cdot q^{j-c}, \; j=c, c+1, c+2, \ldots \qquad \ldots(9{\cdot}43)$$

Obviously, $P(x=j)$ is the probability that one must wait to get c successes (or failures) in j independent trials hence it is also called *binomial waiting time distribution.*

9·19·3 Geometric Distribution

Independent Bernoulli trials are performed until we get *a success.* The probability of success on each trial is p $(0 < p \leqslant 1)$. If j is the required number of trials to get *a* success, then j is called the *geometric random variable* with parameter p. Evidently j is a discrete random variable as it can have any value 1, 2, 3....

Geometric distribution is a special case of negative binomial

distribution and occurs when $c=1$, *i.e.*,

$$P(x=j)={}^{j-1}c_0 \cdot p \cdot q^{j-1}=pq^{j-1}, j=1, 2, 3, \ldots \qquad \ldots(9\cdot44)$$

Equation (9·44) means that if there are $j-1$ failures, followed by a success on jth trial, then the possibility of such a happening is pq^{j-1}.

$P(x=j)$ is a *probability function* since

$P(x=j) \geqslant 0, j=1, 2, 3, \ldots$

$$\sum_{j=1}^{\infty} P(x=j)=\sum_{j=1}^{\infty} pq^{j-1}=p \sum_{j=1}^{\infty} q^{j-1}$$

$$=p \cdot \frac{1}{1-q}=p \cdot \frac{1}{p}=1$$

Properties of geometric distribution are

$$\text{mean} = \frac{1}{p},$$

$$\text{variance} = \frac{q}{p^2},$$

$$\text{standard deviation} = \frac{\sqrt{q}}{p}.$$

9.19.4. Hypergeometric Distribution

Binomial distribution is applicable to an experiment in which the probability of success is same for all trials. However, if it varies from trial to trial, ***hypergeometric distribution*** is more suitable.

From a lot of N items, of which N_1 are good and $N_2(=N-N_1)$ defective, we choose n ($<$N) items at random *without* replacement. Then the probability that the sample contains k items of first category and $n-k$ of the second is given by

$$P(x=k)=\frac{{}^{N_1}C_k \cdot {}^{N_2}C_{n-k}}{{}^{N}C_n}, k=0, 1, 2, \ldots, n. \qquad \ldots(9\cdot45)$$

Properties of hypergeometric distribution are

$$\text{mean}=\frac{nN_1}{N},$$

$$\text{variance}=\frac{N_1N_2 \cdot n(N-n)}{N^2 \cdot (N-1)},$$

$$\text{standard deviation}=\frac{1}{N} \cdot \sqrt{\frac{N_1N_2 \cdot n(N-n)}{N-1}}$$

If there are m categories instead of two (good and defective), the hypergeometric distribution may be generalised to the following form :

$$\frac{{}^{N_1}C_{k1} \cdot {}^{N_2}C_{k2} \ldots {}^{N_m}C_{km}}{{}^{N}C_n} \qquad \ldots(9\cdot46)$$

9.19-5. Multinomial Distribution

It is obtained from generalisation of binomial distribution. Let the sample space of an experiment be divided into s mutually exclusive events $A_1, A_2, \ldots, A_k$ with probabilities $p_1, p_2, \ldots, p_k$; where $p_1+p_2+\cdots+p_k=1$.

Then in n repeated trials, the probability of A_1 occurring n_1 times, A_2 occurring n_2 times ..., A_k occurring n_k times is

$$\frac{n\,!}{n_1\,!\; n_2\,! \ldots n_k\,!} \cdot p_1^{n_1} \cdot p_2^{n_2} \ldots p_k^{n_k}, \qquad \ldots(9\cdot47)$$

where $n_1+n_2+\ldots+n_k=n$.

The above expression is called *multinomial distribution* since its terms are precisely the terms in the expression of $(p_1+p_2+\cdots+p_k)^n$. If $k=2$, this distribution reduces to binomial distribution.

9·19-6. Poisson Distribution

Let x be the random variable which takes non-negative integer values only *i.e.*, $k=0, 1, 2, \ldots$. Then the probability density function

$$P(x=k)=\frac{\lambda^k e^{-\lambda}}{k\,!}, \; k=0, 1, 2, \ldots, \qquad \ldots(9\cdot48)$$

where $\lambda>0$ is called the *Poisson Distribution* with parameter λ.

It is applicable to events in which p is very small and n is large. Since p is small, these events are called rare events. It is also useful when it is possible to prescribe the number of times an event occurs but not the number of times it does not occur. A typical application of the Poisson distribution occurs in analysing queuing problems (Chapter 10).

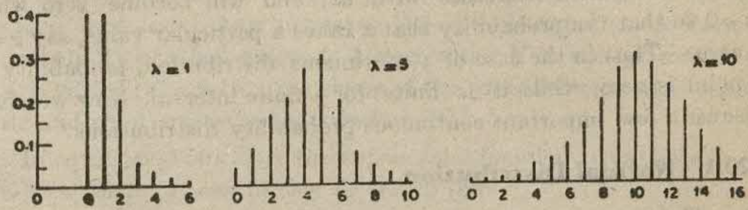
Under the condition that $p \to 0$ as $n \to \infty$ such that $np=\lambda>0$, binomial distribution approximates to Poisson distribution. For binomial distribution,

$$P(x=k)={}^nC_k \left(\frac{\lambda}{n}\right)^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}.$$

It can be shown that as $n \to \infty$,

$$P(x=k)=\frac{\lambda^k \cdot e^{-\lambda}}{k\,!}, \; k=0, 1, 2, \ldots,$$

which is the Poisson distribution. Diagrams of Poisson distribution for various values of λ are given in figure 9·8.



Poisson distributions for different values of λ

Fig. 9·8

Properties of the Poisson distribution are

mean $= \lambda$,

variance $= \lambda$,

standard deviation $= \sqrt{\lambda}$.

This distribution finds application in a wide variety of situations in which some kind of *event occurs repeatedly but haphazardly*. Some of the situations are

1. Number of telephone calls arriving an exchange per unit time.
2. Number of α-particles emitted by a radioactive substance.
3. Number of deaths occurring due to, say, heart disease in a city having large population.
4. Number of typing errors per page in a big text.
5. Number of defects occurring in the long length of cloth being manufactured in a factory.

9·20 Continuous Probability Distributions

In discrete probability distributions, the probability is clustered at certain points. For these distributions,

$$P(x=x_i) = P_i,$$

and $$\Sigma P_i = 1, \text{ where } i = 1, 2, \ldots, n. \qquad \ldots(9\cdot49)$$

In continuous probability distributions, the probability is distributed continuously and uniformly over the whole area of the curve $y=f(x)$. Therefore, to find the probability that x lies between any two values a and b, one has to find the area enclosed between the x-axis, curve $y=f(x)$ and the ordinates $x=a$ and $x=b$. This area is mathematically given by the intergal

$$\int_a^b f(x) \, . \, dx.$$

Similarly, probability that x lies between x and $x+\delta x$ is given by the area of the elementary strip of base δx, constructed near the point x. This area decreases with δx, and will become zero when $\delta x \to 0$ so that the probability that x takes a particular value, say $x=k$ is zero. Thus in the case of a continuous distribution, probability at a point is zero, while it is finite for a finite interval. Now we shall discuss a few important continuous probability distributions.

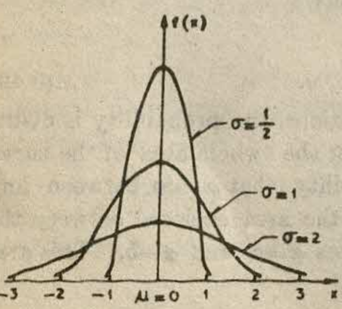
9.20-1 Normal Distribution

The *normal (or Gaussian) distribution or curve* (also called *normal probability curve, probability curve, error curve, etc.*) is the best known continuous distribution and occupies a central position in statistical theory and practice. In nature as well as technology, one often comes across distributions almost similar to the normal distribution. It is most commonly used of all probability laws, firstly, because it occurs most frequently in practical problems and secondly, since it provides an accurate approximation to a large number of probability laws. Errors in measurement are often assumed to have a normal distribution.

The density function of the normal curve is given by

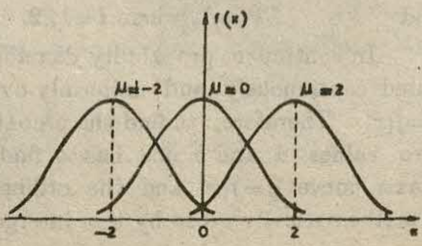
$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty<x<\infty. \qquad (9\cdot50)$$

where μ and σ are the given parameters. Parameter μ can have any real value while σ must always be positive. The form of normal curve depends upon these two parameters and for different values of μ and σ we get different normal curves. Figure 9·9 shows the dependence on σ. The three normal curves have the same value of μ ($\mu=0$) but different values of σ, namely, $\frac{1}{2}$, 1 and 2 respectively.



Normal distribution with μ fixed ($\mu=0$)

Fig. 9·9



Normal distribution with σ fixed ($\sigma=1$)

Fig. 9·10

The curve is symmetric, bell-shaped and centered at μ. The parameter σ controls the relative flatness of the curve. As σ decreases, the curve becomes more sharply peaked and probability of x being close to μ increases. As σ increases, the curve becomes more flatter and probability of x being close to μ decreases. If σ is kept constant and μ is varied (Fig. 9·10), the shape of the curve remains the same but its mid-point moves to the location of μ.

In equation (9·50), since the exponential function is non-negative, $f(x)>0$ for all x. It can further be shown that

$$\int_{-\infty}^{\infty} f(x) \,.\, dx=1,$$

so that $f(x)$ is a density function.

Properties of normal distribution curve are

(*i*) mean $=\mu$,

(*ii*) variance $=\sigma^2$,

(*iii*) standard deviation $=\sigma$.

(*iv*) the curve is symmetrical about the mean. The mean, mode and the median coincide at $x=\mu$. The maximum ordinate (at the mode) is $\frac{1}{\sqrt{2\pi}\sigma}$.

(*v*) Sum (or difference) of independent normal variables also has a normal distribution with mean and variance as the sum (or difference) of the individual variables.

The normal distribution with mean μ and variance σ^2 is denoted by N (μ, σ^2)

The CDF of the normal distribution is given by

$$f(x)=\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y-\mu)^2}{2\sigma^2}} \,.\, dy \,. \qquad \ldots(9{\cdot}51)$$

Unfortunately expression (9·50) and (9·51) cannot be evaluated in closed form. Very exact tables are available for normal distribution and we use them for calculations. They give the value of $F(x)$ as a function of x. These tables are based on the following *standard normal p.d.f.* :

$$\phi(t)=\frac{1}{\sqrt{2\pi}} \,.\, e^{-\frac{t^2}{2}}, \; -\infty<t<\infty, \qquad \ldots(9{\cdot}52)$$

with parameters $\mu=0$ and $\sigma^2=1$. *The standard normal distribution curve* defined by equation (9.52) is shown in figure 9·11.

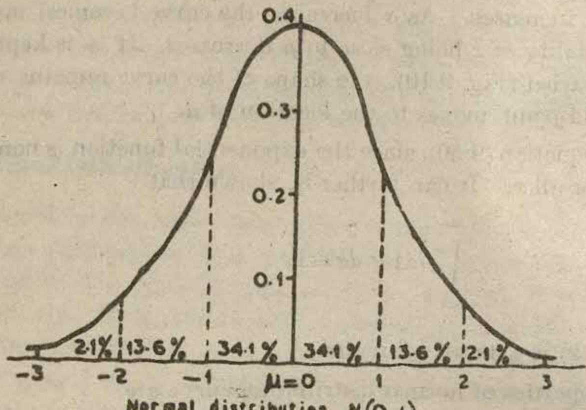


Fig. 9 11

From Fig 9.11, for $-1 \leqslant t \leqslant 1$, we obtain 68.2% of the area under the curve, for $-2 \leqslant t \leqslant 2$, we obtain 95.4% of the area under the curve and for $-3 \leqslant t \leqslant 3$, we get 99.6% of the area under the curve. This means that of the total frequency, 68.2% lies within $\mu \pm \sigma$, 95.4% lies within $\mu \pm 2\sigma$ and 99.6% lies within $\mu \pm 3\sigma$.

The *standard normal CDF* is given by

$$\psi(t)=\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} . dy \qquad ...(9·53)$$

The standard form is obtained from the regular form by making the substitution $t=\frac{x-\mu}{\sigma}$.

Table C.2 at the end of the book gives the areas (between $t=0$ and any positive value of t) of the standard normal distribution curves for values of t varying from 0 to 5·00. It corresponds to equation (9.53).

Normal distribution is the limiting form of the binomial distribution as the number of trials, $n \to \infty$ and neither p nor q is very small. Precisely, if x is the number of successes in n independent trials of an event with p as the probability of success in a single trial, then the binomial variate $\frac{x-np}{\sqrt{npq}}$ is a standard normal variate as $n \to \infty$.

9.20-2. Exponential Distribution

A continuous random variable x, with its *p.d.f.* defined by

$$f(x) = \begin{cases} \mu e^{-\mu x}, \text{ if } x > 0 ; \\ 0, \text{ otherwise,} \end{cases} \qquad \ldots(9{\cdot}54)$$

where parameter $\mu > 0$, is said to have an *exponential distribution*. *P.d.f.* for this distribution is shown in Fig. 9.12.

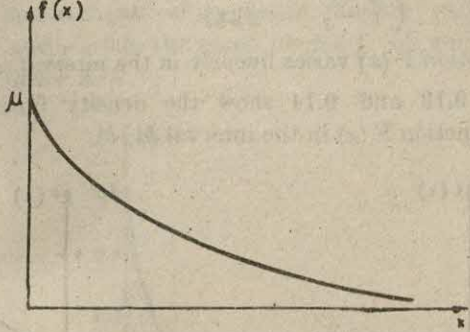


Fig. 9·12

Exponential distribution is often associated with service time in queuing problems. There is distinct analogy between the exponential distribution in continuous case and the geometric distribution in discrete case. For example, a random variable representing the *number* of trials before the first failure in the geometric distribution is analogous to the variable representing *time-to-failure* in the exponential distribution. In the limiting case as $p \to 0$ and inter-trial time$\to 0$, geometric distribution takes the form of exponential distribution.

Another relation of interest exists between the Poisson distribution and the exponential distribution. When the Poisson distribution represents the *number* of failures per unit time, the exponential distribution represents the *time between* two successive failures.

9·20-3 Rectangular Distribution (Uniform or Homogeneous Distribution)

A continuous random variable x is said to have *rectangular* or *uniform distribution* if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, \; a < x < b \; ; \\ 0 \text{ , otherwise,} \end{cases} \qquad \ldots(9{\cdot}55)$$

where $a, b > 0$ are some constants.

This distribution is called uniform distribution because its density function is uniform (constant) in the interval (a, b).

The distribution function F (x) is given by

$$F(x) = \int_{-\infty}^{x} f(x) \cdot dx$$

$$= \begin{cases} 0 & , \quad x < a, \\ \dfrac{x-a}{b-a}, & a \leqslant x < b, \\ 1 & , \quad x > b. \end{cases}$$

The function F (x) varies linearly in the interval (a, b).

Figures 9.13 and 9.14 show the density function $f(x)$ and distribution function F (x) in the interval (a, b).

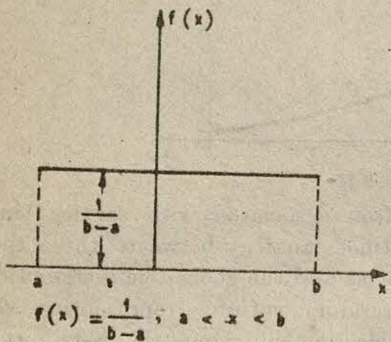


Fig. 9·13

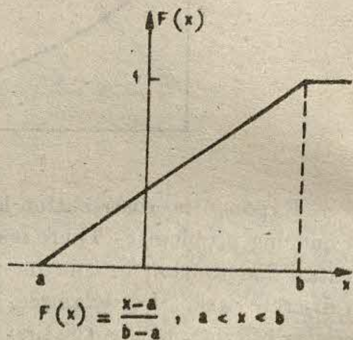


Fig. 9·14

Properties of rectangular distribution are

$$\text{mean} = \frac{a+b}{2},$$

$$\text{variance} = \frac{(b-a)^2}{12},$$

$$\text{standard deviation} = \frac{b-a}{2\sqrt{3}}.$$

Rectangular distribution finds its application in statistical problems.

9·20·4 Gamma Distribution

A continuous random variable x is said to have *gamma* or *Erlang distribution* if it assumes only non-negative values and its *p.d.f.* is given by

$$f(x) = \begin{cases} \dfrac{\mu(\mu x)^{n-1} e^{-\mu x}}{(n-1)!}, & \text{if } 0 < x < \infty ; \\ 0, & \text{otherwise.} \end{cases} \qquad \text{...(9·56)}$$

The parametres μ and n are non-negative parameters. When $n=1$, the above *p.d.f.* is reduced to the exponential *p.d.f.* The gamma distribution is applied to waiting time models in life testing, waiting time until death, etc. If there are n independent and identically distributed exponential random variables, the distribution representing the *summation* of these variables is the gamma distribution. This distribution (sum of exponential random variables) is analogous to the negative binomial (sum of geometric random variables) distribution. For $n=1$ and $\mu=0.5$, the graph for *p.d.f.* of gamma distribution is shown in figure 9.15.

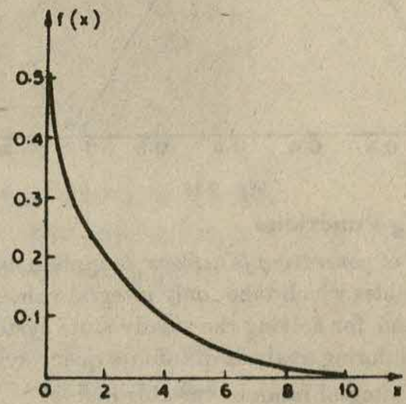


Fig 9·15

As in Poisson distribution, so also in gamma distribution, the mean and variance are equal. This distribution possesses the additive property.

9·20·5 Beta Distribution

A random variable x is said to have *beta distribution* if its density function is given by

$$f(x)=\begin{cases} \dfrac{x^{p-1}\,.\,(1-x)^{q-1}}{\beta(p,q)}, & \text{if } 0<x<1;\ p>0,\ q>0, \\ 0, & \text{if } x\leqslant 0 \text{ and } x\geqslant 1. \end{cases} \quad \ldots(9\cdot57)$$

The p.d.f. of beta distribution for $p=q=2$ is represented graphically in figure 9·16. This type of distribution often occurs when the random variable x is a proportion.

Properties of beta distribution are

$$\text{mean}=\frac{p}{p+q},$$

$$\text{variance}=\frac{p\,(p+1)}{(p+q)\,(p+q+1)},$$

standard deviation $=\sqrt{\dfrac{p(p+1)}{(p+q)(p+q+1)}}$.

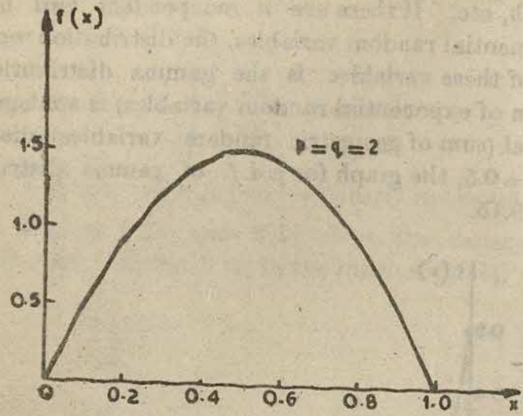


Fig. 9·16

9·21. Generating Functions

The method of *generating functions* is quite helpful in dealing with random variables which take only integral values 0, 1, 2.... It is a powerful method for solving the steady state system of difference equations obtained during analysis of infinite queue systems.

Let x be an integral random variable and let

$$p_n = P(x=n),\ n=0,\ 1,\ 2,\ \ldots \text{with} \sum_{n=0}^{\infty} p_n = 1;$$

then the function G (z) defined by

$$G(z) = \sum_{n=0}^{\infty} p_n \cdot z^n, \qquad \ldots(9\cdot58)$$

is called the *generating function* of the random variable.

EXAMPLE 9·21-1

Find the generating functions of the following sequences :

(*a*) 1, 1, 1. ...,

(*b*) 1, 2, 3, ...,

(*c*) 0, 1, 2, ...

Solution

$$(a) \quad G(z) = \sum_{n=0}^{\infty} p_n z^n = p_0 + p_1 z + p_2 z^2 + \ldots + p_n z^n + \ldots$$

$$=1+z+z^2+\ldots$$
$$=\frac{1}{1-z}.$$

(b) $$G(z)=p_0+p_1z+p_2z^2+\ldots$$
$$=1+2z+3z^2+\ldots$$
$$=\frac{1}{(1-z)^2}.$$

(c) $$G(z)=p_0+p_1z+p_2z^2+\ldots$$
$$=0+z+2z^2+3z^3+\ldots$$
$$=z(1+2z+3z^2+\ldots)$$
$$=\frac{z}{(1-z)^2}.$$

EXAMPLE 9·21·2

Find the generating function for

(a) a binomial distribution,

(b) a Poisson distribution.

Solution

(a) For a binomial distribution,

$$p_k = {}^nC_k p^k (1-p)^{n-k},\ k=0, 1, \ldots, n.$$

$$\therefore \quad G(z) = \sum_{k=0}^{n} {}^nC_k \cdot (pz)^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} {}^nC_k (pz)^k \cdot q^{n-k}$$

$$=(pz+q)^n.$$

(b) For a Poisson distribution,

$$p_k=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0, 1, 2, \ldots .$$

$$\therefore \quad G(z)=\sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot (\lambda z)^k}{k!}=e^{-\lambda} \cdot e^{\lambda z}=e^{-\lambda (1-z)}.$$

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EXERCISES

Section 9.3

1. What is the sample space for the experiment which consists of drawing one ball from an urn containing 8 balls of which 3 are green and 5 are red ? The balls have been numbered 1 through 8.

(*Ans.* $S=\{1, 2, \ldots, 8\}$)

2. For problem 1 define the events as subsets :
A : a green ball is drawn,
B : a red ball is drawn.

(*Ans.* $A=\{1, 2, 3\}$;
$B=\{4, 5, 6, 7, 8\}$)

3. What is the sample space for the experiment which consists of drawing 2 balls with replacement from an urn containing 8 balls ? The balls are numbered 1 through 8.

(*Ans.* $S=\{(x_1, x_2) : x_i=1, 2, \ldots, 8 ;$
$i=1, 2.\}$)

4. For exercise no. 3 define the events as subsets :
A : the first ball is green,
B : the second ball is green,
C : both balls are green.

(*Ans.* $A=\{(x_1, x_2) : x_1=1, 2, 3 ;$
$x_2=1, 2, \ldots, 8\}$,
$B=\{(x_1, x_2) : x_1=1, 2, \ldots, 8 ;$
$x_2=1, 2, 3\}$,
$C=\{(x_1, x_2) : x_i=1, 2, 3 ;$
$i=1, 2\}$)

Section 9.4

5. What is the probability of obtaining 9, 10 and 11 points with 3 dice ?

$$\left(Ans.\ \frac{25}{216}, \frac{27}{216}, \frac{27}{216}\right)$$

6. What is the probability of getting 2 tails and 2 heads when 4 coins are tossed ?

$$\left(Ans.\ \frac{3}{8}\right)$$

Section 9.5-9.10

7. There are 26 persons in a birthday party. What is the probability that at least two of them have the same birthday ?

(*Ans.* 0.60)

8. A coin is so weighted that head is thrice as likely to appear as tail. What is P(H) and P(T) ?

$$\left(Ans.\ \frac{3}{4}, \frac{1}{4}\right)$$

9. An urn contains 3 green and 5 red balls. One ball is drawn, its colour unnoted and laid aside. Then another ball is drawn, find the probability that it is green or red. How does the probability change if the colour of the ball is noted ?

$$\left(Ans.\ \frac{3}{8}, \frac{5}{8}; \frac{2}{7}\right)$$

10. If the probability that A will solve a problem is $\frac{1}{4}$ and the probability that B will solve it is $\frac{3}{4}$, what is the probability that the problem is at all solved ?

$$\left(Ans.\ \frac{13}{16}\right)$$

11. A die is so weighted that all even numbers have the same chance of appearing, all odd numbers have the same chance of appearing, while an even number is twice as likely to appear as any odd number. Find the probability that

(*i*) a prime number appears,

(*ii*) an even number appears,

(*iii*) an odd number appears,

(*iv*) an odd prime number appears.

$$\left(Ans.\ \frac{4}{9}, \frac{2}{3}, \frac{1}{3}, \frac{2}{9}\right)$$

12. A pair of fair dice is rolled once. What is the probability that the sum is equal to each of the integers from 2 to 12 ?

$$\left(\text{Ans. } \begin{array}{cccccccccccc} s & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ p & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} \end{array}\right)$$

13. A die is so loaded that the probability of a particular number appearing is proportional to the number. What is the probability of all single element events ? What is the probability of occurrence of an even number and of a number greater than 4 ?

$$\left(\text{Ans. } \frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21}; \frac{12}{21}; \frac{11}{21}\right)$$

14. Three players A, B and C play a sequence of games. It is so decided that winner of each game scores one point and he who first scores three points is the final winner. A wins the first and third games while B wins the second. What is the probability that C is the final winner ?

$$\left(\text{Ans. } \frac{2}{27}\right)$$

15. An urn contains 1 white and 2 black balls, while another contains 2 white and 1 black ball. One ball is transferred from the first urn into the second, after which a ball is drawn from the second urn. What is the probability that it is black ?

$$\left(\text{Ans. } \frac{5}{12}\right)$$

16. An urn contains a white and b black balls. Balls are drawn one by one until only those of the same colour are left. What is the probability that they are white ?

$$\left(\text{Ans. } \frac{a}{a+b}\right)$$

17. Eight white and 2 black balls are randomly laid out in a row. What is the probability that the two black balls are side by side ? What is the probability that they occupy the end positions ?

$$\left(\text{Ans. } \frac{1}{5}, \frac{1}{45}\right)$$

18. There are 5 boys and 10 girls in a class. Three students are selected at random from the class, one after the other. Find the probability that

(*i*) the first two are girls and the third is a boy,

(*ii*) the first and third are girls and the second is a boy,

(*iii*) the first and third are of the same sex and the second is of opposite sex.

$$\left(Ans.\ (i)\ \frac{10}{15}\cdot\frac{9}{14}\cdot\frac{5}{13}=\frac{15}{91},\quad (ii)\ \frac{10}{15}\cdot\frac{5}{14}\cdot\frac{9}{13}=\frac{15}{91},\quad (iii)\ \frac{15}{91}+\frac{20}{27}=\frac{5}{21}\right)$$

19. An urn contains 10 white and 3 black balls, while another contains 3 white and 5 black balls. Two balls are transferred from the first urn to the second and then one ball is drawn from the latter. What is the probability that it is white?

$$\left(Ans.\ \frac{59}{130}\right)$$

20. A coin is tossed until a head appears, or until it has been tossed 3 times. If the head does not appear on the first toss, find the probability that the coin is tossed 3 times.

$$\left(Ans.\ \frac{1}{2}\right)$$

21. A coin is tossed until a head appears, or until it has been tossed 4 times. If the head does not appear on either of the first two tosses, find the probability that

(*i*) the coin was tossed 4 times,

(*ii*) it was tossed just 3 times.

22. In a particular region it is found that the sex ratio among children is 3 girls to 2 boys. If a family of 5 children is selected at random, calculate the probability that

(*i*) the eldest is a boy and the rest are all girls,

(*ii*) the first, third and fifth children are boys and the remaining girls,

(*iii*) at least two of the children are girls,

(*iv*) the first three children are of one sex and the rest of the other sex.

$$\left(Ans.\ (i)\ \frac{162}{3{,}125}\quad (ii)\ \frac{72}{3{,}125}\quad (iii)\ \frac{2853}{3{,}125}\quad (iv)\ \frac{36}{3{,}125}\right)$$

23. An urn contains 2 white and 5 black balls. A ball is selected at random. If the ball drawn is black, it is replaced and two additional black balls are added to the urn ; if the ball drawn is white, it is neither replaced nor additional balls are added. A ball is then drawn from the urn for the second time. What is the probability that it is black ?

$$\left(\textit{Ans.}\ \frac{50}{63}\right)$$

24. An urn contains a fair coin and a two-headed coin. A coin is selected at random and tossed. If head appears, the other coin is tossed ; if tail appears, the same coin is tossed.

(*i*) Find the probability that head appears on the second toss.

(*ii*) If head appears on the second toss, find the probability that it also appeared on the first toss.

$$\left(\textit{Ans.}\ \frac{5}{8},\ \frac{4}{5}\right)$$

Section 9.14

25. Find the value of c so that the following $f(x)$ is a p.d.f.

$$f(x)=\begin{cases}\dfrac{c}{x^2}, & 10 \leqslant x \leqslant 20\ ; \\ 0, & \text{otherwise.}\end{cases}$$

($Ans.$ $c=20$)

26. The following p.d.f. of the discrete random variable x represents the weekly demand of a certain item :

x	0	1	2	3
$P(x)$	0.15	0.25	0.35	0.45

If the weekly demands are independent and identical, find the p.d.f. for a two-week demand.

10

Queuing Models

The queuing theory or waiting line theory owes its development to A.K. Erlang. He, in 1903, took up the problem on congestion of telephone traffic. The difficulty was that during busy periods, telephone operators were unable to handle the calls the moment they were made, resulting in delayed calls. A.K. Erlang directed his first efforts at finding the delay for one operator and later on the results were extended to find the delay for several operators. The field of telephone traffic was further developed by Molins (1927) and Thornton D-Fry (1928). However, it was only after World War II that this early work was extended to other general problems involving queues or waiting lines.

Waiting lines or queues are omnipresent. Businesses of all types, industries, schools, hospitals, cafeterias, book stores, libraries, banks, post offices, petrol pumps, theatres—all have queuing problems. Further examples of queues, though less apparent are : waiting for a telephone operator to answer, a traffic light to change, the morning mail to be delivered and the like.

Waiting line problems arise either because

(*i*) there is too much demand on the facilities so that we say that there is an excess of waiting time or inadequate number of service facilities.

(*ii*) there is too less demand, in which case there is too much idle facility time or too many facilities.

In either case, the problem is to either *schedule arrivals* or *provide facilities or both* so as to obtain an optimum balance between the costs associated with waiting time and idle time.

Operations research can quite effectively analyse such queuing or congestion phenomena. However, a sound understanding of queuing theory combined with imagination is required to apply the theory to practical situations.

10·1 Applications of Queuing Models

Waiting line or queuing theory has been applied to a wide variety of business situations. All situations where customers are involved such as restaurants, cafeterias, departmental stores, cinema halls, banks, post-offices, petrol pumps, airline counters, patients in clinics, etc., are likely to have waiting lines. Generally, the customer expects a certain level of services, whereas the firm providing service facility tries to keep the costs minimum while providing the required service.

Waiting line theory is also widely used by manufacturing units. It has been popularly used in the area of tool cribs. There is a general complaint from the foremen that their workmen wait too long in line for tools and parts. Though the management wants to reduce the overhead charges, engaging more attendants can actually reduce overall manufaturing costs, since the workers will be working instead of standing in line.

Another problem that has been successfully solved by waiting line theory is the determination of the proper number of docks to be constructed for trucks or ships. Since both dock costs and demurrage costs can be very large, the number of docks should be such that the sum of the two costs is minimized.

Queuing methods have also been used for the problem of machine breakdowns and repairs. There are a number of machines that breakdown individually and at random times. The machines that breakdown form a waiting line for repairs by maintenance personnel and it is required to find the optimum number of repair personnel which makes the sum of the cost of repairmen and the cost of production loss from down time, a minimum.

Queuing theory has been extended to decide wage incentive plans. For example, some workers are asked to operate, say, two machines while the others, four machines. Since the machines are identical, the base rate of payment is same for all workers. However, the incentive bonus for production in excess of quota is half as much per unit for operators with four machines as for those with two machines. Apparently, the arrangement appears to be fair. However, a study of downtime for repairs shows that while the two machines run by one man would have 12 per cent downtime, four machines run by

one man would have 16% downtime. The reason is that two (or more) machines can breakdown at once in the four-machine group which is generally not true for two-machine group. Thus the worker operating four machines would have to operate at a higher efficiency than his counterpart in order to earn the same incentive. The problem was solved by paying the operators of the four-machine group a higher base rate determined by using the probabilities computed from queuing theory. The following examples will further illustrate the fields of application of queuing theory :

EXAMPLE 10·1-1

A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate, find

1. Average number of customers in the system.
2. Average number of customers in queue or average queue length.
3. Average time a customer spends in the system.
4. Average time a customer waits before being served.

EXAMPLE 10.1-2

A person repairing radios finds that the time spent on the radio sets has an exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately Poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?

EXAMPLE 10·1-3

A branch of Punjab National Bank has only one typist. Since the typing work varies in length (number of pages to be typed) and number of copies required, the typing rate is randomly distributed approximating a Poisson distribution with mean service rate of 8 letters per hour. The letters arrive at a rate of 5 per hour during the entire 8-hour work day. If the time of the typist is valued at Rs. 1.50 per hour, determine

1. Equipment utilization.
2. The present time that an arriving letter has to wait.
3. Average system time.
4. Average cost due to waiting and operating the typewriter.

EXAMPLE 10·1-4

The milk plant at a city distributes its products by trucks, loaded at the loading dock. It has its own fleet of trucks plus trucks of a private transport company. This transport company has complained that sometime its trucks have to wait in line and thus

the company loses money paid for a truck and driver that is only waiting. The company has asked the milk plant management either to go in for a second loading dock or discount prices equivalent to the waiting time. The following data is available,

Average arrival rate (all trucks) = 3 per hour,

average service rate = 4 per hour.

The transport company has provided 40% of the total number of trucks. Assuming that these rates are random according to Poisson distribution, determine

1. The probability that a truck has to wait.
2. The waiting time of a truck that waits.
3. The expected waiting time of company trucks per day.

EXAMPLE 10·1·5

Arrival rate of telephone calls at a telephone booth are according to Poisson distribution, with an average time of 9 minutes between two consecutive arrivals. The length of telephone call is assumed to be exponentially distributed, with mean 3 minutes.

(*a*) determine the probability that a person arriving at the booth will have to wait.

(*b*) find the average queue length at any time.

(*c*) the telephone company will install a second booth when convinced that an arrival would expect to have to wait at least four minutes for the phone. Find the increase in flow of arrivals which will justify a second booth.

(*d*) what is the probability that an arrival will have to wait for more than 10 minutes before the phone is free ?

(*e*) what is the probability that he will have to wait for more than 10 minutes before the phone is available and the call is also complete ?

(*f*) find the fraction of a day that the phone will be in use.

EXAMPEE 10·1·6

A mechanic repairs four machines. The mean time between service requirements is 5 hours for each machine and forms an exponential distribution. The mean repair time is 1 hour and also follows the same distribution pattern. Machine downtime costs Rs. 25 per hour and the mechanic costs Rs. 55 per day.

(*a*) find the expected number of operating machines.

(*b*) determine the expected downtime cost per day.

(*c*) would it be economical to engage two mechanics, each repairing only two machines ?

10·2. Introduction

Waiting lines or queues are familiar phenomena, which we observe quite frequently in our daily life. The basic characteristics of a queuing phenomenon are

1. Units arrive, at regular or irregular intervals of time, at a given point called the service centre. For example, trucks arriving a loading station, customers entering a department store, persons arriving a cinema hall, ships arriving a port, letters arriving a typist's desk, etc. All these units are called *entries* or *arrivals* of *customers.*
2. One or more *service channels* or *service stations* or *service facilities* (ticket windows, salesgirls, typists, docks, etc.) are assembled at the service centre. If the service station is empty (free), the arriving customer (s) will be served immediately, if not, the arriving customer (s) will wait in line until the service is provided. Once service has been completed, the customer leaves the system. Whenever we have customers coming to a service facility in such a way that either the customers or the facilities have to wait, we have a queuing problem.

   Figure 10·1 shows the major elements of a queuing system (or delay phenomenon). They are

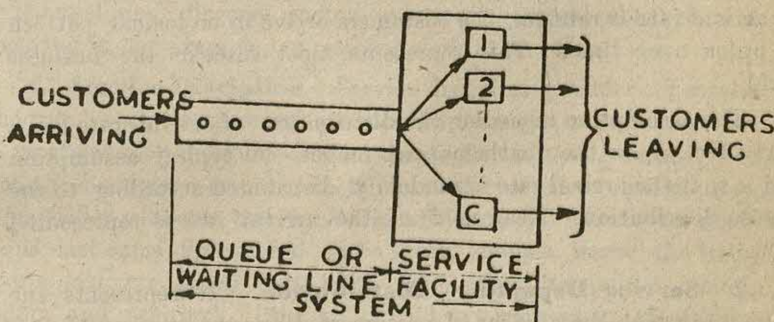

Fig. 10.1.

1. Customer. The arriving unit that requires some service to be performed. As already described, the customer may be persons, machines, vehicles, parts, etc.

2. Queue (Waiting line). The number of customers waiting to be serviced. The queue does not include the customer being serviced.

3. Service Channel. The process or system which is performing the services to the customer. This may be single or multichannel. The number of service channels is denoted by the symbol c.

10·3. Characteristics of Queuing Models

A queuing model is specified completely by six main characteristics :

1. Input or arrival (inter-arrival) distribution
2. Output or departure (service) distribution
3. Service channels
4. Service discipline
5. Maximum number of customers allowed in the system
6. Calling source or population.

1. Arrival Distribution. It represents the *pattern* in which the number of customers arrive at the system. Arrivals may also be represented by the *inter-arrival time*, which is the time period between two successive arrivals.

Arrivals may be separated by equal intervals of time or by unequal but definitely known intervals of time or by unequal intervals of time whose probabilities are known; these are called random arrivals.

The rate at which customers arrive to be serviced, *i.e.*, number of customers arriving per unit of time is called *arrival rate.* When the arrival rate is random, the customers arrive in no logical pattern or order over time. This represents most cases in the business world.

The assumption regarding the distribution of arrival rate has a great effect upon the mathematical model. A typical assumption used is that the arrival rate is randomly distributed according to the Poisson distribution. Mean value of the arrival rate is represented by λ.

2. Service (Departure) Distribution. It represents the *pattern* in which the number of customers leave the system. Departures may also be represented by the *service* (inter-departure) time, which is the time period between two successive services.

Service time may be constant or variable but known, or random (variable with only known probability).

The rate at which one service channel can perform the service, *i.e.*, number of customers served per unit of time is called *service* rate. This rate assumes the service channel to be always busy, *i.e.*, no iple time is allowed.

The assumption regarding the distribution of service rate is equally important in the formulation of queuing model. A typical assumption used is that the service rate is randomly distributed according to exponential distribution. Mean value of service rate is represented by μ. In business problems more cases of uniform service rate will be found than of uniform arrival rates.

3. Service Channels. The queuing system may have a single service channel. Arriving customers may form one line and get serviced, as in a doctor's clinic. The system may have a number of service channels, which may be arranged in parallel or in series or a complex combination of both. In case of parallel channels, several customers may be serviced simultaneously, as in a barber shop. For series channels, a customer must pass successively through all the channels before service is completed, *e.g.*, a product undergoing different processes over different machines. A queuing model is called *one server model*, when the system has one server only and a *multi-server model* when the system has a number of parallel channels each with one server.

Sometimes several service channels may feed into one subsequent service channel; for example, several ticket booths in a theatre may send all the ticket holders to a single ticket collector at the entrance of the theatre. On the other hand, sometimes, a single service channel may disperse customers among several channels that come after it; for example, an enquiry clerk in an office.

4. Service Discipline. Service discipline or order of service is the rule by which customers are selected from the queue for service. The most common discipline is 'first come, first served', according to which the customers are served in the order of their arrival, e.g., cinema ticket windows, railway stations, banks, etc. The other discipline is 'last come, first served', as in a big godown, where the items arriving last are taken out first. Still other disciplines include 'random' and 'priority'. Priority' is said to occur when an arriving customer is chosen for service ahead of some other customers already in the queue. A unit (customer) is said to have 'pre-emptive' priority if it not merely goes to the head of the queue but displaces any unit already being served when it arrives. Provided that the order of service is not related to service time, it does not affect the queue length or average waiting time but it does affect the time an individual customer has to wait. The service discipline, therefore, affects the derivation of equations used for analysis. In this text only the most common service discipline 'first come, first served' will be assumed for further discussion.

5. Maximum Number of Customers allowed in the System. Maximum number of customers in the system can be either finite or infinite. In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system unless the number becomes less than the limiting value.

6. Calling Source or Population. The arrival pattern of the customers depends upon the source which generates them. If there are only a few potential customers, the calling source (population) is called finite. If there are a large number of potential customers (say, over 40 or 50), it is usually said to be infinite. There is still another rule for categorising the source as finite or infinite. A finite source exists when an arrival affects the probability of arrival of potential future customers. For example, a battery of M running machines is a finite source, as far as machine repair situation is concerned. Before any machine is broken, the calling source consists of M potential customers. As soon as a machine is broken, it becomes a customer and hence cannot generate another 'call' until it gets serviced (repaired). An infinite source is said to exist when the arrival of a customer does not affect the rate of arrival of potential future customers.

10·4. Waiting Time and Idle Time Costs

In order to solve a queuing problem, service facility must be manipulated so that an optimum balance is obtained between the cost of waiting time and the cost of idle time.

The cost of waiting customers generally includes either the indirect cost of lost business (because people go somewhere else, buy less than they had intended to, or do not come again in future) or direct cost of idle equipment and persons; for example, cost of truck drivers and equipment waiting to be unloaded or cost of operating an airplane or ship waiting to land or dock. The cost of lost business is not easy to assess. For example, vehicle drivers wanting petrol will avoid pumps having long queues. To determine how much business is lost, some type of experimentation and data collection is required.

The cost of idle service facilities is the payment to be made to the servers (engaged at the facilities), for the period for which they remain idle.

By increasing the investment in labour and equipment (service facilities), waiting time and the losses associated with it can be decreased. It is desirable, then, to obtain the minimum sum of these two costs; costs of investment and operation, and costs due to waiting.

This optimum balance of costs can be obtained by *scheduling* the flow of units requiring service and/or providing proper number of facilities. If the facilities are not under control, flow of units may be scheduled to minimize the sum of waiting time and idle time cost. If the flow is not subject to control, that amount of equipment and personnel be employed which minimizes the overall costs of operation. If both can be controlled, one should schedule the input as well as provide facilities which minimize the overall cost.

10·5. Transient and Steady States of the System

Queuing theory analysis involves the study of system's behaviour over time. If the operating characteristics (behaviour of the system vary with time, it is said to be in *transient state*. Usually a system is transient during the early stages of its operation, when its behaviour still depends upon the initial conditions. However, it is the "long-run" behaviour or the *steady state condition* of the system which is more important. A system is said to be in steady state condition if its behaviour becomes independent of time.

An essential condition for reaching a steady state is that the total elapsed time since the start of the operation must be sufficiently large (theoretically, it should tend to infinity). However, this is not the sufficient condition as the parameters of the system also affect its state.

For example, if the average arrival rate is less than average service rate and both are constant, the system eventually settles down to a steady state and the probability of finding a particular length of queue will be same at any time. If the rates are not constant, the system will not reach a steady state, but it could remain stable. If the arrival rate is greater than service rate, the system cannot attain a steady state (regardless of the length of elapsed time); it is rather unstable, queue length increases steadily with time and theoretically, it could build up to infinity. Such state of the system is called *explosive state*. Evidently, imposing a limit on the maximum length of the queue (so that further arrivals are not accepted) automatically ensures stability. Queuing situations which are unstable for a limited time are common in practice—rush-hour traffic is an example. In this text we shall consider the steady state analysis, transient and explosive states require complex mathematical tools for analysis and will not be touched upon.

10·6. Single-Channel Queuing Theory

A single channel queuing problem results from random arrival

time and random service time at a single service station. The random arrival time can be described mathematically with a probability distribution. The most common distribution found in queuing problems is Poisson distribution. This is used in single channel queuing problems for random arrivals where the service time is exponentially distributed. The sections ahead give the reader an insight into the true nature of operations research--the difficulties of developing OR models, the need for logical assumptions and the utilization of higher mathematics.

10·6·1. Models for Arrival and Service Times

Generally, arrivals do not occur at fixed regular intervals of times but tend to be clustered or scattered in some fashion. A *Poisson distribution* is a discrete probability distribution which predicts the number of arrivals in a given time. The Poisson distribution involves the probability of occurrence of an arrival. Poisson assumption is quite restrictive in some cases. It assumes that arrivals are random and independent of all other operating conditions. The mean arrival rate (*i.e.*, the number of arrivals per unit of time) λ is assumed to be constant over time and is independent of the number of units already serviced, queue length or any other random property of the queue.

Since the mean arrival rate is constant over time, it follows that the probability of an arrival between time t and $t+dt$ is $\lambda.dt$.

Thus probability of an arrival in time $dt=\lambda.dt$. ...(10·1)

The following characteristics of Poisson distribution are written here without proof :

Probability of n arrivals in time $t=\dfrac{(\lambda t)^n.e^{-\lambda t}}{n\,!}$. ...(10·2)

Probability density function of inter-arrival time (time interval between two consecutive arrivals)

$$=\lambda.e^{-\lambda t}. \qquad ...(10·3)$$

Finally, Poisson distribution assumes that the time period dt is very small so that $(dt)^2$, $(dt)^3$, etc.$\to 0$ and can be ignored.

Service time is the time required for completion of a service *i.e.*, it is the time interval between beginning of a service and its completion. The mean service rate is the number of customers served per unit of time (assuming the service to be continuous through the entire time unit), while the average service time $\dfrac{1}{\mu}$ is the time

required to serve one customer. The most common type of distribution used for service times is exponential distribution. It involves the probability of completion of a service. It should be noted that Poisson distribution cannot be applied to servicing because of the possibility of the service facility remaining idle for some time. Poisson distribution assumes fixed time interval of continuous servicing, which can never be assured in all services.

Mean service rate μ is also assumed to be constant over time and independent of number of unit already serviced, queue length or any other random property of the system. Thus probability that a service is completed between t and $t+dt$, provided that the service is continuous $=\mu dt$. ...(10·4)

Under the condition of continuous service, the following characteristics of exponential distribution are written, without proof :

Probability of n complete services in time $t=\frac{(\mu t)^n . e^{-\mu t}}{n\,!}$. ...(10·5)

Probability density function (*p.d.f.*) of interservice time, *i.e.*, time between two consecutive services $=\mu . e^{-\mu t}$. ...(10·6)

10·6·2. Single-Channel Poisson Arrivals with Exponential Service (Infinite-Population Model)

Let us consider a single-channel system with Poisson arrivals and exponential service rates. Both the arrivals and service rates are independent of the number of customers in the waiting line. Arrivals are handled on 'first-come, first served' basis. Also the arrival rate λ is lesser than the service rate μ.

The following mathematical notation (symbols) will be used in connection with queuing models :

n = number of customers in the system (waiting line + service facility) at time t.

λ = mean arrival rate (number of arrivals per unit of time).

μ = mean service rate per busy server (number of customers served per unit of time).

λdt = probability that an arrival enters the system between t and $t+dt$ time interval *i.e.* within time interval dt.

$1-\lambda dt$ = probability that no arrival enters the system within interval dt plus higher order terms in dt.

μ = mean service rate per channel.

μdt = probability of one service completion between t and $t+dt$ time interval *i.e.*, within time interval dt.

$1-\mu.dt$=probability of no service rendered during the interval dt plus higher order terms in dt.

p_n=steady state probability of exactly n customers in the system.

$p_n(t)$=transient state probability of exactly n customers in the system at time t, assuming the system started its operation at time zero.

$p_{n+1}(t)$=transient state probability of having $n+1$ customers in the system at time t.

$p_{n-1}(t)$=transient state probability of having $n-1$ customers in the system at time t.

$p_n(t+dt)$=probability of having n units in the system at time $t+dt$.

L_q=expected (average) number of customers in the queue.

L_s=expected number of customers in the system (waiting+being served).

W_q=expected waiting time per customer in the queue (expected time a customer spends waiting in line).

W_s=expected waiting time per customer in the system.

L_n=expected number of customers waiting in line *excluding* those times when the line is empty *i.e.*, expected number in non-empty queue.

W_n=expected time a customer waits in line if he has to wait at all *i.e.*, expected time in the queue for non-empty queue.

To determine the properties of the single channel system, it is necessary to find an expression for the probability of n customers in the system at time t *i.e.*, $p_n(t)$, for, if $p_n(t)$ is known, the expected number of customers in the system can be calculated. In place of finding an expression for $p_n(t)$, we shall first find the expression for $p_n(t+dt)$.

The probability of n units (customers) in the system at time $t+dt$ can be determined by summing up probabilities of all the ways this event could occur. The event can occur in four mutually exclusive and exhaustive ways.

Table 10·1

Event	*No. of units at time t*	*No. of arrivals in time dt*	*No. of services in time dt*	*No. of units at time t+dt*
1	n	0	0	n
2	$n+1$	0	1	n
3	$n-1$	1	0	n
4	n	1	1	n

Now we compute the probability of occurrence of each of the event, remembering that the probability of a service or arrival is μdt or λdt and $(dt)^2 \to 0$.

$\therefore$ Probability of event 1 = Probability of having n units at time t
× Probability of no arrivals
× Probability of no services

$$= p_n(t).(1-\lambda dt)\,(1-\mu dt)$$
$$= p_n(t)\,[1-\lambda dt-\mu dt+\lambda\mu(dt)^2]$$
$$= p_n(t)\,[1-\lambda dt-\mu dt].$$

Similarly, probability of event $2 = p_{n+1}(t).\,(1-\lambda dt).\,(\mu dt)$

$$= p_{n+1}(t)\,[\mu dt],$$

probability of event $3 = p_{n-1}(t)[\lambda dt].\,(1-\mu dt)$

$$= p_{n-1}(t)[\lambda dt],$$

probability of event $4 = p_n(t).\,(\lambda dt)\,(\mu.dt)$

$$= p_n(t).\,[\lambda.\mu(dt)^2]$$
$$= 0.$$

Note that other events are not possible because of the small value of dt that causes $(dt)^2$ to approach zero (as in event 4).

Since one and only one of the above events can happen, we can obtain $p_n\,(t+dt)$ {where $n>0$} by adding the probabilities of above four events.

$$\therefore\ p_n(t+dt)=p_n(t)\,[1-\lambda dt-\mu dt]+p_{n+1}(t)[\mu dt]+p_{n-1}(t)[\lambda dt]+0]$$

or $$p_n(t+dt)=p_n(t)-p_n(t)[\lambda dt+\mu dt]+p_{n+1}(t)[\mu dt]+p_{n-1}(t)[\lambda dt]$$

or $$\frac{p_n(t+dt)-p_n(t)}{dt}=-(\lambda+\mu).p_n(t)+\mu.p_{n+1}(t).+\lambda p_{n-1}(t).$$

Taking the limit when $dt \to 0$, we get the following differential equation which gives the relationship between p_n, p_{n-1}, p_{n+1} at any time t, mean arrival rate λ and mean service rate μ :

$$\frac{d}{dt}[p_n(t)] = \lambda p_{n-1}(t) + \mu p_{n+1}(t) - (\lambda+\mu)p_n(t), \text{ where } n>0 \quad \ldots(10\cdot7)$$

After solving for $p_n(t+dt)$ where $n>0$, it is necessary to solve for $p_n\ (t+dt)$ where $n=0$ *i.e.*, to solve for $p_0(t+dt)$. If $n=0$, only two mutually exclusive and exhaustive events can occur as shown in table 10·2.

Table 10·2

Event	*No. of units at time t*	*No. of arrivals in time dt*	*No. of services in time dt*	*No. of units at time t+dt*
1	0	0	—	0
2	1	0	1	0

$\therefore$ Probability of event 1 = Probability of having no unit at time t × Probability of no arrivals × Probability of no services

$= p_0(t) \times (1-\lambda dt) \times 1,$

and probability of event 2 = Probability of having one unit at time t × Probability of no arrivals × Probability of one service

$= p_1(t) \times (1-\lambda dt) \times (\mu dt).$

Note that if no units were in the system, the probability of no service would be 1. Probability of having no unit in the line at time $t+dt$ is given by summing up the probabilities of above two events.

$$\therefore \quad p_0(t+dt) = p_0(t).\ (1-\lambda dt) + p_1(t).\ (\mu dt)\ (1-\lambda dt)$$
$$= p_0(t) - p_0(t).(\lambda dt) + p_1(t).\ (\mu dt)$$

or $$p_0(t+dt) - p_0(t) = -p_0(t).\ (\lambda dt) + p_1(t)\ (\mu dt)$$

or $$\frac{p_0(t+dt)-p_0(t)}{dt} = \mu . p_1(t) - \lambda p_0(t).$$

When $dt \to 0$, the differential equation which indicates the relationship between probabilities p_0 and p_1 at any time t, mean arrival rate λ and mean service rate μ, is

$$\frac{d}{dt}[p_0(t)] = \mu p_1(t) - \lambda p_0(t), \text{ where } n=0. \quad \ldots(10\cdot8)$$

Equations (10·7) and (10·8) provide relationships involving the *probability density function* $p_n(t)$ for all values of n but still we do not know the value of $p_n(t)$.

Assuming the steady condition for the system, when the probability of having n units (customers) in the system becomes independent of time, we get

$$p_n(t)=p_n,$$

$$\frac{d}{dt}[p_n(t)]=0.$$

Therefore, for a steady state system the differential equations (10·7) and (10·8) reduce to difference equations (10·9) and (10·10) :

$$0=\lambda p_{n-1}+\mu.p_{n+1}-(\lambda+\mu)p_n, \text{ where } n>0, \qquad \ldots(10\cdot9)$$

$$0=\mu p_1-\lambda p_0, \text{ where } n=0. \qquad \ldots(10\cdot10)$$

From equation (10·10), we have

$$p_1=\frac{\lambda}{\mu}p_0.$$

Putting $n=1$ in equation (10·9), we have

$$0=\lambda p_0+\mu p_2-(\lambda+\mu)p_1.$$

$$\therefore \quad p_2=\frac{\lambda+\mu}{\mu}p_1-\frac{\lambda}{\mu}p_0$$

$$=\frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}p_0\right)-\frac{\lambda}{\mu}p_0$$

$$=\frac{\lambda}{\mu}p_0\left[\frac{\lambda+\mu}{\mu}-1\right]$$

or $$p_2=\left(\frac{\lambda}{\mu}\right)^2 \cdot p_0.$$

Similarly, for $n=2$, equation (10·9) gives

$$p_3=\left(\frac{\lambda}{\mu}\right)^3 \cdot p_0$$

$$\vdots \qquad \vdots$$

For $n=n$, we get $p_n=\left(\dfrac{\lambda}{\mu}\right)^n p_0$, where $n \geqslant 0$. ...(10·11)

Equation (10·11) gives p_n in terms of p_0, λ and μ. Finally, an expression for p_0 in terms of λ and μ must be obtained. The easiest way to do this is to recognize that the probability that the channel is busy is the ratio of the arrival rate and service rate $\left(\dfrac{\lambda}{\mu}\right)$. Thus p_0 is 1 minus this ratio.

i.e. $$p_0=1-\frac{\lambda}{\mu}.\qquad\qquad ...(10\cdot12)$$

Hence $$p_n=\left(\frac{\lambda}{\mu}\right)^n\cdot\left(1-\frac{\lambda}{\mu}\right)\cdot\qquad\qquad ...(10\cdot13)$$

Having known the value of p_n, we can find the various characteristics of the system.

1. Expected number of units in the system (waiting + being served), L_s is obtained by using the definition of an expected value.

$$E(x)=\sum_{i=0}^{i=\infty} x_i p_i$$

$$\therefore \quad L_s=\sum_{n=0}^{n=\infty} n p_n$$

or $$L_s=\sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$=\left(1-\frac{\lambda}{\mu}\right)\cdot\sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left[0\left(\frac{\lambda}{\mu}\right)^0+1\left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2+3\left(\frac{\lambda}{\mu}\right)^3+\dots\right]$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left[0+\frac{\lambda}{\mu}+2\left(\frac{\lambda}{\mu}\right)^2+3\left(\frac{\lambda}{\mu}\right)^3+\dots\right].$$

The seires within brackets is an infinite series of the form 0, a, $2a^2$, $3a^3$, ..., xa^x, ... For such an infinite series, if a is a constant and less than one, the sum is given by the formula

$$S_\infty=\frac{a}{(1-a)^2}$$

$$\therefore \quad L_s=\left(1-\frac{\lambda}{\mu}\right)\left[\frac{\lambda/\mu}{(1-\lambda/\mu)^2}\right]=\frac{\lambda/\mu}{1-\lambda/\mu}=\frac{\lambda}{\mu-\lambda}.\qquad ...(10\cdot14)$$

2. Expected number of units in the queue L_q = Expected number of units in the system − Expected number in service (single server).

$$\therefore \quad L_q=L_s-\frac{\lambda}{\mu}$$

$$=\frac{\lambda}{\mu-\lambda}-\frac{\lambda}{\mu}=\lambda\left[\frac{\mu-\mu+\lambda}{\mu(\mu-\lambda)}\right].$$

$$\therefore \quad L_q=\lambda^2/\mu(\mu-\lambda). \qquad \text{...(10·15)}$$

Note that the expected number in service is 1 times the probability that the service channel is busy *i.e.*, $1.\frac{\lambda}{\mu}$.

3. Expected waiting time per unit in the system (expected time a unit spends in the system)

$$W_s=\frac{\text{Expected number of units in the system}}{\text{Arrival rate}}$$

$$=\frac{L_s}{\lambda}$$

$$=\frac{\lambda}{(\mu-\lambda)\lambda}.$$

$$\therefore \quad W_s=\frac{1}{\mu-\lambda}. \qquad \text{...(10·16)}$$

4. Expected waiting time per unit in the queue W_q = Expected time in system—time in service.

$$W_q=W_s-\frac{1}{\mu}$$

$$=\frac{1}{\mu-\lambda}-\frac{1}{\mu}$$

$$=\frac{\mu-\mu+\lambda}{\mu(\mu-\lambda)}. \qquad \text{...(10·17)}$$

$$\therefore \quad W_q=\frac{\lambda}{\mu(\mu-\lambda)}.$$

5. Expected number of units in a non-empty queue,

$$L_n=\frac{\text{Expected number in queue}}{\text{probability that queue is not empty}}$$

Now, probability of non-empty queue

$$=1-p_0=1-\left(1-\frac{\lambda}{\mu}\right)=\frac{\lambda}{\mu}.$$

$$\therefore \quad L_n=\frac{\frac{\lambda^2}{\mu(\mu-\lambda)}}{\lambda/\mu}=\frac{\lambda}{\mu-\lambda}. \qquad \text{...(10·18)}$$

6. Expected waiting time W_n for a non-empty queue,

$$W_n=\frac{\text{Expected time in queue}}{\text{probability of waiting}}$$

$$=\frac{W_q}{\lambda/\mu}=\frac{\frac{\lambda}{\mu(\mu-\lambda)}}{\lambda/\mu}=\frac{1}{\mu-\lambda} \qquad ...(10\cdot19)$$

7. Probability density function of waiting time (excluding service) distribution :

Let $\psi(w)$ = probability density function of the waiting time distribution.

Then, $\psi(w).dw$ = probability a unit has to wait (on arrival) for a time between w and $w+dw$.

If n is the number of units in the system then

$\psi_n(w).\ dw$ = Probability [$(n-1)$ units are served at time w]
× Probability [one unit is served in time dw]

or $$\psi_n(w).\ dw=\frac{(\mu w)^{n-1}\ .\ e^{-\mu w}}{(n-1)\ !}\times\mu dw.$$

Let W = Waiting time of a unit that has to wait,
such that $w\leqslant W\leqslant w+dw$,

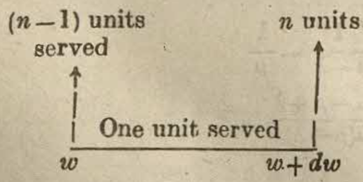

Fig. 10·2

Then, probability density function,

$$\psi(w)\ dw=\text{Prob.}\ (w\leqslant W\leqslant w+dw)$$

$$=\sum_{n=1}^{\infty}p_n\ \psi_n(w).\ dw$$

$$=\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\cdot\left(1-\frac{\lambda}{\mu}\right)\cdot\frac{(\mu w)^{n-1}\ .\ e^{-\mu w}}{(n-1)\ !}.\ \mu dw$$

$$=\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right).\ (\mu dw)\ .\ e^{-\mu w}\ \sum_{n=1}^{\infty}\frac{\left(\frac{\lambda}{\mu}\ .\ \mu w\right)^{n-1}}{(n-1)\ !}$$

$$=\lambda\left(1-\frac{\lambda}{\mu}\right).\,dw\,.\,e^{-\mu w}\,.\,\sum_{n=1}^{\infty}\frac{(\lambda w)^{n-1}}{(n-1)\,!}$$

$$=\lambda\left(1-\frac{\lambda}{\mu}\right).\,e^{-\mu w}\,.\,dw.\left[1+\frac{(\lambda w)}{1\,!}+\frac{(\lambda w)^2}{2\,!}+\ldots\right]$$

$$=\lambda\left(1-\frac{\lambda}{\mu}\right).\,e^{-\mu w}\,.\,dw\,.\,e^{\lambda w}.$$

$$\therefore\quad \psi(w).dw=\lambda\left(1-\frac{\lambda}{\mu}\right).\,e^{-(\mu-\lambda)}\,w\,.\,dw,\;w>0. \qquad \ldots(10\cdot 20)$$

$$\text{Obviously}\int_0^{\infty}\psi(w)\,dw=\lambda\;\left(\frac{\mu-\lambda}{\mu}\right).\int_0^{\infty}e^{-(\mu-\lambda)w}.\,dw$$

$$=\frac{\lambda}{\mu}\cdot(\mu-\lambda)\,.\left[\frac{1}{-(\mu-\lambda)}\,e^{-(\mu-\lambda)\,w}\right]_0^{\infty}$$

$$=\frac{\lambda}{\mu}\;(\mu-\lambda)\left[0+\frac{1}{\mu-\lambda}\right]$$

$$=\frac{\lambda}{\mu}. \qquad \ldots(10\cdot 21)$$

Note that the case for which W=0 has been excluded in equation (10·20).

Thus probability [W=0] = Probability [no unit in the system]

$$=p_0=1-\frac{\lambda}{\mu}. \qquad \ldots(10\cdot 22)$$

8. Probability density function of waiting time (waiting+service) an arrival spends in the system :

Probability density function of waiting +service time (for busy period)

$$=\frac{\psi(w)}{\int_0^{\infty}\psi(w).\,dw}$$

$$=\frac{\lambda\cdot\left(1-\frac{\lambda}{\mu}\right)\cdot e^{-(\mu-\lambda)\,w}}{\int_0^{\infty}\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}\,.\,dw}$$

$$=\frac{\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)\,w}}{\lambda/\mu}=(\mu-\lambda)\,.\,e^{-(\mu-\lambda)\,w}.$$

$$\ldots(10\cdot 23)$$

EXAMPLE 10.6.2.1.

Solve example 10.1-1.

Solution. Arrival rate $\lambda=\frac{9}{5}=1\cdot8$ customers/minute,

service rate $\mu=\frac{10}{5}=2$ customers/minute.

1. Average number of customers in the system

(Equation 10.14), $L_s=\frac{\lambda}{\mu-\lambda}=\frac{1\cdot8}{2-1\cdot8}=\mathbf{9}$.

2. Average number of customers in the queue

(Equation 10.15) $L_q=\frac{\lambda^2}{\mu(\mu-\lambda)}=\frac{\lambda}{\mu}\cdot\frac{\lambda}{(\mu-\lambda)}$

$$=\frac{1\cdot8}{2}\times\frac{1\cdot8}{2-1\cdot8}=\mathbf{8\cdot1}.$$

3. Average time a customer spends in the system,

$$W_s=\frac{1}{\mu-\lambda}=\frac{1}{2-1\cdot8}=\textbf{5 minutes}.$$

4. Average time a customer spends in the queue,

$$W_q=\frac{\lambda}{\mu}\left(\frac{1}{\mu-\lambda}\right)=\frac{1\cdot8}{2}\left(\frac{1}{2-1\cdot8}\right)=\textbf{4·5 minutes}.$$

EXAMPLE 10.6.2.2.

Solve example 10·1-2.

Solution. Arrival rate $\lambda=\frac{15}{8\times60}=\frac{1}{32}$ unit/minute,

service rate $\mu=\frac{1}{20}$ units/minute.

Number of jobs ahead of the set brought in = Average number of jobs in the system,

$$L_s=\frac{\lambda}{\mu-\lambda}=\frac{1/32}{1/10-1/32}$$

$$=\frac{1/32}{\frac{12}{32\times20}}=\frac{1}{32}\times\frac{32\times20}{12}=\mathbf{5/3}.$$

Number of hours for which the repairman remains busy in an 8-hour day

$$=8.\ \frac{\lambda}{\mu}=8\times\frac{1/32}{1/20}=8\times\frac{20}{32}=\textbf{5 hours}$$

$\therefore$ Time for which repairman remains idle in an 8-hour day

$$=8-5=\textbf{3 hours}.$$

EXAMPLE 10.6-2.3

Solve example 10·1.3.

Solution. Arrival rate = 5 per hour,

service rate = 8 per hour.

1. Equipment utilization, $\rho = \frac{\lambda}{\mu} = 5/8 = 0{\cdot}625$.

$\therefore$ Equipment is in use **62·5%** of the time.

2. The per cent time an arriving letter has to wait = per cent time the equipment remains busy = **62·5%** of time.

3. Average system time, $W_s = \frac{1}{\mu-\lambda} = \frac{1}{8-5} = \frac{1}{3}$ hr.

$= 20$ minutes.

4. Average cost per day = Number of letters typed per day × average cost per letter

$= (8 \times 5) \times$ average cost per letter

$= (8 \times 5) \times$ average time per letter × Rs./hour

$= (8 \times 5) \times 1/3 \times 1{\cdot}5$

$=$ **Rs. 20**.

EXAMPLE 10.6-2.4

Solve example 10.1-4.

Solution. 1. The probability that a truck has to wait for service = utilization factor $= \rho = \frac{\lambda}{\mu} = 3/4 = \mathbf{0{\cdot}75}$.

2. The waiting time of truck that waits

$$= W_n = \frac{1}{\mu-\lambda} = \frac{1}{4-3} = \frac{1}{1} = \mathbf{1\ hour}.$$

Total expected waiting time of company trucks per day = Trucks day × % of company trucks × expected waiting time per truck

$$= (3 \times 8) \times (0{\cdot}40) \times \frac{\lambda}{\mu(\mu-\lambda)}$$

$$= 24 \times 0.4 \times \frac{3}{4(4-3)}$$

$$= 24 \times 0{\cdot}4 \times \frac{3}{4} = \mathbf{7{\cdot}2\ hours/day}.$$

EXAMPLE 10.6-2.5

Solve example 10.1-5.

Solution. Here arrival rate $\lambda = 1/9$ per minute,

service rate $\mu = 1/3$ per minute.

(*a*) Probability that a person will have to wait

$$=\frac{\lambda}{\mu}=\frac{1/9}{1/3}=1/3=0{\cdot}33.$$

(*b*) Average queue length $=\frac{\mu}{\mu-\lambda}=\frac{1/3}{1/3-1/9}=\frac{1/3}{2/9}$

$$=1/3\times 9/2=1{\cdot}5 \text{ persons.}$$

(*c*) Average waiting time in the queue $=\frac{\lambda_1}{\mu(\mu-\lambda_1)}$.

$$\therefore \quad 4=\frac{\lambda_1}{1/3\ (1/3-\lambda_1)}$$

or $$1/9-\frac{\lambda_1}{3}=\frac{\lambda_1}{4} \quad \text{or} \quad \lambda_1\times 7/12=1/9.$$

$$\therefore \quad \lambda_1=\frac{12}{7\times 9}=\frac{4}{21} \text{ arrivals/minute.}$$

(*d*) Probability [waiting time $>$ 10]

$$=\int_{10}^{\infty}\left(1-\frac{\lambda}{\mu}\right).\ \lambda\ .\ e^{(\lambda-\mu)w}\ .\ dw$$

$$=\lambda\left(1-\frac{\lambda}{\mu}\right)\left[\frac{e^{(\lambda-\mu)w}}{\lambda-\mu}\right]_{10}^{\infty}$$

$$=\frac{\lambda(\mu-\lambda)}{\mu}\left[0-\frac{1}{\lambda-\mu}\ .\ e^{(\lambda-\mu)10}\right]$$

$$=\frac{\lambda}{\mu}\ .\ e^{(\lambda-\mu)10}$$

$$=\frac{1/9}{1/3}\ .\ e^{(1/9-1/3)\times 10}=1/3\ e^{-20/9}=1/30.$$

(*e*) Probability [time in system $>$ 10]

$$=\int_{10}^{\infty}(\mu-\lambda)\ .\ e^{(\lambda-\mu)w}\ .\ dw$$

$$=e^{10(\lambda-\mu)}=e^{-20/9}=0{\cdot}1.$$

(*f*) The expected fraction of a day that the phone will be in

use

= expected fraction of any other time interval that the phone is in use.

$$=\lambda/\mu=\frac{1/9}{1/3}=1/3=0.33$$

10.6.3. Single-Channel, Finite-Population Model with Poisson Arrivals and Exponential Service Times

In some cases, units arrive from a limited pool of potential customers. Once a unit joins the queue, there is one less unit which could arrive and, therefore the probability of an arrival is lowered. When a unit is served, it rejoins the pool of potential customers, and the probability of an arrival is, thereby, increased. As a rule of thumb, if the population is less than 40, the equations for a finite population should be used.

Although the concepts are same as those for infinite population, some of the terms are different and the equations required for analysis are different. One distinct difference is that the probability of an arrival depends upon the number of potential customers available to enter the system. Thus if the total customers' population is M and n represents the number of customers already in the queuing system, any arrival must come from $M-n$ number that is not yet in the system.

Now, if $1/\lambda$ is the mean time between service needs for any unit *i.e.*, mean time between arrivals for a given customer, then λ is the probability that a customer will require service during time interval dt. Now, if λ is the probability of a specific unit requiring service and there are $M-n$ customers not in the queuing system, then the probability of a customer requiring service is $(M-n)\ \lambda$. Note that it is still assumed that dt is too small for the probability of two or more arrivals to be significant.

To determine the properties of this system, it is necessary to find an expression for the probability of n customers in the system at time t *i.e.*, $p_n(t)$, for if, $p_n(t)$ is known, the expected number of customers in the system and other such like properties can be found. In place of finding an expression for $p_n(t)$, we shall initially find an expression for $p_n(t+dt)$.

The probability of n units in the system at time $t+dt$ can be determined by summing up probabilities of all the ways this event could occur. The event can occur in three mutually exclusive and exhaustive ways.

Table 10·3

Event	*No. of units at time t*	*No. of arrivals in time dt*	*No. of services in time dt*	*No. of units at time t+dt*
1	n	0	0	n
2	$n+1$	0	1	n
3	$n-1$	1	0	n

Probability of event $1 = p_n(t).\ [1-(M-n)\ \lambda\ dt]\ (1-\mu\ dt)$

$$= p_n(t).\ - p_n(t).\ (M-n)\ \lambda\ dt - p_n(t).\ \mu dt.$$

Probability of event $2 = p_{n+1}(t).\ [1-M-n-1)\ \lambda\ dt]\ (\mu dt)$

$$= p_{n+1}(t).\ \mu\ dt.$$

Probability of event $3 = p_{n-1}(t).\ [(M-n+1)\ \lambda\ dt]\ (1-\mu\ dt]$

$$= p_{n\ 1}(t).\ (M-n+1)\ \lambda\ dt.$$

$$\therefore \quad p_n(t+dt) = p_n(t) - p_n(t).\ (M-n)\ \lambda\ dt - p_n(t).\ \mu\ dt + p_{n+1}(t).\ \mu dt + p_{n-1}(t)\ (M-n+1).\ \lambda dt$$

or
$$\frac{p_n(t+dt) - p_n(t)}{dt} = -p_n(t).\ [\lambda\ (M-n)+\mu] + p_{n+1}(t).\ \mu + p_{n-1}(t)\ [(M-n+1)\ \lambda].$$

Taking the limit when $dt \to 0$, we get

$$\frac{d}{dt}\ [p_n(t)] = -p_n(t)\ [(M-n)\lambda + \mu] + p_{n+1}(t).\ \mu + p_{n-1}(t).\ [M-n+1)\ \lambda].$$

Assuming a steady state condition for the system when

$p_n(t) = p_n$ and $\frac{d}{dt}\ [p_n(t)] = 0$, we get

$$0 = -p_n\ [(M-n)\ \lambda+\mu] + p_{n+1}\ (\mu) + p_{n-1}\ [(M-n+1)\ \lambda].$$

$$\therefore\ p_{n+1} = p_n \left[\ (M-n)\ \frac{\lambda}{\mu} + 1\ \right] - p_{n-1}\ (M-n+1)\ \frac{\lambda}{\mu}. \quad ...(10.24)$$

This is a general expression for p_{n+1} as a function of p_n and p_{n-1}. Now we find an expression for p_n in terms of p_0, λ, μ and M.

When $n=0,\ p_0 = p_0,$

when $n=1,\ p_1 = p_0.\ M \left(\frac{\lambda}{\mu}\right),$

when $n=2,\ p_2 = p_1 \left[\frac{(M-1)\ \lambda}{\mu} + 1\right] - p_0\ (M-1+1)\ \frac{\lambda}{\mu}$

(From equation 10.24)

$$= p_0 . \text{M} \left(\frac{\lambda}{\mu}\right)\left[\frac{(\text{M}-1)\,\lambda}{\mu}+1\right] - p_0 . \text{M}\left(\frac{\lambda}{\mu}\right)$$

$$= p_0 \,\text{M}\left(\frac{\lambda}{\mu}\right)\left[\frac{(\text{M}-1)\lambda}{\lambda}+1-1\right]$$

$$= p_0 . \frac{\lambda}{\mu} . \text{M}. (\text{M}-1). \frac{\lambda}{\mu}$$

$$= p_0 \left(\frac{\lambda}{\mu}\right)^2 . \text{M}\,(\text{M}-1),$$

....

....

$$p_n = p_0 \left(\frac{\lambda}{\mu}\right)^n . \text{M}. (\text{M}-1)\,(\text{M}-2)...(\text{M}-n+1)$$

$$= p_0 \left(\frac{\lambda}{\mu}\right)^n . \frac{\text{M}\,!}{(\text{M}-n)\,!} = p_0 . \frac{\text{M}\,!}{(\text{M}-n)\,!} - \left(\frac{\lambda}{\mu}\right)^n . \qquad ...(10.25)$$

Now find P_0 in terms of λ, μ and M.

$$\sum_{n=0}^{n=\text{M}} p_n = \sum_{n=0}^{n=\text{M}} p_0 . \frac{\text{M}\,!}{(\text{M}-n)\,!}\left(\frac{\lambda}{\mu}\right)^n = 1.$$

1. $\therefore$ probability of an empty system,

$$p_0 = \frac{1}{\displaystyle\sum_{n=0}^{n=\text{M}} \frac{\text{M}\,!}{(\text{M}-n)\,!)}\left(\frac{\lambda}{\mu}\right)^n} . \qquad ...(10.26)$$

2. Therefore, from equation (10.25) and equation (10.26), we have, probability of n customers in the system,

$$p_n = \frac{\dfrac{\text{M}\,!}{(\text{M}-n)\,!}\left(\dfrac{\lambda}{\mu}\right)^n}{\displaystyle\sum_{n=0}^{n=\text{M}} \frac{\text{M}\,!}{(\text{M}-n)\,!}\left(\frac{\lambda}{\mu}\right)^n} = p_0 . \frac{\text{M}\,!}{(\text{M}-n)\,!}\left(\frac{\lambda}{\mu}\right)^n . \qquad ...(10.27)$$

3. Expected number of customers in the system,

$$L_s = \sum_{n=0}^{n=\text{M}} n\, p_n = \text{M} - \frac{\mu}{\lambda}\,(1-P_0). \qquad ...(10.28)$$

4. Expected number of customers in the queue,

$$L_q = \text{M} - \frac{\lambda+\mu}{\lambda}\,(1-p_0). \qquad ...(10.29)$$

EXAMPLE 10.6-3.1

Solve example 10.1-6.

Solution

This situation involves finite population.

Arrival rate $\lambda=\frac{1}{5}=0.2$,

service rate $\mu=\frac{1}{1}=1$.

Let us first find the probability of an empty system,

$$p_0=\frac{1}{\sum_{n=0}^{n=4}\frac{4!}{(4-n)!}\left(\frac{.2}{1}\right)^n}$$

$$=\frac{1}{1+4(\cdot 2)+(4\times 3)(\cdot 2)^2+(4\times 3\times 2)\times(\cdot 2)^3+(4\times 3\times 2\times 1)(\cdot 2)^4}$$

$=0{\cdot}4$.

(*a*) Expected number of broken-drown machines in the system,

$$L_s=M-\frac{\mu}{\lambda}(1-p_0)$$

$$=4-\frac{1}{\cdot 2}(1-\cdot 4)=4-5\times\cdot 6=4-3=1.$$

$\therefore$ Expected number of operating machines in the system $=4-1=3$.

(*b*) Expected down time cost per day (assuming an 8-hour day) $=8\times$expected number of broken-down machines $\times$ Rs. 25 per hour.

$=8\times 1\times 25=$Rs. 200 per day.

(*c*) When there are two mechanics each serving two machines, $M=2$.

$$\therefore\quad p_0=\frac{1}{\sum_{n=0}^{n=2}\frac{2!}{(2-n)!}\left(\frac{\cdot 2}{1}\right)^n}=\frac{2}{1+2(\cdot 2)+2\times 1(\cdot 2)^2}$$

$$=\frac{1}{1.48}=0{\cdot}68.$$

It is assumed that each mechanic with his two machines constitutes a separate system with no interplay. Expected number of machines in the system $=M-\frac{\mu}{\lambda}=(1-p_0)$

$$=2-\frac{1}{\cdot 2}(1-\cdot 68)=0{\cdot}4.$$

$\therefore$ Expected down time/day $= 8 \times \cdot 4 \times$ number of mechanics

$$= 8 \times \cdot 4 \times 2 = 6.4 \text{ hr/day.}$$

$\therefore$ Total cost with two mechanics

$$= \text{Rs. } 2 \times 55 + \text{Rs. } 6.4 \times 25$$
$$= \text{Rs. } (110 + 160) = \text{Rs. } 270 \text{ per day.}$$

Total cost with one mechanic $=$ Rs. $(55+200) =$ Rs. 255 per day. Hence use of two mechanics is not economical.

10·7. Multi-Channel Queuing Theory

Multi-channel queuing theory treats the condition in which there are several service stations in parallel and each element in the waiting line can be served by more than one station. Each service facility is prepared to deliver the same type of service. The new arrival selects one station without any external pressure. When a waiting line is formed, a single line usually breaks down into shorter lines in front of each service station. The arrival rate λ and service rate μ are mean values from Poisson distribution and exponential distribution respectively. Service discipline is first-come, first served and customers are taken from a single queue *i.e.*, any empty channel is filled by the next customer in line.

Let $n =$ number of customers in the system,

$p_n =$ probability of n customers in the system,

$c =$ number of parallel service channels,

$\lambda =$ arrival rate of customers,

$\mu =$ service rate of individual channel.

When $n < c$, there is no queue because all arrivals are being serviced, whereas when $n \geqslant c$, a queue will be formed because the service demanded by the arrival (s) is greater than the capabilities of the service stations. The first case presents no problem, whereas the second does.

To determine the properties of the multi-channel system, it is necessary to find an expression for the probability of n customers in the system at time t *i.e.*, $p_n(t)$ when $n < c$ as well as when $n \geqslant c$.

(*i*) **when n<c.** As a simplification, first let $c = 2$.

Let us first find $p_0(t+dt)$. It can occur only in two exclusive and exhaustive ways :

Table 10·4

Event	*No. of units at time t*	*No. of arrivals in time dt*	*No. of services in time dt*	*No. of units at time t+dt*
1	0	0	—	0
2	1	0	1	0

$$p_0\,(t+dt)=p_0\,(t)\,.\,(1-\lambda\,.\,dt)+p_1\,(t)\,.\,(1-\lambda dt)\,.\,(\mu\, dt)$$

$$=p_0\,(t)-p_0\,(t)\,.\,\lambda dt+p_1\,(t)\,.\,(\mu dt).$$

$$\therefore\quad \frac{p_0\,(t+dt)-p_0\,(t)}{dt}=\mu p_1\,(t)-\lambda p_0\,(t).$$

Taking the limit when $dt \to 0$, $\frac{d}{dt}\left[\; p_0\,(t) \;\right]=\mu\,.\,p_1\,(t)-\lambda p_0\,(t).$

Considering the steady state system, $0=\mu p_1-\lambda p_0.$

$$\therefore\quad p_1=\frac{\lambda}{\mu}\,p_0. \qquad \text{...(10.30)}$$

Now, let us consider the ways in which $p_1\,(t)$ can occur. There are three exclusive ways :

Table 10.5

Event	*No. of units at time t*	*No. of arrivals in time dt*	*No. of services in time dt*	*No. of units at time (t+dt)*
1	0	1		1
2	1	0	0	1
3	2	0	1	1

$$\therefore\quad p_1\,(t+dt)=p_0\,(t)\,.\,(\lambda\,.\,dt)\,.\,(1-\mu dt)+p_1\,(t)\,.\,(1-\lambda dt)$$

$$(1-\mu dt)+p_2\,(t)\,.\,(1-\lambda dt)\,.\,(2\mu.\,dt).$$

Note that if both channels are filled, the probability of one service is $\mu dt+\mu dt=2\mu dt.$

$$\therefore\quad p_1\,(t+dt)=p_0\,(t)\,.\,\lambda dt+p_1\,(t)\,.\,[1-\lambda dt-\mu dt]+p_2\,(t)\,.\,2\mu\, dt$$

or
$$\frac{p_1\,(t+dt)-p_1\,(t)}{dt}=\lambda\,.\,p_0\,(t)-p_1\,(t)\,.\,(\lambda+\mu)+p_2\,(t)\,.\,2\mu$$

Taking the limit when $dt \to 0$, $\frac{d}{dt}\left[p_1(t) \right] = \lambda \cdot p_0(t)$

$$-(\lambda+\mu) \cdot p_1(t) + 2\mu \cdot p_2(t).$$

Considering a steady state system, $0 = \lambda \cdot p_0 - (\lambda+\mu)\, p_1 + 2\mu p_2$.

$$\therefore \quad p_2 = \frac{\lambda+\mu}{2\mu} p_1 - \frac{\lambda}{2\mu} \cdot p_0 \qquad \ldots(10.31)$$

$$= \frac{\lambda+\mu}{2\mu} \cdot \left(\frac{\lambda}{\mu} p_0 \right) - \frac{\lambda}{2\mu} p_0$$

$$= \frac{\lambda}{2\mu} p_0 \left[\frac{\lambda+\mu}{\mu} - 1 \right] = \frac{p_0}{2} \left(\frac{\lambda}{\mu} \right).$$

Similarly, $p_3 = \frac{\lambda+2\mu}{3\mu} p_2 - \frac{\lambda}{3\mu} p_1$

$$= \frac{\lambda+2\mu}{3\mu} \cdot \frac{p_0}{2} \left(\frac{\lambda}{\mu} \right)^2 - \frac{\lambda}{3\mu} \cdot \left(\frac{\lambda}{\mu} p_0 \right)$$

$$= \left(\frac{\lambda}{\mu} \right)^2 \frac{p_0}{3} \left[\frac{\lambda+2\mu}{2\mu} - 1 \right]$$

$$= \left(\frac{\lambda^2}{\mu} \right) \cdot \frac{p_0}{3} \cdot \frac{\lambda}{2\mu} = \frac{p_0}{3 \cdot 2} \left(\frac{\lambda}{\mu} \right)^3.$$

Generalizing,

$$p_n = \frac{p_0}{n!} \left(\frac{\lambda}{\mu} \right)^n \text{ for } n < c \text{ i.e., } n = 0, 1, 2, 3, \ldots, c-1. \qquad \ldots(10.32)$$

(*ii*) **when n>2**, the three exclusive and exhaustive ways for having n units at time $t+dt$ are :

Table 10.6

Event	*No. of units at time t*	*No. of arrivals in time dt*	*No. of services in time dt*	*No. of units at time t+dt*
1	n	0	0	n
2	$n+1$	0	1	n
3	$n-1$	1	0	n

$$\therefore \quad p_n(t+dt) = p_n(t) \cdot [(1-\lambda dt)(1-2\mu dt)]$$
$$+ p_{n+1}(t) \cdot [(1-\lambda dt) \cdot (2\mu dt)]$$
$$+ p_{n-1}(t) [(\lambda \cdot dt)(1-2\mu dt)]$$
$$= p_n(t) [1 - \lambda dt - 2\mu dt] + p_{n+1}(t)(2\mu dt) + p_{n-1}(t) \cdot (\lambda \cdot dt).$$

$$\therefore \quad \frac{p_n(t+dt)-p_n(t)}{dt} = -(\lambda+2\mu)p_n(t)+2\mu p_{n+1}(t)+\lambda \cdot p_{n-1}(t).$$

Considering the limit when $dt \to 0$,

$$\frac{d}{dt}\left[p_n(t)\right] = -(\lambda+2\mu)\,p_n(t)+2\mu p_{n+1}(t)+\lambda \cdot p_{n-1}(t).$$

Considering a steady state system,

$$0=-(\lambda+2\mu)p_n+2\mu p_{n+1}+\lambda p_{n-1}$$

or $$p_{n+1}=\frac{\lambda+2\mu}{2\mu}\,p_n-\frac{\lambda}{2\mu}\,p_{n-1}.$$

This can be generalized to c channels.

Thus for c channels, $p_{n+1}=\dfrac{\lambda+c\mu}{c\mu}\,p_n-\dfrac{\lambda}{c\lambda}\,p_{n-1}$ ...(10·34)

or $$p_n=\frac{\lambda+c\mu}{c\mu}\,p_{n-1}-\frac{\lambda}{c\mu}\,p_{n-2}>c \text{ for } n\geqslant c+1$$

or $$p_n=\frac{\lambda+(n-1)\mu}{n\mu}\,p_{n-1}-\frac{\lambda}{n\mu}\,p_{n-2},\ n=2,3,\ldots,c. \qquad \text{...(10·35)}$$

Let $n=c$

$$\therefore \quad p_c=\frac{\lambda+(c-1)\mu}{c\mu}\,p_{c-1}-\frac{\lambda}{c\mu}\,p_{c-2}$$

$$=\frac{\lambda+(c-1)\mu}{c\mu}\left[\frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^{c-1}p_0\right]-\frac{\lambda}{c\mu}\cdot\left[\frac{1}{(c-2)!}\left(\frac{\lambda}{\mu}\right)^{c-2}\cdot p_0\right]$$

$$=\frac{p_0}{c(c-2)!}\left(\frac{\lambda}{\mu}\right)^{c-1}\left[\frac{\lambda+(c-1)\mu}{\mu(c-1)}-1\right]$$

$$=\frac{p_0}{c(c-2)!}\left(\frac{\lambda}{\mu}\right)^{c-1}\cdot\frac{\lambda}{\mu(c-1)}=\frac{p_0}{c!}\left(\frac{\lambda}{\mu}\right)^{c}. \qquad \text{...(10·36)}$$

Let $n=c+1$

$$\therefore \ p_{c+1}=\frac{\lambda+c\mu}{c\mu}\,p_c-\frac{\lambda}{c\mu}\,p_{c-1}$$

$$=\frac{\lambda+c\mu}{c\mu}\cdot\frac{p_0}{c!}\left(\frac{\lambda}{\mu}\right)^{c}-\frac{\lambda}{c\mu}\cdot\frac{p_0}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^{c-1}$$

$$=\frac{p_0}{c!}\left(\frac{\lambda}{\mu}\right)^{c}\cdot\left(\frac{\lambda+c\mu}{c\mu}-1\right)$$

$$=\frac{p_0}{c!}\left(\frac{\lambda}{\mu}\right)^{c}\cdot\frac{\lambda}{c\mu}$$

$$=\frac{p_0}{c!\,c}\left(\frac{\lambda}{\mu}\right)^{c+1} \qquad \text{...(10·37)}$$

Let $n = c+2$

$$\therefore p_{c+2} = \frac{\lambda + c\mu}{c\mu} p_{c+1} - \frac{\lambda}{c\mu} p_c$$

$$= \frac{\lambda + c\mu}{c\mu} \cdot \left[\frac{p_0}{c!.c}\left(\frac{\lambda}{\mu}\right)^{c+1}\right] - \frac{\lambda}{c\mu}\left[\frac{p_0}{c!}\left(\frac{\lambda}{\mu}\right)^{c}\right]$$

$$= p_0 \cdot \left(\frac{\lambda}{\mu}\right)^{c+1} \cdot \frac{1}{c!.c}\left(\frac{\lambda + c\mu}{c\mu} - 1\right)$$

$$= p_0 \cdot \left(\frac{\lambda}{\mu}\right)^{c+1} \cdot \frac{1}{c!.c} \cdot \left(\frac{\lambda}{c\mu}\right) = \frac{p_0}{c!\,c^2}\left(\frac{\lambda}{\mu}\right)^{c+2} \quad \ldots(10\cdot38)$$

$$\text{Thus } p_n = \frac{p_0}{c!\,c^{n-c}} \cdot \left(\frac{\lambda}{\mu}\right)^n, n \geq c. \quad \ldots(10\cdot39)$$

Now p_0 must be expressed in terms of c, μ and λ. Then the values of p_n and p_0 can be used to develop the other equations. Since sum of all possible probabilities must be equal to 1,

$$\sum_{n=0}^{n=\infty} p_n = 1.$$

$\because$ p_n has been expressed for two cases, $n<c$ (*i.e.*, $n \leq c-1$) and $n \geq c$;

$$\sum_{n=0}^{n=c-1} \frac{p_0}{n!} \cdot \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{n=\infty} \frac{p_0}{c!\,c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n = 1.$$

$$\therefore p_0 \left\{\sum_{n=0}^{n=c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left[\sum_{n=c}^{n=\infty} \left(\frac{\lambda}{\mu}\right)^n \cdot \right]\frac{1}{c^{n-c}}\right\} = 1. \quad \ldots(10\cdot40)$$

The second term within brackets is an infinite series. This needs to be expressed as a limit.

$$\text{Let } A = \sum_{n=c}^{n=\infty} \frac{(\lambda/\mu)^n}{c^{n-c}}$$

$$= \frac{(\lambda/\mu)^c}{c^0} + \frac{(\lambda/\mu)^{c+1}}{c^1} + \frac{(\lambda/\mu)^{c+2}}{c^2} + \ldots$$

$$= (\lambda/\mu)^c + \frac{(\lambda/\mu)^{c+1}}{c} = \frac{\lambda/\mu)^{c+2}}{c} + \ldots$$

$$= (\lambda/\mu)^c \left[1 + \frac{(\lambda/\mu)}{c} + \left(\frac{\lambda/\mu}{c}\right)^2 + \ldots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^c \cdot \left[\frac{1}{1 - \frac{\lambda}{\mu c}}\right]$$

$$= \frac{\mu c}{\mu c - \lambda} \cdot \left(\frac{\lambda}{\mu}\right)^c.$$

$\therefore$ From equation (10·40),

$$p_0=\frac{1}{\sum\limits_{n=0}^{n=c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\cdot\frac{\mu c}{\mu c-\lambda}} \quad \ldots(10\cdot41)$$

Now the various properties of the multi-channel system can be found out.

1. The expected number of units in the system,

$$L_s=\sum_{n=0}^{n=\infty} n\, p_n$$

or
$$L_s=\sum_{n=0}^{n=c-1} n\cdot\frac{1}{n!}\left(\frac{\lambda}{\mu}\right) p_0+\sum_{n=c}^{n=\infty}\frac{n}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n p_0$$

$$=\sum_{n=0}^{n=\infty}\frac{n\lambda}{n!\,\mu}\,p_0-\sum_{n=c}^{n=\infty}\frac{n}{n!}\left(\frac{\lambda}{\mu}\right)p_0+\sum_{n=c}^{n=\infty}\frac{n}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n p_0. \quad \ldots(10\cdot42)$$

Using a similar method of evaluating an infinite series, this reduces to

$$L_s=\frac{\lambda\mu\,(\lambda/\mu)^c}{(c-1)!\,(c\mu-\lambda)^2}\,p_0+\frac{\lambda}{\mu}. \quad \ldots(10\cdot43)$$

2. The expected queue length, $L_q=L_s-$mean number being served

$$=L_s-c\cdot\frac{\lambda}{c\mu}=L_s-\frac{\lambda}{\mu}. \quad \ldots(10\cdot44)$$

3. Expected waiting time in the queue,

$$W_q=L_q\cdot\frac{1}{\lambda}=\frac{\mu\cdot(\lambda/\mu)^c\cdot p_0}{(c-1)!\,(c\mu-\lambda)^2}. \quad \ldots(10\cdot45)$$

4. Expected waiting time in the system,

$$W_s=W_q+\frac{1}{\mu}=\frac{\mu\cdot(\lambda/\mu)^c\cdot p_0}{(c-1)!\,(c\mu-\lambda)^2}+\frac{1}{\mu}. \quad \ldots(10.46)$$

EXMAPLE 10.7-1

In example 10.1·3, there is assumed a possibility of installing two type-writers or renting a better and faster machine. The data is given below.

Table 10·7

	Service rate μ *per hour*	*Daily rental cost (Rs.)*
Present typewriter	8	2·50
Proposed typewriter	12	4·50

Suggest the better alternative.

Solution

Total cost per day of the present typewriter

=rental cost+lost time cost

=Rs. 2·50+Rs. 20·00=Rs. 22·50.

For proposed typewriter:

Average system time, $W_s=\frac{1}{\mu-\lambda}=\frac{1}{12-5}=\frac{1}{7}$ hours.

Lost time cost per day $=(8\times 5)\times\frac{1}{7}\times 1{\cdot}50=$ Rs. $\frac{60}{7}$

=Rs. 8·57.

Total cost per day=rental cost+lost time cost

=Rs. 4·50+Rs. 8·57=Rs. 13·07.

To calculate the waiting time in the system for two machines, first let us find p_0.

$$p_0=\frac{1}{\sum_{n=0}^{n=c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\frac{c\mu}{c\mu-\lambda}}$$

$$=\frac{1}{\sum_{n=0}^{n=1}\frac{1}{n!}\left(\frac{5}{8}\right)^n+\frac{1}{2!}\left(\frac{5}{8}\right)^2\cdot\frac{2\times 8}{2\times 8-5}}$$

$$=\frac{1}{\frac{1}{0!}\left(\frac{5}{8}\right)^0+\frac{1}{1!}\left(\frac{5}{8}\right)^1+\frac{1}{2!}\left(\frac{5}{8}\right)^2\cdot\frac{16}{11}}$$

$$=\frac{1}{1+\frac{5}{8}+\frac{25}{64}\times\frac{8}{11}}$$

$$=\frac{1}{1+\frac{5}{8}+\frac{25}{88}}=\frac{1}{\frac{88+55+25}{88}}$$

$$=\frac{88}{168}=\frac{11}{21}.$$

Expected waiting time in the system,

$$W_s=\frac{\mu(\lambda/\mu)^c\, p_0}{(c-1)!(c\mu-\lambda)^2}+\frac{1}{\mu}$$

$$=\frac{8\times(5/8)^2\times\frac{11}{21}}{(2-1)!(2\times 8-5)^2}+\frac{1}{8}$$

$$=\frac{\frac{25}{8}\times\frac{11}{21}}{11\times 11}+\frac{1}{8}$$

$$=\frac{25}{88\times 21}+\frac{1}{8}$$

$$=\frac{25+231}{88\times 21}=\frac{256}{88\times 21}=\frac{32}{231}.$$

$\therefore$ Total cost per day = rental cost of two typewriters + cost of lost time

$$=\text{Rs.}\left(2\times 2{\cdot}50+8\times 5\times\frac{32}{231}\times 1{\cdot}50\right)$$

$$=\text{Rs.}\left(5+\frac{640}{77}\right)=\text{Rs.}\ (5+8{\cdot}31)$$

$$=\text{Rs.}\ 13{\cdot}31.$$

$\therefore$ It is slightly less expensive to use a single proposed better (faster) machine.

EXAMPLE 10·7·2

Solve example 10·1·4 assuming two equal sized docks.

Solution

1. The probability that a truck has to wait for service = probability p_0 that there are two or more trucks already in the system, where

$$p_c=\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\cdot\frac{c\mu}{c\mu-\lambda}\cdot p_0,$$

where
$$p_0=\frac{1}{\sum_{n=0}^{n=c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\frac{c\mu}{c\mu-\lambda}}$$

$$= \frac{1}{\sum_{n=0}^{n=1} \frac{1}{n!}(3/4)^n + \frac{1}{2!}(3/4)^2 \cdot \frac{2\times 4}{2\times 4-3}}$$

$$= \frac{1}{\frac{1}{0!}(3/4)^0 + \frac{1}{1!}(3/4)^1 + \frac{1}{2}\times\frac{9}{16}\times\frac{8}{5}}$$

$$= \frac{1}{1+\frac{3}{4}+\frac{9}{20}} = \frac{1}{\frac{20+15+9}{20}} = \frac{20}{44} = \frac{5}{11}.$$

$$\therefore \qquad p_c = \frac{1}{2!}\left(\frac{3}{4}\right)^2 \frac{2\times 4}{2\times 4-3}\times\frac{5}{11} = \frac{1}{2}\times\frac{9}{16}\times\frac{8}{5}\times\frac{5}{11} = \frac{9}{44}$$

$\therefore$ **The probability that a truck must wait is 0·205.**

2. **The waiting time of a truck that waits is,**

$$W_n = \frac{\text{Expected waiting time of a truck, } W_q}{\text{Probability that a truck actually has to wait, } p_c}$$

$$\therefore \qquad W_n = \frac{\mu\cdot\left(\frac{\lambda}{\mu}\right)^c \cdot p_0}{(c-1)!(c\mu-\lambda)^2} \times \frac{1}{p_c}$$

$$= \frac{4(3/4)^2}{1!(2\times 4-3)^2} \times \frac{5}{11} \times \frac{44}{9}$$

$$= \frac{4}{1}\times\frac{9}{16}\times\frac{1}{25}\times\frac{5}{11}\times\frac{44}{9} = \frac{1}{5} = 0{\cdot}2.$$

3. Total expected waiting time of company trucks per day

$$= \text{Trucks/day} \times \% \text{ of company trucks} \times \text{Expected waiting time/truck}$$

$$= (3\times 8)\times(0{\cdot}40)\times W_q$$

$$= 24\times 0{\cdot}40\times(p_c\times W_n)$$

$$= 24\times 0{\cdot}40\times\frac{9}{44}\times 0{\cdot}2$$

$$= 0{\cdot}393.$$

10·8. Monte Carlo Technique Applied to Queuing problems*

The Monte Carlo technique is quite useful for analysing waiting-line problems which are difficult or impossible to be analysed mathematically. *Simulated sampling methods*, for example, are quite helpful when the first-come, first-served assumption is not valid for a particular queuing problem. In may cases, the observed distributions for arrival times and service times cannot be fitted to certain mathematical distribution (Poisson and exponential distribution) and Monte Carlo approach is the only hope under such situations. Similarly, multichannel queuing, in which departures

*For further details the reader is referred to Chapter 13 of this book.

from one queue form the arrivals for another, is another difficult area which can be easily handled by Monte Carlo technique.

Simulated sampling methods consist of replacing the actual universe of items by its theoretical counterpart, which is a universe described by some assumed probability distribution. A random number table is then used for sampling from this theoretical population. Such simulated sampling methods are called the *Monte Carlo Methods.*

The Monte Carlo approach has many advantages over the ordinary sampling method of just looking at the actual situation and forming a history of arrivals, services, queue lengths and waiting times, etc. Firstly, with a digital computer, this approach can develop many months or years of data in only a few minutes. Secondly, it allows manipulation of those factors which can be controlled. For example, we can readily assess the effect of adding one or more service stations without actually having to install them. Similarly, changes in queue discipline can be tried out experimentally on paper, without any disruption of the actual process

To draw an item at random from a universe described by the probability density $f(x)$, one proceeds as follows :

(*i*) plot the cumulative probability function

$$y = F(x) = \int_{-\infty}^{x} f(u).\ du.$$

(*ii*) select a random decimal between 0 and 1 (up to as many decimal places as desired) by means of a table of random numbers.

(*iii*) project horizontally the point on the Y-axis corresponding to this random decimal, until the projection line intersects the curve $y = F(x)$.

(*iv*) write down the value of x corresponding to the point of intersection. This value is taken as the sample value of x.

Illustration. A firm has a single-channel service station having the following characteristics based on empirical data : the mean time between arrivals, A_m is 6 minutes; mean service time, S_m is 5·5 minutes and arrival and service time distributions are given in figures 10·3 and 10·4 respectively.

In figures 10·3 and 10·4 the probabilities of values of A_x and S_x are shown. The cumulative arrival and service time distributions are determined by summing up the individual probabilities, starting from left to right. The arrival and service times are then obtained from these cumulative distributions (figures 10·5 and 10·6) along with a table of random numbers.

Let us assume that the queuing process starts at 9·00 A.M. and continues for approximately 2 hours. An arrival is served immediately if the service facility is free. If not, the arrival will wait in a queue. Further, assume that units are served on 'first-come, first-served basis'.

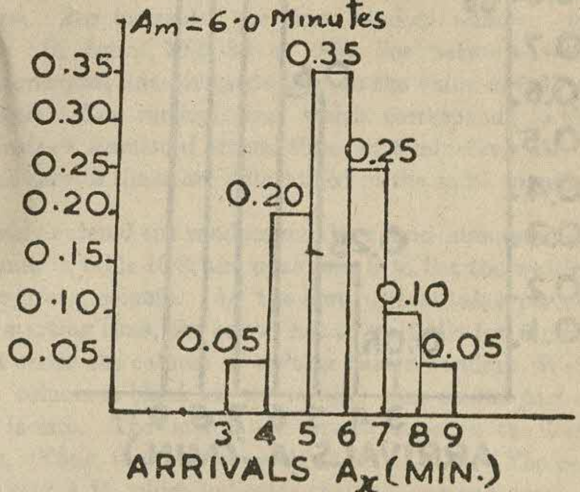

Fig. 10.3.

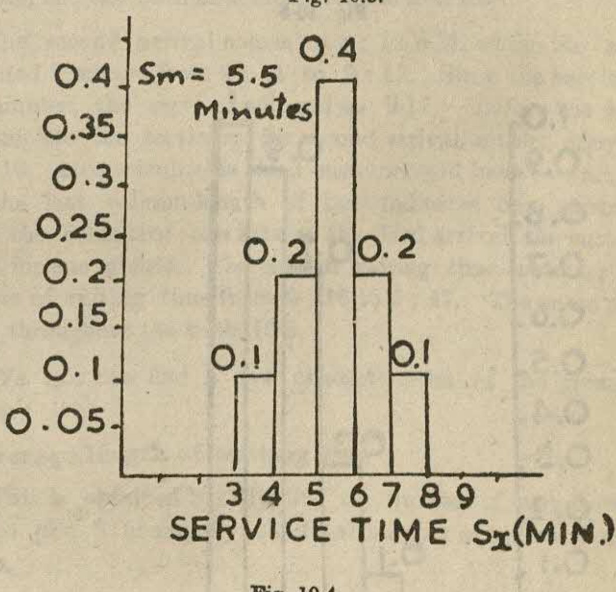

Fig. 10·4

The arrival and service times are shown in table 10·8. These times have been obtained from table C-1 (in appendix) of random numbers (a group of numbers which occur in no order with no one number more

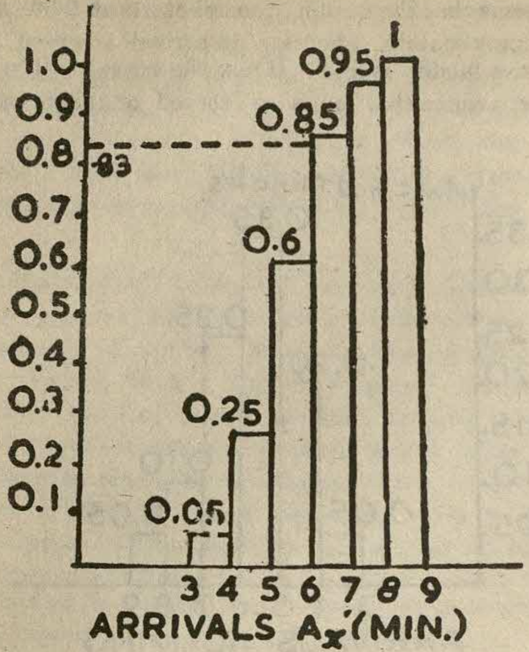

Fig. 10·5

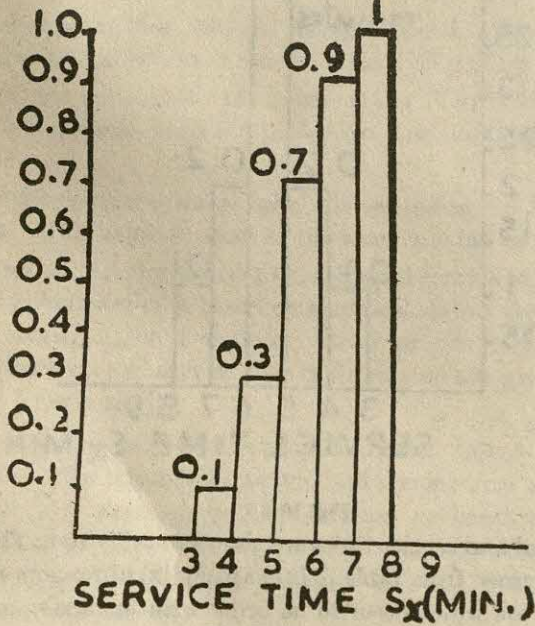

Fig. 10·6.

likely to occur than any other number). The random numbers for arrival times have been taken from the second last group column (first two digits) and the service times are taken from the last group column (first two digits). These random numbers are related to figure 10·5 for arrival time distribution and figure 10·6 for service time distribution. For example, the first random number for arrival time is 83. In figure 10·5, 83 or 0·83 lies between 0·6 and 0·85. Draw a horizontal line corresponding to the value of 0·83 along the vertical axis. The vertical line which corresponds to the value of 0·83, indicates a simulated arrival time of 6 minutes. All simulated arrival and service times are determined in the same manner.

Having entered the random numbers and simulated arrival and service times in table 10·8, the next step is to list the waiting time in the appropriate column. As the first arrival takes place 6 minutes after the starting time, the server has to wait idle for 6 minutes; this is entered under the column of waiting time-attendant. Waiting time-customer column is blank as the waiting time on the part of the first customer is zero. The simulated service time for the first arrival is 5 minutes. Thus the service ends at 9 : 11 A.M. The next arrival comes at 9·12 A.M. which indicates that no one has waited in line. Therefore, the last column is blank for the first line.

The second arrival comes at 9 : 12 A.M. while the attendant has waited 1 minute from 9 : 11 to 9 : 12. Since the service time is of 5 minutes, the service will end at 9·17. Before the attendant can complete the servicing for second arrival, a third arrival comes at 9 : 16, as a result this third customer will have to wait. Therefore, the last column-length of line indicates one person in line. Before the attendant can service the third arrival, the customer has to wait for one minute. The column waiting time-customer indicates 1 minute of waiting time from 9 : 16 to 9 : 17. The same procedure is used throughout the table 10·8.

We can now find a few characteristics of the given queuing problem :

1. Average length of waiting line

This is obtained by dividing the number of customers in waiting line (for 2 hours) by the total number of arrivals $= 9/22 = 0{\cdot}41$ persons.

2. Average time a customer waits before being served

This is obtained by dividing the total waiting time by the total number of arrivals $= 14/22 = 0{\cdot}64$ minute.

Table 16·8
Simulation of Arrival Time, Service Time and Waiting Time

Random Number	Time till next arrival (minutes)	Arrival Time (A.M.)	Service Begins (A.M.)	Random Number	Service Time (Minutes)	Service Ends (A.M.)	Waiting Time Attendant (Minutes)	Waiting Time Customer (Minutes)	Length of Line
83	6	9 : 06	9 : 06	46	5	9 : 11	6	--	—
70	6	9 : 12	9 : 12	64	5	9 : 17	1	—	1
06	4	9 : 16	9 : 17	09	3	9 : 20	—	1	—
12	4	9 : 20	9 : 20	48	5	9 : 25	—	—	—
59	5	9 : 25	9 : 25	97	7	9 : 32	—	—	1
46	5	9 : 30	9 : 32	22	4	9 : 36	—	2	1
54	5	9 : 35	9 : 36	29	4	9 : 40	—	1	1
04	3	9 : 38	9 : 40	01	3	9 : 43	—	2	—
51	5	9 : 43	9 : 43	40	5	9 : 48	—	—	—
99	8	9 : 51	9 : 51	75	6	9 : 57	3	—	—
84	6	9 : 57	9 : 57	10	4	10 : 01	—	—	—
81	6	10 : 03	10 : 03	09	3	10 : 06	2	—	—
15	4	10 : 07	10 : 07	70	6	10 : 13	1	—	1
36	5	10 ; 12	10 : 13	41	5	10 : 18	—	1	1
12	4	10 : 16	10 : 18	40	5	10 : 23	—	2	1
54	5	10 : 21	10 : 23	37	5	10 : 28	—	2	—
97	8	10 : 29	10 : 29	21	4	10 : 33	1	—	—
00	9	10 : 38	10 : 38	38	5	10 : 43	5	—	—
49	5	10 : 43	10 : 43	14	4	10 : 47	—	—	—
44	5	10 : 48	10 : 48	32	5	10 : 53	1	—	1
13	4	10 : 52	10 : 53	60	5	10 : 58	—	1	1
23	4	10 : 56	10 : 58	31	5	11 ; 03	—	2	—
22	116				103		20	14	9

3. Average time a customer spends in the system

$$\text{Average service time} = \frac{\text{Total service time}}{\text{Number of arrivals}}$$

$$= 103/22 = 4.68 \text{ minutes.}$$

$\therefore$ Average time a person spends in the system

$$= \text{Average service time} + \text{Average waiting time}$$

$$= 4.68 + 0.64 = 5.32 \text{ minutes.}$$

4. Would it be economical to add another attendant

This can be ascertained by comparing the cost of one attendant plus customer waiting time to the cost of two attendants and no waiting time. (Simulated analysis for the two attendants indicates no customer waiting time). If the cost of one attendant is Rs. 6 per hour and that of customer waiting time is Rs. 12 per hour, we get the following table :

Table 10.9

Two hours (approx.) period	*One attendant*	*Two attendants*
Customer waiting time $\left(\text{Rs. } \frac{14 \times 12}{60}\right)$	Rs. 2·80	—
Attendant's cost (2 hrs. $\times$ Rs. 6)	Rs. 12·00	Rs. 24·00
Total cost (of 2 hrs. approx. period)	Rs. 14·80	Rs. 24·00

Hence it is not economical to add the second attendant.

10·9 Additional Examples

EXAMPLE 10.9-1

A drive-in bank window has a mean service time of 2 minutes, while the customers arrive at a rate of 20 per hour. Assuming that these represent rates with a Poisson distribution,

(*a*) what percentage of time will the teller be idle ?

(*b*) after driving up, how long will it take the average customer to wait in line and be served ?

(c) what fraction of customers will have to wait in line ?

EXAMPLE 10·9-2

An airport can accommodate three air-planes in 2 minutes for either take-off or landing. If this rate is Poisson, what is the mean time between arrivals (for landing or departure) to ensure that the average waiting time is 5 minutes or less ? Assume an exponential distribution on the time between arrivals.

EXAMPLE 10·9-3

At a certain airport, it takes exactly 5 minutes to land an airplane, once it is given the signal to land. Although incoming planes have scheduled arrival times, the wide variability in arrival times produces an effect which makes the arrival of incoming planes as Poisson at an average rate of 6 per hour. This produces occasional stack-ups at the airport which can be dangerous and costly. How much time will a pilot expect to spend circling the field waiting to land ?

(*Ans.* $2\frac{1}{2}$ *minutes*)

EXAMPLE 10·9-4

In the production shop of a company, the breakdown of the machines is found to be Poisson with an average rate of 3 machines per hour. Break-down time of one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairmen. One of the repairmen is slow but cheap, the other fast but expensive. The slow-cheap repairman demands Rs. 20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast-expensive repairman demands Rs. 30 per hour, and will repair machines exponentially at an average rate of 6 per hour. Which repairman should be hired ?

(*Ans. Fast-expensive repairman*)

EXAMPLE 10·9-5

An insurance company has three claim adjusters in its branch office. People with claims arrive in a Poisson fashion, at an average rate of 30 per 8-hour day. The time spent by an adjuster with a claimant is found to have exponential distribution with mean service time 30 minutes. If claimants are attended in the order of their arrival, find

(a) how many hours a week an adjuster expects to spend with claimants ?

(b) how much time, on the average, does a claimant spend in the branch office ?

EXAMPLE 10·9·6

A telephone company plans to install telephone booths in a new locality. It is decided that a person should not have to wait more than 10 per cent of the times he tries to use a phone. The demand for use is estimated to be Poisson with an average of 30 per hour. The average phone call has an exponential distribution with a mean time of 5 minutes. How many phone booths should be installed ? (*Ans. Six*)

EXAMPLE 10·9·7

A supermarket has two sales girls at the counter. If the service time for each customer is exponential with average of 3 minutes and if people arrive in a Poisson fashion at the rate of 12 per hour,

(a) what is the probability for having to wait for service ?

(b) what is the expected percentage of idle time for each sales girl ?

(c) if the customers have to wait, what is the expected length of their waiting time ?

10·10. Bibliographic Notes

There are many books on the theory of queues. Of these, only three are mentioned here. An excellent book by W. Feller [4], in addition to exposition of the probability calculus, contains a discussion of some of the special cases, such as, theory of queues.

Book by P. Morse [6] comprises not only an elementary exposition of the queuing theory, but also many applications written in clear language.

A more advanced exposition of this theory, dealing also with its application to technology is given in the book by A. Khinchin [7].

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REPLACEMENT

Replacement problems fall into two categories depending upon the life pattern of the equipment involved *i.e.*, whether the equipment wears out or becomes obsolete with time (because of constant use or new technological developments) or suddenly fails.

For items that wear out, the problem is to balance the cost of new equipment against the cost of maintaining efficiency on the old and/or that due to the loss of efficiency. Though no general solution is possible, models have been constructed and solutions have been derived using simplified assumptions about the conditions of the problem.

For items that fail, the problem is to determine which items to replace and how often to replace so as to minimize the sum of the following costs :

(*i*) cost of equipment,

(*ii*) cost of replacing the unit,

(*iii*) cost associated with failure of the unit in terms of loss of earnings due to equipment breakdown.

One extreme policy is to replace items only when they fail. It, no doubt, minimizes equipment cost but costs of individual replacement and failure may be very high. Other extreme policy is to replace all units when or before the first one fails. It minimizes failure cost and also replacement cost due to economy of mass replacement but increases equipment cost. The optimum policy usually lies between the two extremes.

Life spans of items that fail are probabilistic. A lot of work has been done to find probability distribution of failure of various items as function of time.

11

Replacement Models

11·1. Introduction

All industrial and military equipment gets worn with time and usage and it functions with decreasing efficiency. For example, a machine requires higher operating cost, a transport vehicle such as a car or air plane requires more and more maintenance cost, a railway time-table becomes more and more out of date with the passage of time. The ever increasing repair and maintenance cost necessitates the replacement of the equipment. However, there is no sharp, clearly defined time which indicates the need for this replacement. The replacement policy, in this case, consists of calculating the increased operating cost, maintenance cost, forced idle time cost together with cost of the replacing new equipment.

A separate but similar problem involves the replacement of items such as electric bulb, radio tubes, etc. of equipment which does not deteriorate with time but suddenly fails. The problem, in this case, is of finding which items to replace and whether or not to replace them in a group and, if so, when. The objective is to minimize the sum of the cost of the item, cost of replacing the item and the cost associated with failure of item.

There is still another situation in which replacement becomes necessary. This is *obsolescence* due to new discoveries and better design of the equipment. The equipment needs replacement not because it no longer performs to the designed standards, but because more modern equipment performs higher standards. For example, an equipment may have an economic life of 20 years, yet it may become obsolete after 10 years because of better technical developments.

11·2. Replacement of Items that Deteriorate *i.e.*, Whose Maintenance Costs Increase with Time

Quite often the repair and maintenance costs of items increase with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one. Since both of these costs tend to increase with time, they are grouped while analysing a problem.

If these costs decrease or remain constant with time, the best policy is never to replace the item. However, this condition is hardly met with in practice. If these costs fluctuate with time, the item should be replaced only when they are increasing, of course, the analysis becomes more involved.

Generally, *all* costs that depend upon the *choice or age* of the equipment must be taken into account while analysing the decision of its replacement. However, in special situations, certain costs may not be considered. For example, costs (such as labour cost, electric cost, etc.) that do not change with the *age* of the equipment may not be included in calculations.

Now we shall consider a few cases of items that deteriorate with time and it will be assumed that suitable expressions for maintenance costs are available.

11.2·1. Replacement of Items whose Maintenance and Repair Costs Increase with Time, Ignoring Changes in the Value of Money During the Period.

Let us first consider a simple situation which consists of minimizing the average annual cost of an equipment whose maintenance cost is a function increasing with time and whose scrap value is constant. As the time value of money is not to be considered, the interest rate is zero and the calculations can be based on *average annual cost.*

Case 1. When time 't' is a continuous variabl

Let C = Capital cost of the item,

S = Scrap value of the item,

T_{ave} = Average annual total cost of the item,

n = Number of years the item is to be in use,

$f(t)$ = Operating and maintenance cost of the item at time t.

It is desired to find the value of n that minimizes T.

Annual cost of the item at any time t = capital cost − scrap value + maintenance cost at time t.

Now total maintenance cost incurred during n years $= \int_0^n f(t).dt.$

$\therefore$ Total cost incurred during n years, $T = C - S + \int_0^n f(t).dt.$

$\therefore$ Average annual total cost incurred on the item,

$$T_{ave} = \frac{1}{n}\left(C - S + \int_0^n f(t).dt\right) \qquad \text{...(11·1)}$$

Now we shall find that value of n for which T_{ave} is minimum. Differentiating equation (11·1) w.r.t. n, we get

$$\frac{d}{dn}(T_{ave}) = -\frac{1}{n^2}(C-S) - \frac{1}{n^2}\int_0^n f(t)dt + \frac{1}{n}f(n).$$

For $\frac{d}{dn}(T_{ave}) = 0$, we have

$$f(n) = \left[C - S + \int_0^n f(t)dt\right] = T_{ave}. \qquad \text{...(11·2)}$$

Thus the item should be replaced when the average annual cost to date becomes equal to the current maintenance cost. Using this result we can decide when to replace an item provided an explicit expression is given for the maintenance and repair costs.

Case 2 : When time 't' is a discrete variable

Tabular method is used in this case. It has the advantage of being a simpler method. The examples below explain this method.

EXAMPLE 11·2·1 :

The maintenance cost and resale value per year of a machine whose purchase price is Rs. 7,000 is given below.

Table 11·1

Year	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	900	1,200	1,600	2,100	2,800	3,700	4,700	5,900
Resale value in Rs.	4,000	2,000	1,200	600	500	400	400	400

When should the machine be replaced ?

Solution

Capital cost C = Rs. 7,000. Let it be profitable to replace the machine after n years. Then n should be determined by the minimum value of T_{ave}. Values of T_{ave} for various years are computed in table 11·2.

Table 11·2

(1) Years of service	(2) Resale value	(3) Purchase price —resale value	(4) Annual maintenance cost	(5) Summation of maintenance cost	(6) Total cost	(7) Average annual cost
(n)	(S)	$(C\text{-}S)$	$f(t)$	$\sum_0^n f(t)$	$\left[C-S\sum_0^n f(t)\right]$	$\frac{1}{n}\left[C-S+\sum_0^n f(t)\right]$
1	4,000	3,000	900	900	3,900	3,900
2	2,000	5,000	1,200	2,100	7,100	3,550
3	1,200	5,800	1,600	3,700	9,500	3,166.67
4	600	6,400	2,100	5,800	12,200	3,050
5	500	6,500	2,800	8,600	15,100	**3,020**
6	400	6,600	3,700	12,300	18,900	3,150
7	400	6,600	4,700	17,000	23,600	3,371.14
8	400	6,600	5,900	22,900	29,500	3,687.50

Table 11·3

(1) Years of service	(2) Resale value	(3) Purchase price —resale value	(4) Annual maintenance cost	(5) Summation of maintenance cost	(6) Total cost (3)+(5)	(7) Average annual cost
(n)	(S)	($C-S$)	$f(t)$	$\sum_{0}^{n} f(t)$		$\frac{(6)}{(1)}$
1	Zero	9,000	200	200	9,200	9,200
2	Zero	9,000	2.200	2,400	11,400	5,700
3	Zero	9,000	4,200	6,600	15,600	5,200
4	Zero	9,000	6,200	12,800	21,800	5,450
5	Zero	9,000	8,200	21,000	30,000	6,000

We observe from the table that average annual cost is minimum (Rs. 3,020) at the end of 5th year. Hence the machine should be replaced at the end of 5 years of service.

EXAMPLE 11·2-2

(*a*) Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine ? Assume that the machine has no resale value when replaced and that future costs are not discounted.

(*b*) Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You have now a machine of type A which is one year old. Should you replace it with B, and if so, when ?

[*Delhi 1968, Agra M. Stat. 1974*]

Solution

(*a*) Let us assume that the machine A has no resale value when replaced. The average annual cost is computed in table 11·3.

From table 11·3 we find that machine A should be replaced at the end of 3 years and the average yearly cost of owning and operating the machine at this time of replacement is Rs. 5,200.

(*b*) The average annual cost for machine B is computed in table 11·4.

Table 11.4 indicates that machine B should be replaced at the end of 5 years. Moreover, since the lowest average cost of Rs. 4,000 for machine B is less than the lowest average cost of Rs. 5,200 for machine A, machine A can be replaced by machine B.

Now we have to determine as to when machine A should be replaced. *Machine A should be replaced when the cost for next year of running this machine becomes more than the average yearly cost for machine B.*

Now total cost of machine A in the first year=Rs. 9,200,

total cost of machine A in the second year

=Rs. 11,400—Rs. 9,200=Rs. 2,200,

total cost of machine A in the third year=Rs. 4,200,

total cost of machine A in the fourth year=Rs. 6,200.

As the cost of running machine A in third year (Rs. 4,200) is more than the average yearly cost for machine B (Rs. 4,000) ; machine A should be replaced at the end of two years *i.e.*, one year after it is one year old.

Table 11·4

(1) *Years of service* (n)	(2) *Resale value* (S)	(3) *Purchase price −resale value* (C—S)	(4) *Annual maintenance cost* $f(t)$	(5) *Summation of maintenance cost* $\sum_{0}^{n} f(t)$	(6) *Total cost* (3)+(5)	(7) *Average annual cost* $\frac{(6)}{(1)}$
1	Zero	10,000	400	400	10,400	10,400
2	Zero	10,000	1,200	1,600	11,600	5,800
3	Zero	10,000	2,000	3,600	13,600	4,533.33
4	Zero	10,000	2,800	6,400	16,400	4,100
5	Zero	10,000	3,600	10,000	20,000	4,000
6	Zero	10,000	4,400	14,400	24,400	4,066.67

EXAMPLE 11.2-3

(*a*) An auto-rickshaw driver finds from his previous records that the cost per year of running an auto-rickshaw whose purchase price is Rs. 7,000 is as given below.

Year	1	2	3	4	5	6	7	8
Running cost Rs	1,100	1,300	1,500	1,900	2,400	2,900	3,500	4,100
Resale price (Rs)	3,100	1,600	850	475	300	300	300	300

At what age is a replacement due ?

(*b*) Another person has three auto-rickshaws of the same purchase price and cost of running each as in part (*a*). Two of these vehicles are 2 years old and the third one is 1 year old. He is considering a new type of auto-rickshaw with 50% more capacity than one of the old ones and at a unit price of Rs. 9,000. He estimates that the running costs and resale price for the new vehicle will be as follows :

Year	1	2	3	4	5	6	7	8
Running cost (Rs.) :	1,300	1,600	1,900	2,500	3,200	4,100	5,100	6,200
Resale price (Rs.) :	4,100	2,100	1,100	600	400	400	400	400

Assuming that the loss of flexibility due to fewer vehicles is of no importance, and that he will continue to have sufficient work for three of the old vehicles, what should his policy ?

Solution

(*a*) The average annual cost for old auto-rickshaw is computed in table 11.5.

Table 11·5

(*1*) *Years of service* (*n*)	(*2*) *Resale value* (*S*)	(*3*) *Purchase price — resale value* (*C—S*)	(*4*) *Annual maintenance cost*	(*5*) *Summation of maintenance cost* $f t \sum_{0}^{n} f(t)$	(*6*) *Total cost* (*3*)+(*5*)	(*7*) *Average annual cost* $\frac{(6)}{(1)}$
1	3,100	3,900	1,100	1,100	5,000	5,000
2	1,600	5,400	1,300	2,400	7,800	3,900
3	850	6,150	1,500	3,900	10,050	3,350
4	475	6,525	1,900	5,800	12,325	3,081
5	300	6,700	2,400	8,200	14,900	2,980
6	300	6,700	2,900	11,100	17,800	2,967
7	300	6,700	3,500	14,600	21,300	3,043
8	300	6,700	4,100	18,700	25,400	3.175

Thus the old auto-rickshaw should be replaced at the end of 6th year.

(*b*) Now let us compute the average annual cost of the new auto-rickshaw of larger capacity. This is done in table 11.6.

Table 11.6

(*1*) *Years of service* (n)	(*2*) *Resale value* (S)	(*3*) *Purchase price—resale value* (C-S)	(*4*) *Annual maintenance cost* $f(t)$	(*5*) *Summation of maintenance cost* $f(t)$	(*6*) *Total cost* (3)+(5)	(*7*) *Average annual cost* $\frac{(6)}{(1)}$
1	4,100	4,900	1,300	1,300	6,200	6,200
2	2,100	6,900	1,600	2,900	9,800	4,900
3	1,100	7,900	1,900	4,800	12,700	4,233
4	600	8,400	2,500	7,300	15,700	3,925
5	400	8,600	3,200	10,500	19,100	**3,820**
6	400	8,600	4,100	14,600	23,200	3,867
7	400	8,600	5,100	19,700	28,300	4,043
8	400	8,600	6,200	25,900	34,500	4,312

As the new auto-rickshaw has 50% more capacity than the old one, the minimum average annual cost of Rs. 3,820 for the former is equivalent to Rs. $3,820 \times 2/3 =$ Rs. 2,547 for the latter. Since this amount is less than Rs. 2,967 for it, the latter *will be replaced* by the new auto-rickshaw.

Having decided to replace the old vehicle by the new one, we now will determine as to when this replacement should be made. For uniformity we assume that all the three old auto-rickshaws will be replaced by two new larger ones. The new vehicles will be purchased when the cost for the next year of running the three old vehicles becomes more than the average annual cost of the two new ones.

Total annual cost of one smaller auto-rickshaw during first year

=Rs. 5,000,

annual cost of one smaller auto-rickshaw during second year

=Rs. 7,800—Rs. 5,000=Rs. 2,800,

annual cost of one smaller auto-rickshaw during third year

=Rs. 2,250,

annual cost of one smaller auto-rickshaw during fourth year

=Rs. 2,275,

annual cost of one smaller auto-rickshaw during fifth year

$$= \text{Rs. } 2,575,$$

annual cost of one smaller auto-rickshaw during sixth year

$$= \text{Rs. } 2,900, \text{ and so on.}$$

Total cost during next first year for two smaller vehicles aged two years and one vehicle aged one year $= 2 \times 2,250 + 2,800 = \text{Rs. } 7,300$.

Similarly, total cost during next second year

$$= 2 \times 2,275 + 2,250 = \textbf{Rs. 6,800},$$

total cost during next third year

$$= 2 \times 2,575 + 2,275 = \text{Rs. } 7,425,$$

total cost during next fourth year

$$= 2 \times 2,900 + 2,575 = \text{Rs. } 8,375, \text{ and so on.}$$

But minimum average cost for two new vehicles $= 2 \times 3,820 = \text{Rs } 7,640$.

As the total cost of old vehicles during next second year is less than the minimum average cost of the new vehicles and becomes more only in the next third year, the old auto-rickshaws should be replaced by the new larger ones *in the next third year* of their life.

11.2-2. Replacement of Items Whose Maintenance Costs Increase With Time and Value of Money also Changes With Time.

As the money value changes with time, we must calculate the *present value or present worth* of the money to be spent a few years hence. If it is the interest rate (i may also be considered as the *rate of inflation* or the sum of the rates of *interest and inflation*) per year, a rupee invested at present will be equivalent to $(1+i)$ a year hence, $(1+i)^2$ two years hence, and $(1+i)^n$ in n years time. In other words, making a payment of one rupee after n years is equivalent to paying $(1+i)^{-n}$ now. The quantity $(1+i)^{-n}$ is called the present worth or present value of one rupee spent n years from now.

$$\text{Present value of a rupee spent } n \text{ years hence} = (1+i)^{-n}$$
$$= v^n,$$

where $v = (1+i)^{-1} = \dfrac{1}{1+i}$ is called *discount rate* and is always less than unity.

In order to find the optimal policy of replacement *i.e.*, when a manufacturer should replace a machine on which he is working, let us assume that the machine is replaced after n years. Let C be the purchase price of the machine and $R_1, R_2, \ldots, R_n$ be the running costs in 1st, 2nd, ..., nth year respectively. Assuming that scrap

value of the machine is zero and that all payments are made at the beginning of each year, the present worth of expenditure in n years is

$$P_n = C + R_1 + vR_2 + v^2R_3 + \ldots + v^{n-1}.R_n \,. \quad \ldots(11.3)$$

Thus P_n is the amount of money required *now* to pay all future costs of acquiring and operating the machine assuming that it is to be replaced after n years.

Now P_n increases as n increases which means that the present worth, if the machine is replaced after $n+1$ years is greater than if it is replaced after n years. Thus for any additional amount spent we get an extra year's service. We are, therefore, interested in finding some function of *the replacement interval* which allows for this.

In order to do so, let us assume that the manufacturer invests the amount P_n by borrowing money at the interest rate i and repays it off in fixed annual payments, each of value x, throughout the life of the machine. Thus after n years he will have paid off the total cost P_n of the machine.

The present worth of fixed annual payments, each of value x, for n years is

$$x + vx + v^2x + \ldots + v^{n-1}x = \frac{1-v^n}{1-v}x.$$

Since this is equal to the sum P_n borrowed,

$$P_n = \frac{1-v^n}{1-v}x,$$

or $$x = \frac{1-v}{1-v^n}P_n. \quad \ldots(11.4)$$

Thus the best period to replace the machine is the period n which minimizes $x = \frac{1-v}{1-v^n}P_n$. However, since $(1-v)$ is a positive constant the period at which to replace the machine is the period n which minimizes the function $F_n = \frac{P_n}{1-v^n}$. Since n can have only discrete values, method of finite differences (appendix B) can be used to calculate its optimal value. By this method, n will be optimal *i.e.*, F_n will be minimum if

$$\Delta F_{n-1} < 0 < \Delta F_n \quad \ldots(11.5)$$

Now $$\Delta F_n = F_{n+1} - F_n \quad \ldots(11.6)$$

$$= \frac{P_{n+1}}{1-v^{n+1}} - \frac{P_n}{1-v^n}$$

$$= \frac{(1-v^n)\,P_{n+1} - (1-v^{n+1})P_n}{(1-v^{n+1})(1-v^n)}$$

$$=\frac{1}{(1-v^{n+1})(1-v^n)}\Big[(P_{n+1}-P_n)+(v^{n+1}P_n-v^n P_{n+1})\Big] \quad ...(11.7)$$

Further, $P_{n+1}=(C+R_1+vR_2+...+v^{n-1}R_n)+v^n R_{n+1}$

$=P_n+v^n R_{n+1}$.

$\therefore$ From equation (11·7) we get

$$\triangle F_n=\frac{1}{(1-v^{n+1})(1-v^n)}[(v^n R_{n+1})+v^{n+1} P_n-v^n\{P_n+v^n R_{n+1}\}]$$

$$=\frac{1}{(1-v^{n+1})(1-v^n)}[v^n R_{n+1}(1-v^n)-v^n P_n(1-v)]$$

$$=\frac{v^n(1-v)}{(1-v^{n+1})(1-v^n)}\left[\frac{1-v^n}{1-v}R_{n+1}-P_n\right] \quad ...(11·8)$$

$$=\text{a positive constant.}\left[\frac{1-v^n}{1-v}R_{n+1}-P_n\right]$$

$\therefore$ F_n has always the same sign as the quantity in brackets.

$\therefore$ From inequation (11.5), n will be optimal if

$$\frac{1-v^{n-1}}{1-v}R_n-P_{n-1}<0<\frac{1-v^n}{1-v}R_{n+1}-P_n. \quad ...(11.9)$$

From inequation (11.9), we have

$$\frac{1-v^n}{1-v}R_{n+1}-P_n>0$$

or $$R_{n+1}>P_n\cdot\frac{1-v}{1-v^n}$$

or $$R_{n+1}>P_n\Big/\frac{1-v^n}{1-v}$$

or $$R_{n+1}>\frac{C+R_1+vR_2+v^2R_3+...+v^{n-1}R_n}{1+v+v^2+...+v^{n-1}} \quad ...(11·10a)$$

or $$R_{n+1}>\frac{C+\sum_{r=1}^{n}R_r v^{r-1}}{\sum_{r=1}^{n}v^{r-1}} \quad ...(11·10\,b)$$

or next periods cost > *weighted average* of previous costs, since the expression on the R.H.S. of inequation (11·10 a) is the weighted average of all costs upto and including period $n-1$. The weights $1, v, v^2, ..., v^{n-1}$ are the *discount factors* applied to the costs in each period.

The other part of inequation (11·9) can, similarly, be expressed as

$$R_{n+1} < \frac{C + R_1 + vR_2 + v^2R_3 + \ldots + v^{n-1} R_n}{1 + v + v^2 + \ldots + v^{n-1}} \qquad (11.11a)$$

or

$$R_{n+1} < \frac{C + \sum_{r=1}^{n} R_r\, v^{r-1}}{\sum_{r=1}^{n} v^{r-1}} \qquad \ldots(11{\cdot}11b)$$

From expressions (11·10*b*) and (11.11*b*) we conclude that

(*a*) *The machine should be replaced if the next period's cost is greater than the weighted average of previous costs.*

(*b*) *The machine should not be replaced if the next period's cost is less than the weighted average of previous costs.*

The corresponding value of the minimum annual payment x is obtained from equation (11.4) as

$$x = \frac{1-v}{1-v^n} P_n.$$

Further, if x_1 and x_2 are the minimum annual payments for two machines A and B, A will be preferred if $x_1 > x_2$ and *vice versa.*

It may be noted that the replacement policy of section 11·2-1 in which money value is ignored is a special case of this section. As interest rate $i \to 0$, the discount rate $v \to 1$ and expression (11.10*a*) reduces to

$$R_{n+1} > \frac{C + R_1 + R_2 + \ldots + R_n}{1 + 1 + 1 + \ldots\ n \text{ times}}$$

or $\quad R_{n+1} > \dfrac{P_n}{n}$, which is identical to equation (11.2).

In actual practice, this type of replacement problem may be further complicated by the prevailing tax laws. A discussion of tax laws is beyond the scope of this book, but in any real problem the effect of taxes has got to be taken into account.

EXAMPLE 11.2-4

The yearly cost of two machines A and B, when money value is neglected is shown in table 11.7. Find their cost patterns if money value is 10% per year and hence find which machine is more economical.

Table 11-7

Year	1	2	3
Machine A (Rs.)	1,800	1,200	1,400
Machine B (Rs.)	2,800	200	1,400

Solution. When the value of money is 10% per year, the discount rate

$$v=\frac{1}{1+0\cdot10}=\frac{1}{1.1}=0\cdot9091.$$

The discounted cost patterns for machines A and B are shown in table 11.8.

Table 11.8

Year	1	2	3	*Total cost (Rs.)*
Machine A (Discounted cost in Rs.)	1,800	$1,200\times0\cdot9091$ $=1,090.90$	$1,400\times0.9091^2$ $=1,157\cdot04$	4,047.94
Machine B (Discounted cost in Rs.)	2,800	200×0.9091 $=181\cdot82$	$1,400\times0.9091^2$ $=1,157.04$	4,138.86

As total cost for machine A is less than that for machine B, machine A is more economical.

EXAMPLE 11.2-5

A machine costs Rs. 500. Operation and maintenance costs are zero for the first year and increase by Rs. 100 every year. If money is worth 5% every year, determine the best age at which the machine should be replaced. The resale value of the machine is negligibly small. What is the weighted average cost of owning and operating the machine ?

Solution. Discount rate,

$$v=\frac{1}{1+r}=\frac{1}{1+0\cdot05}=0\cdot9524.$$

To find the best replacement age, we enter the calculations in a table. Table 11·9 represents these calculations.

From table 11·9 we find that

$$200<217\cdot66<300,$$

where 200 is the running cost of 3rd year and 300 is that of 4th year. Therefore, the machine should be replaced after *third year.* The weighted average cost of owning and running the machine is *Rs. 217.66.*

Table 11.9

(1) Year of Service (r)	(2) Maintenance cost (R_r)	(3) Discount factor (v^{r-1})	(4) Discounted cost ($R_r . v^{r-1}$)	(5) $C+\sum_{r=1}^{n} R_r v^{r-1}$	(6) $\sum_{r=1}^{n} v^{r-1}$	(7) $\dfrac{C+\sum_{r=1}^{n} R_r v^{r-1}}{\sum_{r=1}^{n} v^{r-1}}$	
1	0	1.0000	0.00	500.00	1.0000	500.00	
2	100	0.9524	95.24	595.24	1.9524	304.98	
3	200	0.9070	181.40	776.64	2.8594	**217.66**	**Replace**
4	300	0.8638	259.14	1,035.78	3.7232	278.26	
5	400	0.8227	329.08	1,364.86	4.5459	300.25	

Machine A **Table 11.10**

(1) Year of Service (r)	(2) Running cost (R_r)	(3) Discount factor v^{r-1}	(4) Discounted cost $R_r\, v^{r-1}$	(5) $C+\sum_{r=1}^{n} R_r v^{r-1}$	(6) $\sum_{r=1}^{n} v^{r-1}$	(7) $\frac{(5)}{(6)}$
1	400	1.0000	400.00	2,900.00	1.0000	2,900.00
2	400	0.9091	363.64	3,263.64	1.9091	1,709.45
3	400	0.8264	330.56	3,594.20	2.7355	1,313.84
4	400	0.7513	300.52	3,894.72	3.4868	1,116.93
5	400	0.6830	273.20	4,167.92	4.1698	999.50
6	500	0.6209	310.45	4,478.37	4.7907	948.23
7	600	0.5645	338.70	4,817.07	5.3552	899.40
8	700	0.5132	359.24	5,176.31	5.8684	881.92
9	800	0.4665	373.20	5,549.51	6.3349	**875.86 Replace**
10	900	0.4241	381.69	5,931.20	6.7590	877.35

Machine B **Table 11.11**

(1) Year of service (r)	(2) Running cost (R_r)	(3) Discount factor (v^{r-1})	(4) Discounted cost ($R_r\, v^{r-1}$)	(5) $C+\sum_{r=1}^{n} R_r\, v^{r-1}$	(6) $\sum_{r=1}^{n} v^{r-1}$	(7) $\frac{(5)}{(6)}$
1	600	1.0000	600.00	1,850 00	1.0000	1,850.00
2	600	0.9091	545.46	2,395.46	1.9091	1,254.75
3	600	0.8264	495.84	2,891.30	2.7355	1,056 95
4	600	0.7513	450.78	3,342.08	3.4868	958.49
5	600	0.6830	409.80	3,751.88	4.1698	899.77
6	600	0.6209	372.54	4,124.42	4.7907	860.92
7	700	0.5645	395.15	4,519.57	5.3552	843.96
8	800	0.5132	410.56	4,930.13	5.8684	**840.11 Replace**
9	900	0.4665	419.85	5,349.98	6.3349	844 52
10	1,000	0.4241	424.10	5,774.08	6.7590	854.28

EXAMPLE 11.2-6.

A manufacturer is offered two machines A and B. A has cost price of Rs. 2,500, its running cost is Rs. 400 for each of the first 5 years and increases by Rs. 100 every subsequent year. Machine B, having the same capacity as A, costs Rs. 1,250, has running cost of Rs. 600 for 6 years, increasing by Rs. 100 per year thereafter. If money is worth 10% per year, which machine should be purchased ? Scrap value of both machines is negligibly small.

Solution. As money is worth 10% per year, the discount rate for both machines is

$$v=\frac{1}{1+r}=\frac{1}{1+0\cdot10}=0\cdot9091.$$

The calculations for machines A and B are entered in tables 11·10 and 11·11 respectively.

From table 11·10 we conclude that for machine A

$$800<875\cdot86<900,$$

where 800 is the running cost during 9th year and 900 is that in 10th year. Hence machine A should be replaced *after 9th year.*

Similarly from table 11·11 for machine B find that

$$800<840\cdot11<900,$$

where 800 is the running cost in 8th year and 900 is that in 9th year. Hence machine B should be replaced *after 8th year.*

Further, since the weighted average cost in 9 years of machine A is Rs. 875·86 and weighted average cost in 8 years of machine B is Rs. 840·11, it is advisable to purchase *machine B.*

EXAMPLE 11.2-7

A scooter costs **Rs. 6,000** when new. The running cost and salvage value (sale price) at the end of the year is given in table 11.12. If the interest rate is 10% per year and running costs are assumed to have occurred at mid year, find when the scooter should be replaced.

Table 11.12

Year	1	2	3	4	5	6	7
Running cost Rs.	1,200	1,400	1,600	1,800	2,000	2,400	3,000
Salvage value Rs.	4,000	2,666	2,000	1,500	1,000	600	600

Solution

As the interest rate is 10% per year, discount rate

$$v=\frac{1}{1+0\cdot10}=0\cdot9091.$$

Table 11·13

1	2	3	4	5	6	7	8	9	10	11	12
Year of service (r)	*Salvage value* S_r	*Running cost (mid-year)* (R'_r)	*Running cost (start of the year)* ($R_r = v^{1/2}R_r'$)	(v^{r-1})	(v^r)	*Discounted running cost* ($R_r v^{r-1}$)	*Discounted salvage value* ($S_r v^r$)	$C+\sum_{r=1}^{n} R_r v^{r-1}$	$C+\sum_{r=1}^{n} R_r v^r - S_r v^r$	$\sum_{r=1}^{n} v^r$	$\left(\frac{10}{11}\right)$
1	4,000	1,200	1,144·2	1·0000	0·9091	1,144·2	3,636·4	7,144·2	3,507·8	0·9091	3,859·0
2	2,666	1,400	1,334·8	0·9091	0·8264	1,211.6	2,203·2	8,355·8	6,152·6	1·7355	3,544·2
3	2,000	1,600	1.525·6	0·8264	0·7513	1,260·8	1,502·6	9,616·6	8,114·0	2·4868	3,262·6
4	1,500	1,800	1,716·2	0·7513	0·6830	1,289.2	1,024·5	10,905.8	9,881·3	3·1698	3,117·2
5	1,000	2,000	1,871.0	0·6830	0·6209	1,302·4	620·9	12,208·2	11,587·3	3·7902	**3056·6 Replace**
6	600	2,400	2,288·4	0·6209	0·5645	1,420·8	338·7	13,629·0	13,290·3	4.3552	3,117·0

The running costs given in table 11.12 occur at mid-year and we can discount them to the start of the year by multiplying by $v^{1/2} = \sqrt{0.9091} = 0.95346$. The remaining calculations are shown in table 11.13. Costs have been calculated upto first decimal place only. It follows from the table that the scooter should be replaced after 5 years.

11.3. Replacement of Items that Fail Suddenly

In the previous section we considered replacement of items that deteriorate with time resulting in increasing maintenance and operation costs. The optimal lives of the items were found by balancing the increased running costs against decreased depreciation. However, there are many real life situations in which items do not deteriorate with time but fail suddenly. A system usually consists of a large number of low cost items that are increasingly liable to failure with age (*e.g.*, failure of some resistor in a radio, television, computer etc.). Sometimes, the failure of an item may cause a complete breakdown of the system. The costs of failure, in such a case will be quite higher than the cost of the item itself. For example, a tube or a condenser in an aircraft costs little, but its failure may result in total collapse of the aircraft. Similarly, failure of an industrial equipment such as a pump in a refinery may close down the entire system and may cause heavy losses due to loss in production, idle labour, wastage and other damages.

It is, therefore, quite important to know in advance as to when the failure is likely to take place so that item can be replaced before it actually fails. Rigorous inspection may be required to detect imminent failures and once they are detected, *preventive replacement* may be quite economical. Quite often, however, it may not be possible to predict the time of failure by direct inspection. In such cases, the time of failure can be predicted from the probability distribution of failure time obtained from past experience. The problem, then, is to find the optimal value of time t which minimizes the total cost involved in the system.

Two types of replacement policies are considered when dealing with such situations :

1. *Individual replacement policy* in which an item is replaced immediately after it fails.
2. *Group replaced policy* in which all items are replaced, irrespective of whether they have failed or not, with a provision that if any item fails before the optimal time, it may be individually replaced.

11.3-1. Individual Replacement Policy

The probability distribution of failure time (or survivals at different ages) can be obtained from *mortality tables* or *life tables* for a particular item on a large sample basis. Let us imagine a large population which is subjected to a given mortality curve for a long time period. Let all deaths (part failures) be immediately replaced by births (replacements) and let there be no other entries or exits. The problem is to determine the *rate of deaths* that occur during a certain time period.

Let us assume that each item in the system fails just before time $t=k$, where k is an integral constant so that no item survives beyond time $k+1$.

Let $f(t)$ denote the number of births (replacements) at time t and p_x denote the probability of an item dying (failing) just before attaining the age $x+1$. Obviously, $\sum_{x=0}^{k} p_x=1$. As $f(t-x)$ denotes the number of births at time $t-x$, these newly borns attain the age x at time t and their *a- priori* probability of dying just before time $t+1$ is p_x. Consequently, the number of deaths of these survivors just before time $t+1$ is $p_x \cdot f(t-x)$.

$\therefore$ Total number of deaths just before time $t+1$

$$= \sum_{x=0}^{k} f(t-x) \cdot p_x. \qquad \ldots(11.12)$$

Since deaths are immediately replaced by births, this quantity equals the births at time $t+1$.

$$\therefore \quad f(t+1)= \sum_{x=0}^{k} f(t-x) \cdot p_x, \text{ where } t=k, k+1, k+2, \ldots \qquad \ldots(11.13)$$

Equation (11.13) is a difference equation in t. Let its trial solution be

$$f(t)=A\, \alpha^t, \text{ where A is a constant.}$$

The difference equation (11.13) becomes

$$A\, \alpha^{t+1} = \sum_{x=0}^{k} A\, \alpha^{t-x} \cdot p_x.$$

Dividing both sides by A α^{t-k}, we get

$$\alpha^{k+1}=\sum_{x=0}^{k} \alpha^{k-x} \cdot p_x.$$

$$\therefore \quad \alpha^{k+1}-\sum_{x=0}^{k} \alpha^{k-x} \cdot p_x=0$$

or $$\alpha^{k+1}-[\alpha^k.p_0+\alpha^{k-1}.p_1+\alpha^{k-2}\,p_2+\dots+p_k]=0 \qquad \dots(11.14)$$

Now we know that the sum of all probabilities is unity.

i.e., $$\sum_{x=0}^{k} p_x=1$$

or $$1-\sum_{x=0}^{k} p_x=0. \qquad \dots(11.15)$$

Comparing equations (11.14) and (11.15) we find that one root of the polynomial (11.14) is $\alpha_0=1$. However, since the polynomial (11.14) has exactly $k+1$ roots, let its remaining roots be $\alpha_1, \alpha_2, \dots, \alpha_k$.

The most general solution of this polynomial has the form

$$f(t)=A_0\,\alpha_0{}^t+A_1\,\alpha_1{}^t+A_2\,\alpha_2{}^t+\dots+A_k\,\alpha_k{}^t$$
$$=A_0+A_1\,\alpha_1{}^t+A_2\,\alpha_2{}^t+\dots+A_k\,\alpha_k{}^t,$$

where $A_0, A_1, A_2, \dots, A_k$ are constants which can be determined from the age distribution at some given point in time. Further, since absolute value of all remaining roots is less than unity, *i.e.*,

$$|\alpha_i|<1 \text{ for all } i,$$
$$\text{as } t\to\infty, \quad \lim_{t\to\infty} f(t)=A_0,$$

which means that the number of *deaths* (*or births*) *per unit time is constant at A_0*.

Let P(x) denote the probability that an item survives beyond x time units. Then

$$P(x)=1-\text{probability that the item dies before } x \text{ time units}$$
$$=1-[p_0+p_1+\dots+p_{x-1}]$$
$$=1-\sum_{i=0}^{x-1} p_i.$$

Clearly, probability of surviving at $x=0$ *i.e.*, P(0)=1.

Now, since birth and death rates have settled to a constant rate A_0, the number of survivors of age x also becomes stable at $A_0\,P(x)$. Moreover, as deaths are always immediately replaced by births, the size N of the total population remains the same, so that

$$N = A_0 \sum_{x=0}^{k} P(x)$$

or

$$A_0 = \frac{N}{\sum_{x=0}^{k} P(x)} \qquad \ldots(11.16)$$

Finally, we have to show that the denominator in (11.16) can be interpreted as the mean age at death. For this we write

$$\sum_{x=0}^{k} P(x).\,1 \text{ as } \sum_{x=0}^{k} P(x).\,\triangle x. \quad [\because \triangle x = (x-1) - x = 1]$$

Now from the knowledge of difference operator $\triangle$ we have

$$\sum_{x=a}^{b} f(x).\,\triangle g(x) = f(b+1).\,g(b+1) - f(a).\,g(a) - \sum_{x=a}^{b} g(x+1).\,\triangle f(x).$$

$$\therefore \quad \sum_{x=0}^{k} P(x) = \sum_{x=0}^{k} P(x).\,\triangle x$$

$$= (k+1)\,P(k+1) - 0.P(0) - \sum_{x=0}^{k} (x+1).\,\triangle P(x)$$

$$= (k+1).\,P(k+1) - \sum_{x=0}^{k} (x+1).\,[P(x+1) - P(x)]$$

$$= (k+1).\,P(k+1) - \sum_{x=0}^{k} (x+1)\left[\left\{1 - \sum_{i=0}^{x} p_i\right\} - \left\{1 - \sum_{i=0}^{x-1} p_i\right\}\right]$$

$$=(k+1).\ P(k+1)-\sum_{x=0}^{k}(x+1)\ (-p_x)$$

$$=\sum_{x=0}^{k}(x+1).\ p_x \text{ (since P }(k+1)=0\text{, by assumption)}$$

= mean age at death.

$\therefore$ From equation (11.16) we have

$$A_0=\frac{\text{N, the size of population}}{\text{mean age at death}}.$$

Hence we conclude that *the age distribution ultimately becomes stable and the ultimate rate of death is obtained by dividing the size of population by the mean age at death.*

11.3-2. Group Replacement Policy

Quite often a system consists of a large number of identical, low cost items which are more and more likely to fail with time. It may be economical to replace all such items at fixed intervals. Such a policy of replacement is called *group replacement policy* and is particularly suitable when the cost of individual item is comparatively small. An important example is of replacing the street light bulbs.

Thus under this policy we replace all items at fixed interval 't' whether they have failed or not and at the same time go on replacing failed items as and when they fail. The problem is to determine an optimum group replacement time interval.

Let N : The total number of items in the system,

N_t : Number of items that fail during time t,

C(t) : The total Cost of group replacement after a time t, so that average cost per unit time is $\frac{C(t)}{t}$,

C_1 : Cost of replacing an item when all the items in that group are replaced simultaneously,

C_2 : Cost of replacing an individual item on failure.

Then, clearly

$$C(t)=C_1N+C_2\ [N_1+N_2+\ldots+N_{t-1}].$$

$\therefore$ Average cost per unit time $=\frac{C(t)}{t}=F(t)$

$$=\frac{C_1\ N+C_2\ [N_1+N_2+\ldots+N_{t-1}]}{t} \quad \ldots(11.17)$$

Now optimum group replacement time 't' will be that period which minimizes the average cost per unit time.

The condition for minimum $F(t)$ is

$$\triangle F(t-1) < 0 < \triangle F(t). \quad \text{...(11.18)}$$

Now $\triangle F(t) = F(t+1) - F(t)$

$$= \frac{C(t+1)}{t+1} - \frac{C(t)}{t}$$

$$= \frac{C(t)+C_2 N_t}{t+1} - \frac{C(t)}{t}$$

$$= \frac{t\{C(t)+C_2 N_t\} - C(t)\{t+1\}}{t(t+1)}$$

$$= \frac{tC_2 N_t - C(t)}{t(t+1)}$$

or $$\triangle F(t) = \frac{C_2 N_t - \frac{C(t)}{t}}{t+1}, \quad \text{...(11·19)}$$

which must be greater than zero for minimum $F(t)$.

i.e., $$\frac{C_2 N_t - \frac{C(t)}{t}}{t+1} > 0$$

or $$C_2 N_t > \frac{C(t)}{t}. \quad \text{...(11·20)}$$

Similarly from $\triangle F(t-1) < 0$, we can prove that

$$C_2 N_{t-1} < \frac{C(t-1)}{t-1}. \quad \text{...(11·21)}$$

From equations (11·20) and (11·21) we get the following group replacement policy :

1. *Group replacement should be made at the end of t^{th} period if the cost of individual replacement for the t^{th} period is more than the average cost per unit time through the end of t periods.*
2. *Group replacement should not be made at the end of t^{th} period if the cost of individual replacement for the t^{th} period is less than the average cost per unit time through the end of t periods.*

EXAMPLE 11·3·1

The following mortality rates have been observed for a certain type of light bulbs :

End of week	1	2	3	4	5	6
Probability of failure to date	0·09	0·25	0·49	0·85	0·97	1·00

There are a large number of such bulbs whi ch are to be kept in

working order. If a bulb fails in service, it costs Rs. 3 to replace but if all the bulbs are replaced in the same operation, it can be done for only Rs. 0·70 a bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. (*a*) What is the best interval between group replacements ? (*b*) At what group replacement price per bulb, would a policy of strictly individual replacement become preferable to the adopted policy ?

Solution. Let the total number of bulbs in use be 1,000. Let p_i be the probability that a new light bulb fails during the *i th* week of its life.

Thus we have

$$p_1 = 0{\cdot}09,$$
$$p_2 = 0{\cdot}25 - 0{\cdot}09 = 0{\cdot}16,$$
$$p_3 = 0{\cdot}49 - 0{\cdot}25 = 0{\cdot}24,$$
$$p_4 = 0{\cdot}85 - 0{\cdot}49 = 0{\cdot}36,$$
$$p_5 = 0{\cdot}97 - 0{\cdot}85 = 0{\cdot}12,$$
$$p_6 = 1{\cdot}00 - 0{\cdot}97 = 0{\cdot}03.$$

Since the sum of all probabilities is unity, all probabilities higher than p_6 must be zero *i.e.*, $p_7 = p_8 = p_9$, etc. $= 0$. Thus all light bulbs are sure to burn out by *the* 6th week.

Further, we assume

(*i*) that light bulbs which fail during a week are replaced just before the end of that week.

(*ii*) that the *actual* percentage of failures during a week for a sub-population of bulbs with the same age is the same as the *expected* percentage of failures during the week for that sub-population.

Let N_i represent the number of replacements made at the end of *i th* week when all the 1,000 bulbs are new initially. Then we have

$$N_0 = N_0 \qquad = 1{,}000,$$
$$N_1 = N_0\, p_1 = 1{,}000 \times 0{\cdot}09 \qquad = 90,$$
$$N_2 = N_0\, p_2 + N_1\, p_1 = 1{,}000 \times 0{\cdot}16 + 90 \times 0{\cdot}09 \qquad = 168,$$
$$N_3 = N_0\, p_3 + N_1\, p_2 + N_2\, p_1 = 1{,}000 \times 0{\cdot}24 + 90 \times 0{\cdot}16 + 168 \times 0{\cdot}09 = 269,$$
$$N_4 = N_0\, p_4 + N_1\, p_3 + N_2\, p_2 + N_3\, p_1 = 1{,}000 \times 0{\cdot}36 + 90 \times 0{\cdot}24 + 168 \times 0{\cdot}16 + 269 \times 0{\cdot}09 = {}^{*}432,$$

$$N_5=N_0\,p_5+N_1\,p_4+N_2\,p_3+N_3\,p_2+N_4\,p_1=1{,}000\times0{\cdot}12$$
$$+90\times0.36+168\times0{\cdot}24+269\times0{\cdot}16+432\times0{\cdot}09=274,$$

$$N_6=N_0\,p_6+N_1\,p_5+N_2\,p_4+N_3\,p_3+N_4\,p_2+N_5\,p_1$$
$$=1{,}000\times0{\cdot}03+90\times0{\cdot}12+168\times0{\cdot}36+269\times0{\cdot}24$$
$$+432\times0{\cdot}16+274\times0{\cdot}09=260,$$

$$N_7=0+N_1\,p_6+N_2\,p_5+N_3\,p_4+N_4\,p_3+N_5\,p_2+N_6\,p_1$$
$$=900\times0{\cdot}03+168\times0{\cdot}12+269\times0{\cdot}36+432\times0{\cdot}24$$
$$+274\times0{\cdot}16+260\times0{\cdot}09=291,$$

and so on.

Thus we find that the number of bulbs failing each week increases till the 4th week, then decreases and again increases from 7th week. Thus N_i will continue to oscillate till the system attains a steady state.

$$\text{Average life of light bulbs}=\sum_{i=1}^{6} ip_i$$
$$=1\times0{\cdot}09+2\times0{\cdot}16+3\times0{\cdot}24+4\times0{\cdot}36$$
$$+5\times0{\cdot}12+6\times0{\cdot}03$$
$$=3{\cdot}35.$$

$$\therefore\ \text{Average number of failures per week}=\frac{1000}{3{\cdot}35}=299.$$

$\therefore$ Cost of individual replacement of bulbs

$$=\text{Rs. }3\times299=\text{Rs. }897.$$

Since the replacement of all the 1,000 bulbs in one operation costs Rs. 0·70 per bulb and replacement of an individual bulb costs Rs. 3, the total cost of replacement is

End of week	*Total cost of group replacement* (*Rs.*)	*Average cost per week* (*Rs.*)
1.	$1{,}000\times0{\cdot}70+90\times3=970$	970·00
2.	$1{,}000\times0{\cdot}70+3(90+168)=1{,}474$	937·00
3.	$1{,}000\times0{\cdot}70+3(90+168+269)$ $=2{,}281$	760·33

(*a*) As the average minimum cost is in the 2nd week, it is optimal to have a group replacement after every *two weeks*.

(*b*) Let Rs. x be the group replacement price per bulb. Then

$$\text{Rs. } 897 < \frac{1{,}000\, x + 3(90+168)}{2} \quad \therefore\ x > \text{Rs. } 1{\cdot}02.$$

EXAMPLE 11·3·2

(*a*) At time zero, all items in a system are new. Each item has a probability p of failing immediately before the end of first month of life and a probability $q(=1-p)$ of failing immediately before the end of the second month (*i.e.*, all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures $f(x)$ at the end of month x is given by

$$f(x) = \frac{N}{1+q}\left[1-(-q)^{x+1}\right],$$

where N is the number of items in the system.

(*b*) If the cost per item of individual replacement is C_1 and the cost per item of group replacement is C_2, find the condition under which

(*i*) a group replacement policy at the end of each month is the most profitable.

(*ii*) a group replacement policy at the end of every other month is the most profitable.

(*iii*) no group replacement policy is better than a policy of pure individual replacement.

[*Delhi M.Sc.* (*Math.*) *1971, 76*; *Meerut M.Sc.* (*Stat.*) *1970*]

Solution

(*a*) Let N_i be the number of items expected to fail at the end of ith month.

$\therefore$ N_1 = Number of items expected to fail at the end of 1st month

$= N.p$

$= N(1-q)$,

N_2 = Number of items expected to fail at the end of 2nd month

$= Nq + N_1 p$

$= Nq + N_1(1-q)$

$= Nq + N(1-q)^2$

$= N(1-q+q^2)$,

N_3 = Number of items expected to fail at the end of 3rd month

$$=N_1q+N_2p$$
$$=N(1-q)q+N(1-q+q^2)(1-q)$$
$$=N(1-q+q^2-q^3)$$
$$N_n=N[1-q+q^2-q^3+\dots+(-q)^n].$$

$\therefore$ $f(x)=$ Number of items expected to fail at the end of month x

$$=N[1-q+q^2-q^3+\dots+(-q)^x]$$
$$=N\left[\frac{1-(-q)^{x+1}}{1-(-q)}\right]=\frac{N}{1+q}\left[1-(-q)^{x+1}\right]$$

(*b*) Average life of an item $=1.p+2p=1(1-q)+2q=1+q$.

$\therefore$ Average number of failures $=\dfrac{N}{1+q}$.

$\therefore$ Cost of individual replacement $=\dfrac{N}{1+q}.C_1$.

(*i*) For a group replacement policy at the end of each month, the cost of replacement is NC_2.

This replacement policy is most profitable if

$$\frac{N}{1+q}C_1>NC_2.$$

(*ii*) For a group replacement policy at the end of every other month, the cost of replacement is $NC_2+N_1C_1=NC_2+NpC_1$. Therefore the average cost per month is $\dfrac{NC_2+NpC_1}{2}$.

This replacement policy is most profitable if

$$\frac{N}{1+q}C_1>\frac{NC_2+NpC_1}{2}.$$

(*iii*) No group replacement policy is better than a policy of pure individual replacement if

$$NC_2>\frac{N}{1+q}C_1,$$

and

$$\frac{NC_2+NpC_1}{2}>\frac{N}{1+q}C_1.$$

11.4 Staffing Problems

Problems concerning recruitment and promotion of staff can sometimes be analysed in a manner similar to that used in replacement problems in industry. The next two examples will make the idea clear.

EXAMPLE 11.4-1.

An airline requires 250 assistant hostesses, 350 hostesses and 50 supervisors. Girls are recruited at age 21 and, if in service, they retire at age 60. Given the 'life' table (table 11·14), determine

(*i*) How many girls should be recruited each year ?

(*ii*) At what age promotions should take place ?

Table 11.14

Age (years)	*No. in service*	*Age (years)*	*No. in service*
21	1,000	41	120
22	700	42	112
23	500	43	105
24	400	44	100
25	300	45	92
26	260	46	88
27	230	47	80
28	210	48	72
29	195	49	65
30	180	50	60
31	170	51	53
32	165	52	45
33	160	53	40
34	155	54	32
35	150	55	26
36	145	56	20
37	140	57	18
38	135	58	15
39	130	59	10
40	125	60	—

Solution

If 1,000 girls had been recruited each year for the past 39 years, the total number of them serving upto the age of 59 years $=6{,}603$. Total number of girls required in the airline

$$=250+350+50=650.$$

(*i*) $\therefore$ No. of girls to be recruited every year in order to maintain a strength of $650 = \frac{1,000}{6,603} \times 650 = \mathbf{98}$ (approx.).

(*ii*) Let the assistant hostesses be promoted at the age x. Then upto age $x-1$ year, number of assistant hostesses required$=250$. Now out of 650 girls, 250 are assistant hostesses; therefore out of 1,000, their number is $250/650 \times 1,000 = 385$ (approx.).

From table 11.14, this number is available upto the age of 24 years.

$\therefore$ Promotion of assistant hostesses is due in the *25th year.*

Now out of 650 girls, 350 are hostesses. Therefore, if we recruit 1,000 girls, the no. of hostesses will be $\frac{350}{650} \times 1,000 = 538$ (approx.).

$\therefore$ Total number of assistant hostesses and hostesses in a recruitment of $1,000 = 385 + 538 = 923$.

$\therefore$ No. of supervisors required$=1,000-923=77$.

From table 11.14, this number is available upto the age of 47 years.

$\therefore$ Promotion of hostesses is due in the *48th year.*

EXAMPLE 11.4-2

A faculty in a college is planned to rise to a strength of 50 staff members and then to remain at that level. The wastage of recruits depends upon their length of service and is as follows :

Year	:	1	2	3	4	5	6	7	8	9	10
Total % left up to the end of year	:	5	35	56	65	70	76	80	86	95	100

(*i*) Find the number of staff members to be recruited every year.

(*ii*) If there are seven posts of Head of Deptt. for which length of service is the only criterion of promotion, what will be average length of service after which a new entrant should expect promotion ?

Solution

Let us assume that the recruitment is 100 per year. Then, when the equilibrium is attained, the distribution of length of service of staff members will be as follows :

Table 11·15

Year	*No. of staff members*
0	100
1	95
2	65
3	44
4	35
5	30
6	24
7	20
8	14
9	5
10	0
Total	432

(*i*) Thus if 100 staff members are recruited every year, the total number of staff members after 10 years of service $=432$.

To maintain a strength of 50, the number to be recruited every year$=\frac{100}{432}\times 50=11\cdot 6$.

In the calculations above we have assumed that those staff members who completed x years' service but left before $x+1$ years' service, actually left immediately before completing $x+1$ years. If it is assumed that they left immediately after completing x years' service, the total number will become $432-100=332$ and the required intake will be $\frac{50\times 100}{332}=15$. In actual practice they may leave at any time in the year so that reasonable number of recruitments per year$=\frac{11\cdot 6+15}{2}=13$ (nearly).

(*i*) With 50 staff members in the faculty, the distribution of the completed length of service is as given in table 11·16.

Table 11.16

Year	*No. of staff members*
0	$\frac{50\times 100}{432}=11.6\simeq 12$
1	$\frac{50\times 95}{432}=11$
2	8
3	5
4	4
5	4
6	3
7	2
8	2
9	0
10	0

Thus the promotion of a newly recruited staff member will be due after completing 5 years and before putting in 6 years of service ($\because$ No. of staff members having 5 years' service is $3+2+2=7$).

11·5 Miscellaneous Replacement Problems

There are many replacement situations which do not clearly fall under the categories discussed in the previous sections. We shall consider two such problems here.

EXAMPLE 11·5·1

A piece of equipment either can fail completely, so that it has to be scrapped (no salvage value), or may suffer a minor defect which can be repaired. The probability that it will not have to be scrapped before age t is $f(t)$ and the conditional probability that it will need a repair in the instant t to $t+dt$, knowing that it was in running order at age t, is $r(t).\,dt$. The probability of a repair or complete failure is dependent only on the age of the equipment, and not on the previous repair history.

Each repair costs Rs. C, and complete replacement costs Rs. K. For some considerable time, the policy has been to replace only on failure.

(*a*) Derive a formula for the expected cost per unit time of the present policy of replacing only on failure.

(*b*) It has been suggested that it might be cheaper to scrap equipment at some fixed age T, thus avoiding the higher risk of repairs with advancing age. Show that the expected cost per unit time of such policy is

$$\frac{C\displaystyle\int_0^T f(u).\,r(u).\,du+K}{\displaystyle\int_0^T f(u).\,du}$$

[*Kuru. M.Sc. (Math.) 1975*]

Solution

Since $f(t)$ is the probability that the equipment will not have to be scrapped before age t, $\displaystyle\int_0^\infty f(t).\,dt=1$.

Further, it is given that the probability that equipment will need a repair in the interval t and $t+dt$, knowing that it was running

at age t is

$$r(t).\, dt.$$

(*a*) The probability that equipment will need a repair between age u and $u+du$ is

$$f(u).\, r(u).\, du$$

$$\therefore \quad \text{Expected cost of repair} = C\int_0^{\infty} f(u).\, r(u).\, du,$$

$$\text{and total expected cost} \qquad = \frac{C\int_0^{\infty} f(u).\, r(u).\, du + K}{\int_0^{\infty} f(u).\, du}$$

$$= C\int_0^{\infty} f(u).\, r(u).\, du + K \;;\; \text{since} \int_0^{\infty} f(u).\, du = 1.$$

(*b*) If the equipment is scrapped at a fixed age T, the expected cost of repair is

$$\int_0^{T} f(u).\, r(u).\, du,$$

and the total expected cost up to age T is

$$C\int_0^{T} f(u).\, r(u).\, du + K.$$

$\therefore$ Expected cost per unit time is given by

$$F(t) = \frac{C\int_0^{T} f(u).\, r(u)\, du + K}{\int_0^{T} f(u).\, du}.$$

EXAMPLE 11·5·2

Automobile batteries are manufactured by a firm at a factory cost of Rs. 20 each. The mortality table for the battery life is given in table 11.17. The batteries are covered under a guarantee policy such that if a battery fails during the first month after purchase, full

price, of the new battery is refunded ; a failure in second month carries a refund of 19/20 of the full price, in third month 18/20 and so on till the 20th month, during which a failure carries a refund of 1/20 of the full price. What should be the break-even selling price of the batteries ?

Table 11·17

Month	*Probability of failure in next month*	*Month*	*Probability of failure in next month*
0	0.05	11	0.01
1	0.00	12	0.01
2	0.00	13	0.01
3	0.00	14	0.01
4	0.00	15	0.015
5	0.00	16	0.020
6	0.00	17	0.025
7	0.00	18	0.030
8	0.00	19	0.035
9	0.00	20 and above	0.785
		Total	1,000
10	0.00		

Solution

Let p_i be the probability that a new battery will fail during $i+1$st month after purchase and let X be the break-even price.

$\therefore$ Average refund for a battery that fails is

$$Y=\sum_{i=0}^{19}\frac{20-i}{20}\,Xp_i.$$

$$\therefore \quad Y=X\Big[0.05\times 1+0{\cdot}00\Big\{\frac{19}{20}+\frac{18}{20}+\frac{17}{20}+\frac{16}{20}+\frac{15}{20}+\frac{14}{20}+\frac{13}{20}+\frac{12}{20}+\frac{11}{20}\Big\}+0.01\Big\{\frac{10}{20}+\frac{9}{20}+\frac{8}{20}+\frac{7}{20}\Big\}$$

$$+0.015\times\frac{6}{20}+0.020\times\frac{5}{20}+0.025\times\frac{4}{20}+0.030\times\frac{3}{20}+0.035\times\frac{2}{20}+7.85\times\frac{1}{20}\Big]$$

$$=X\Big[0.05+\frac{0.34}{20}+\frac{0.09}{20}+\frac{0.10}{20}+\frac{0{\cdot}10}{20}+\frac{0.09}{20}+\frac{0.07}{20}+\frac{0.785}{20}\Big]$$

$$=X\Big[\frac{2.575}{20}\Big]=0.12875\ X.$$

Now the break-even price X is the sum of the factory cost and the expected refund.

$\therefore \quad X=20+0.12875\ X$

or $\quad X=\text{Rs. } 23$ (approx.).

11·6 Renewal Theory

If the probability distribution of failure time of items is known, one can solve replacement problems analytically. The mathematical technique used to solve these replacement problems is called *renewal theory*. This theory treats the life of the item as a random variable. In statistical terminology, a *renewal* takes place when an old item is replaced by a new one or it is repaired so that the probability density function (p.d.f.) of its future life time is same as that of the new item. We give below the basic principle bypassing the details of a renewal process; readers interested in the basic theory itself may consult Cox. [3].

Renewal Density Theorem

Let the life of each item be denoted by random variable x_i, $i=1, 2, 3, \ldots$ and let each item be replaced immediately on failure. Then, if we start using the first item at time $t=0$, the nth will start being used at time $t=x_1+x_2+\ldots+x_{n-1}$. Let x_i be identically and independently distributed with probability distribution F_x and let the probability of event x be p_x.

The probability that a renewal occurs during a small interval $(t, t+dt)$ is called *replacement rate* at time t, where t is measured from the start of use of the first item. It is denoted by $h(t) \cdot dt$ and is called *renewal density function.*

Thus $h(t) \cdot dt$ = probability that a renewal occurs in time interval $(t, t+dt)$, and this probability = prob. that first machine fails in time interval $(t, t+dt)$

+ Prob. that second machine fails in time interval $(t, t+dt)+\ldots$

$\therefore \quad h(t)=f_1(t)+f_2(t)+\cdots$

$$=\sum_{r=1}^{\infty} f_r(t), \qquad \ldots(11\cdot22)$$

where $f_1(t)$ = density function of a random variable with probability distribution F_x,

must be replaced or retreaded when it wears smooth. However, retreading is possible only if the tyre walls have not deteriorated. For the data given in table 11.18, find the average cost per thousand km. if

(*a*) the old tyre is always replaced by a new tyre,

(*b*) the old tyre is retreaded whenever possible. Of course, a tyre can be retreaded only once.

Table 11·18

Age (thousands of km.) :	10	11	12	13	14	15	16
Probability that tyre becomes smooth :				0·15	0·15	0·25	0.25
Probability that smooth trye can be retreaded :				0·8	0·8	0·7	0·7
Probability of failure of a retreaded trye :	0·1	0·15	0·2				

Solution

Let us assume that smoothness occurs at the middle of km. intervals. Then the average age in thousands of km. at which a new tyre becomes smooth is

$$13{\cdot}5\times0{\cdot}15+14{\cdot}5\times0{\cdot}15+15{\cdot}5\times0{\cdot}25+16{\cdot}5\times0{\cdot}25=12{\cdot}2.$$

Thus the cycle time is 12·2 and the cost per cycle is Rs. 200.

$\therefore$ Average cost per thousand km. $=\dfrac{200}{12{\cdot}2}=$ Rs. 16·40.

Similarly, the average age in thousands of km. at which the tyre must be retreaded is $10{\cdot}5\times0{\cdot}1+11.5\times0{\cdot}15+12{\cdot}5\times0{\cdot}2=5{\cdot}275$.

$\therefore$ Average cost of retreading per thousand km.

$$=\frac{65}{5{\cdot}275}=\text{Rs. }12{\cdot}32.$$

In order to find the cost per thousand km. of new tyre and a retreaded one, whenever possible, we must find the total average life of both.

Now total average life of both = Average life of a new tyre + probability of using a retreaded tyre $\times$ average life of retreaded tyre.

Now probability of using a retreading tyre

= Probability of smoothness at each age $\times$ probability that the tyre can be retreaded.

$=0{\cdot}15\times0{\cdot}8\times0{\cdot}15\times0{\cdot}8+0{\cdot}25\times0{\cdot}7+0{\cdot}25\times0{\cdot}7$

$=0{\cdot}24+0{\cdot}35=0{\cdot}59.$

Total average life till we again have new tyre
$=12.2+0.59\times 5{\cdot}275$
$12{\cdot}2+3{\cdot}11=15{\cdot}31$.

Expected cost over the cycle $=$ Rs. $(200+0.59\times 65)=$ Rs. 238.35.

$\therefore$ Average cost per thousand km. if we retread whenever possible

$$=\text{Rs. } \frac{238.35}{15.31}=\text{Rs. } 15.57.$$

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EXERCISES

1. Explain how the theory of replacement is used in the following problems :

   (*i*) Replacement of items whose maintenance cost varies with time.

   (*ii*) Replacement of items that fail completely.

   (*Bombay M. Com. 1974*)

2. Find the optimum replacement policy which minimizes the total of all future discounted costs for an equipment which costs Rs. a and which needs maintenance costs of Rs. $C_1, C_2, \ldots, C_n$, etc. $(C_{n+1}>C_n)$ during the first year, second year, etc., respectively ; and further d is the depreciation value for unit of money during a year.

Section 11 2·1

3. A firm is thinking of replacing a particular machine whose cost price is Rs. 12,200.. The scrap price of this machine is only Rs. 200. The maintenance costs are found to be as follows :

Year	1	2	3	4	5	6	7	8
Maintenance Cost (Rs.)	200	500	800	1,200	1,800	2,500	3,200	4,000

Determine when the firm should get the machine replaced.

(*Ans.* 7th year)

4. A fleet owner finds from his past records that the costs per year of running a truck whose purchase price is Rs. 6,000 are as given below.

Year	1	2	3	4	5	6	7
Running Cost (in Rs.)	1,000	1,200	1,400	1,800	2,300	2,800	3,400
Resale value (in Rs.)	3,000	1,500	750	375	200	200	200

Determine at what age is the replacement due ?

[*Delhi M. Sc. (Math.) 1969*]

(*Ans.* End of 5th year)

5. Following table gives the running costs per year and resale prices of a certain equipment whose purchase price is Rs. 5,000 :

Year	1	2	3	4	5	6	7	8
Running costs (in Rs.)	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,000
Resale value (in Rs.)	3,500	2,500	1,700	1,200	800	500	500	500

At what year is the replacement due ?

[*Bombay M. Com. 1975*]

(*Ans.* End of 4th year)

Section 11 2·2

6. Purchase price of a machine is Rs. 3,000 and its running cost is given in the table below. If the discount rate is 0·90, find at what age the machine should be replaced.

Year	1	2	3	4	5	6	7
Running cost (Rs.)	500	600	800	1,000	1,300	1,600	2,000

(*Ans.* After 5 years)

7. The cost of a new machine is Rs. 5,000. The maintenance cost of nth year is given by $R_n = 500\ (n-1)$; $n = 1,\ 2, \ldots$. Suppose that the discount rate per year is 0.05. After how many years,

it will be economical to replace the machine by a new one ?

[*Agra M.* (*Stat.*) *1974*]

(*Ans.* After 9 years)

8. A truck is priced at Rs. 60,000 and running costs are estimated at Rs. 6,000 for each of the first four years, increasing by Rs. 2,000 per year in the fifth and subsequent years. If money is worth 10% per year, when should the truck be replaced ? Assume that the truck will eventually be sold for scrap at a negligible price.

[*Delhi M.B.A. 1974, 1976*]

(*Ans.* After 9 years)

9. It is required to find the optimal replacement time of a certain type of equipment. The initial cost of equipment is C. Salvage value and repair cost are given by $S(t)$ and $R(t)$ respectively. The cost of capital is r per cent and T is the time period of replacement cycle.

   (*i*) Show that the present value of all future costs associated with a policy of equipment replacement after time T is

$$\left(\frac{1}{1-e^{-rt}}\right)\left[C-s(t).\,e^{-rt}+\int_0^T R(t).\,e^{-rt}\,dt\right].$$

   (*ii*) The optimal value of T is given by

$$R(t)-S'(t)+S(t).\,r=r^k/(1-e^{-rt}),$$

   where k is the present value of the cycle.

[*I.S.I. Dip. 1976*]

10. If you wish to have a return of 10% per annum on your investment, which of the following plans would your prefer ?

	Plan A	*Plan B*
	(*in rupees*)	
Ist cost :	2,00,000	2,50,000
Scrap value after 15 years:	1,50,000	1,80,000
Excess of annual revenue : over annual disbursement	25,000	30,000

[*Meerut M. Com. 1973*]

(*Ans.* Plan A)

Sections 11.3 to 11.6

11. A large population is subject to a given mortality curve for a long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Show that the age distribution becomes stable and that the number of deaths per unit time becomes constant.

12. The following mortality rates have been observed for a certain type of light bulbs.

Week :	1	2	3	4	5
Per cent failing: by week end	10	25	50	80	100

There are 1,000 bulbs in use and it costs Rs. 2 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously, it would cost 50 paise per bulb. It is proposed to replace all the bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced ?

[*Meerut M.Sc. (Math.) 1973*; *Delhi M.Sc. (Math.) 1972*]

(*Ans.* Every two weeks)

13. The probability P_n of failure just before age n is shown below. If individual replacement costs Rs. 1.25 and group replacement costs Rs. 0.50 per item, find the optimal group replacement policy.

n :	1	2	3	4	5	6	7	8	9	10	11
p_n :	0.01	0.03	0.05	0.07	0.10	0.15	0.20	0.15	0.11	0.08	0.05

[*Bombay M. Com. 1975*]

(*Ans.* After every 6 weeks)

14. The following failure rates have been observed for a certain type of light bulbs :

End of week	*Probability of failure to date*
1	0.05
2	0.13
3	0.25
4	0.43
5	0.68
6	0.88
7	0.96
8	1.00

The cost of replacing an individual failed bulb is Rs. 1.25. The decision is made to replace all bulbs simultaneously at fixed intervals, and also to replace individual bulbs as they fail in service. If the cost of group replacement is 30 paise per bulb, what is the best interval between group replacements ? At what group replacement price per bulb would a policy of strictly individual replacement become preferable to the adopted policy ?

[*Madras I.I.T. (M. Tech.) 1978, Bombay B.Sc. (Stat.) 1974, 75*]

(*Ans.* After every third week, 49 paise.)

15. A computer contains 10,000 resistors. When any resistor fails, it is replaced. The cost of replacing a resistor individually is Re. 1 only. If all the resistors are replaced at the same time,

the cost per resistor would be reduced to 35 paise. The per cent surviving at the end of month t is given below.

Month	:	0	1	2	3	4	5	6
Per cent surviving at the end of month	:	100	97	90	70	30	15	0

What is the optimum replacement plan ? [*I.S.I. Dip. 1971*]

(*Ans.* Replace every 3 months)

16. Truck tyres which fail in service can cause expensive accidents. A failure in service is estimated to cost Rs. 2,000 excluding the cost of replacing the blown tyre. A new tyre costs Rs. 800 and has the mortality shown in the table below. If the tyres are to be replaced after covering a certain fixed number of km. or on failure, whichever occurs first, determine the replacement policy that minimizes the average cost per km.

Truck Tyre mortality

Age at failure (*km.*)	*Probability of failure*
$\leqslant$10,000	0.000
10,001 — 13,000	0.035
13,001 — 16,000	0.083
16,001 — 19,000	0.190
19,001 — 22,000	0.475
22,001 — 25,000	0.217
	1.000

[**Hint :** Assume that failure takes place at the exact ages 11,500, 14,500, 17,500, etc.]

17. A certain piece of equipment is very difficult to adjust. During a period when no adjustment is made, the running cost increases linearly with time at the rate of Rs. b per hour. The running cost immediately after an adjustment is not known precisely until the adjustment has been made. Before the adjustment, the resulting running cost x is deemed to be a random variable x with density function $f(x)$. If each replacement costs Rs. k, when should replacement be made ?

[*Meerut M.Sc. (Math.) 1970*]

18. If the life of electric light bulbs follows the distribution

$$f(x)dx=\lambda e^{-\lambda x}.\,dx \qquad (\lambda>0;\ 0\leqslant x<\infty),$$

find the renewal density $h(t)$ after the end of the period $(0, t)$.

If there are M points in a building, how many bulbs would you expect to replace within a period (t_1, t_2), where $t_2>t_1$?

INVENTORY CONTROL

More operations research has been directed toward inventory control than toward any other area in business, military and industry. That is why much more number of models are available for inventory control than for any other.

It was in 1915 that F.W. Harris developed an economic-lot-size equation. This equation minimized the inventory carrying and setup costs for a known and constant demand. Research work in this area was taken up by mathematicians, economists and industrial engineers and by 1950 a lot of literature was available. However, most of the tools and techniques being currently used in inventory control area have been developed in the recent past only.

The models described in the current chapter are suited to specific inventory problems only. We shall start our discussion with very simple situations and move towards more complex ones. It should, however, be borne in mind that no one of the models developed here can be applied *in toto* to any specific situation. But once the method of constructing the model is clearly understood, one can make the necessary modifications to take care of the specific situation one comes across.

12

Inventory Models

12.1. Introduction

This chapter presents the kind of analysis which develops mathematical models of inventory processes. Efforts will be made to develop not a single general model but a wide variety of models each for a specific situation.

An *Inventory* consists of usable but idle resources such as men, machines, materials or money. When the resource involved is a material, the inventory is also called '*stock*'. An inventory problem is said to exist if either the resources are subjected to control or if there is at least one such cost that decreases as inventory increases. The objective is to minimize total (actual or expected) cost. However, in situations where inventory affects demand, the objective may also be to maximize profit.

12.2. Necessity for Maintaining Inventory

Though inventory of materials is an idle resource (since the materials lie idle and are not to be used immediately), almost every business must maintain it for efficient and smooth running of its operations. If an enterprise has no inventory of materials at all, on receiving a sales order it will have to place order for purchase of raw materials, wait for their receipt and then start production. The customer will thus have to wait for a long time for the delivery of the goods and may turn to other suppliers resulting in loss of business for the enterprise. Maintaining an inventory is necessary because of

the following reasons :

1. It helps in smooth and efficient running of an enterprise.
2. It provides service to the customer at a short notice.
3. In the absence of inventory, the enterprise may have to pay high prices because of piece-meal purchasing. Maintaining of inventory may earn price discount because of bulk purchasing.
4. It reduces product costs since there is an added advantage of batching and long, uninterrupted production runs.
5. It acts as a buffer stock when raw materials are received late and shop rejections are too many.
6. Process and movement inventories (also called pipeline stocks) are quite necessary in big enterprises where significant amounts of times are required to tranship items from one location to another.
7. It helps in maintaining economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.

However, too often inventories are wrongly used as a substitute for management. For example, if there are large finished goods inventories, inaccurate sales forecasting by marketing deptt. may never be apparent. Similarly, a production foreman who has large in-process inventories may be able to hide his poor planning since there is always something to manufacture. Furthermore, inventory means unproductive 'tied up' capital of the enterprise. Maintenance of inventory costs additional money to be spent on personnel, equipment, insurance, etc. Thus excess inventories are not at all desirable.

This necessitates *controlling* the inventories in the most useful way.

12.3. Variables in an Inventory Problem

There are two types of variables used in an inventory model :

(*i*) Controlled variables

(*ii*) Uncontrolled variables.

Controlled Variables

Variables that may be controlled separately or in combination are :

1. *The quantity acquired i.e. how much.* The quantity may be acquired by purchase, production or some other means. This may be fixed separately for every type of resource or collectively for all items.

2. *The frequency or timing of acquisition i.e., how often or when.* Either both or only one of these variables may be under the direct control of the decision maker.

3. *The completion stage of stocked items.* The more finished the stock items, the lesser the delay in meeting demands but the higher the cost of holding them in stock. The less finished the stock items, the longer the time in meeting demands but lesser the cost of holding in stock.

Most inventory models deal with only the first two control variables and only these two will be considered in the discussion that follows.

Uncontrolled Variables

The following are the variables that may be uncontrolled :

1. *Holding costs.* They are assumed to vary directly with the size of inventory as well as the time the item is held in stock. Various components of holding cost are :

(*a*) *Invested Capital Cost.* This is, by far, the most important component. It means an interest charge; careful investigation is required to determine its rate.

(*b*) *Record-keeping and Administrative cost.* There is no use of keeping stocks unless one can easily know whether or not the required item is in stock. This signifies the need of keeping funds for record-keeping and necessary administration.

(*c*) *Handling Costs.* These include all costs associated with movement of stock, such as cost of labour, overhead cranes, gantries and other machinery used for this purpose.

(*d*) *Storage Costs.* These consist of rent for space or depreciation and interest if the space is owned.

(*e*) *Depreciation, Deterioration and Obsolescence Costs.* They are especially important for fashion items or items undergoing chemical changes during storage.

(*f*) *Taxes and Insurance Costs.* All these costs require careful study and generally amount to 1% to 2% of the invested capital per month.

2. *Shortage Costs or Stock-out Costs.* These costs are associated with either a delay in meeting demands or the inability to meet it at all. Therefore, shortage costs are usually interpreted in two ways. In case the unfilled demand can be filled at a late stage (backlog case), these costs are proportional to quantity that is short as well as the delay time. They represent less of goodwill and cost of idle equipment. In case the unfilled demand is lost (no backlog case), these costs become proportional to only the quantity that is short. These result in cancelled orders, lost sales, profit and even the business itself.

3. *Setup Costs.* These include the fixed cost associated with placing of an order or setting up a machinery before starting production. They include costs of purchase, requisition, followup, receiving the goods, quality control, etc. Also called *order costs* or *replenishment costs*, they are assumed to be independent of the quantity ordered or produced.

4. *Purchase Price or Production Costs.* Purchase price per unit item is affected by the quantity purchased due to *quantity discounts or price breaks.* Production cost per unit item depends upon the length of production runs, for long uninterrupted production runs it is lower because of greater efficiency of men and machines. The order quantity, therefore, must be suitably modified to take advantage of these price discounts.

5. *Salvage Costs or Selling Price.* When the demand for an item is affected by its quantity in stock; the decision model of the problem depends upon the profit maximization criterion and includes the revenue from the sale of the item. Salvage costs are generally combined with the storage costs and not considered independently.

6. *Demand.* It is the number of items required per period. It may not be the number of items sold, since some demand may remain unfilled due to shortage or delays.

The demand pattern of an item may be either deterministic or probabilistic. It is deterministic if the quantities required in future periods are known exactly. Further, the known demand may be fixed or variable with time. Such demands are called *static* and *dynamic* demands respectively.

The demand is probabilistic if the requirement over a time period is not known with certainty but can be described by a known probability distribution. A probabilistic demand may be either stationary or non-stationary over time.

Further, the demand over a given period may be satisfied *instantaneously* at the beginning of the period or *uniformly* during the period.

7. *Lead Time or Delivery Lag.* It is the time between placing an order and its receipt in stock. Lead time may be deterministic or probabilistic. If the lead time is zero, there is no need to order in advance. If it is not zero but known with certainty, one has to order in advance by an amount equal to lead time, provided the demand is also known. If the demand or lag time are known only probabilistically, an order should be placed only after considering the *expected costs* of holding and shortage over the lead time.

8. *Number of Items.* An inventory system may comprise of more than one item. This aspect is especially important in case of some interaction between different items *e.g.*, different items competing for a limited space.

9. *Stock Replenishment.* It may occur instantaneously or uniformly. Instantaneous replacement occurs when the items are purchased from outside while uniform replenishment takes place when the items are manufactured in the enterprise. In general, a system may operate with positive lead time as well as with uniform stock replenishment.

10. *Time Horizon.* It is the time period over which the size of inventory will be controlled. It depends upon the demand of the item and may be finite or infinite.

The above variables are the basic characteristics to be considered while analysing an inventory problem. However, of all, demand is the most important characteristic since it determines the manner of analysing and solving inventory problem.

12.4. Classification of Inventory Models

Inventory Models

- Elementary Models
  - Deterministic
    - Model 1 (No shortage permitted)
      - 1 (*a*) Demand rate uniform, production rate infinite.
      - (*b*) Demand rate non-uniform production rate infinite.
      - (*c*) Demand rate uniform, production rate finite.
    - Model 2 (shortage permitted)
      - (*a*) Demand rate uniform, production rate infinite.
      - (*b*) Demand rate uniform, production rate infinite, time interval fixed.
      - (*c*) Demand rate uniform, production rate finite.
  - Probabilistic
    - 3 (*a*) Instantaneous demand, setup cost zero, stock levels discrete, lead time zero.
      - (*b*) Same as 3 (*a*) except that stock levels are continuous.
    - 4 (*a*) Continuous demand, setup cost zero, stock levels discrete, lead time zero.
      - (*b*) Same as 4 (*a*) except that stock levels are continuous.
    - 5 (a) Continuous demand, setup cost zero, stock levels discrete, lead time significant.
      - (*b*) Same as 5(*a*) except that stock levels are continuous.
- Model with Price breaks
- Models with restrictions

12.5. Inventory Models with Deterministic Demand

It is extremely difficult to formulate a single general inventory model which takes into account all variations in real systems. In fact, even if such a model were developed, it may not be analytically solvable. Thus inventory models are usually developed for some specific situations.

In this section we shall deal with situations in which demand is assumed to be fixed and completely known. Models for such situations are called *Economic lot size models* or *Economic order quantity models.*

12.5-1. Model 1 (a) (Demand Rate Uniform, Production Rate Infinite)

This is one of the simplest inventory models. A manufacturer has order to supply goods at a uniform rate R per unit time. Hence demand is fixed and known. No shortages are allowed, consequently, the cost of shortage is infinity. He starts a production run every t time units, where t is fixed; and the setup cost per production run is C_3. Production time is negligible *i.e.*, production rate is infinite so that replacement is instantaneous (lead time is zero). The holding cost is assumed to be proportional to the amount of inventory as well as the time inventory is held. Thus the cost of holding inventory I for time T is C_1IT, where C_1 is the cost of holding one unit in inventory for a unit of time. The cost coefficients C_1, C_2 and C_3 are assumed to be constants. The manufacturer's problem is to determine.

(*i*) How frequently he should make a production run.

(*ii*) How many units should be made per production run.

This model is illustrated schematically in figure 12.1

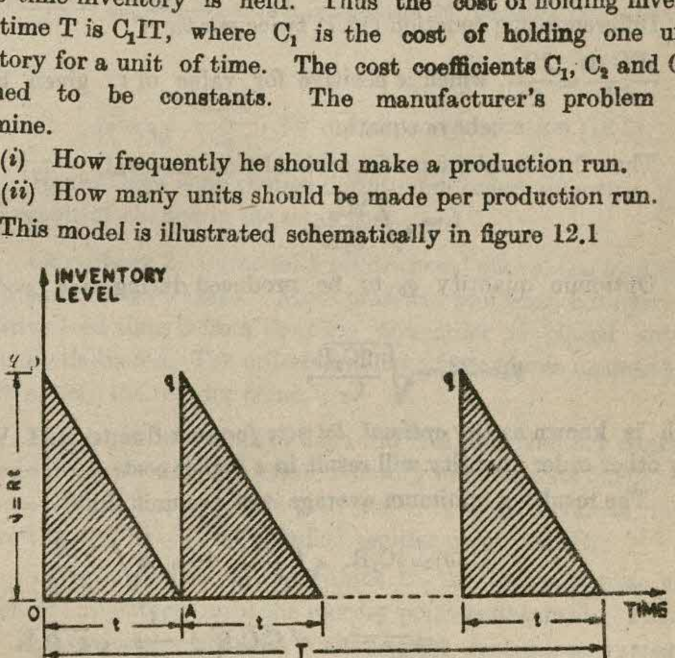


Figure 12.1. Inventory situation for model 1 (a).

If a production run is made at intervals t, a quantity $q=Rt$ must be produced in each run. Since the stock in small time dt is $Rtdt$, the stock in time period t will be

$$\int_0^t Rt.\ dt=\tfrac{1}{2}Rt^2=\tfrac{1}{2}qt=\text{Area of inventory triangle OAP.}$$

$\therefore$ Cost of holding inventory per production run $=\frac{1}{2}C_1Rt^2$.

Setup cost per production run $=C_3$.

$\therefore$ Total cost per production run $=\frac{1}{2}C_1Rt^2+C_3$.

$\therefore$ Average total cost per unit time, $C(t)=\frac{1}{2}C_1Rt+\frac{C_3}{t}$. ...(12.1)

C will be minimum if $\frac{dC(t)}{dt}=0$ and $\frac{d^2C(t)}{dt^2}$ is positive.

Differentiating equation (12.1) w.r.t. 't',

$$\frac{dC(t)}{dt}=\tfrac{1}{2}C_1R-\frac{C_3}{t^2}=0, \text{ which gives}$$

$$t=\sqrt{\frac{2C_3}{C_1R}}.$$

Differentiating equation (12.1) twice w.r.t., 't',

$\frac{d^2C(t)}{dt^2}=\frac{2C_3}{t^3}$, which is positive for value of t given by the above equation.

Thus C (t) is minimum for optimal time interval,

$$t_0=\sqrt{\frac{2C_3}{C_1R}}. \qquad ...(12.2)$$

Optimum quantity q_0 to be produced during each production run,

$$q_0=Rt_0=\sqrt{\frac{2C_3R}{C_1}}, \qquad ...(12.3)$$

which is known as the *optimal lot size formula* due to R.H. Wilson. Any other order quantity will result in a higher cost.

The resulting minimum average cost per unit time,

$$C_0(q)=\tfrac{1}{2}C_1R.\ \sqrt{\frac{2C_3}{C_1R}}+C_3.\ \sqrt{\frac{C_1R}{2C_3}}$$

$$=\frac{1}{\sqrt{2}}.\ \sqrt{C_1C_3R}+\frac{1}{\sqrt{2}}\sqrt{C_1C_3R}$$

$$=\sqrt{2C_1C_3R} \qquad ...(12.4)$$

Equation (12.1) can be written in an alternative form by replacing t by q/R as

$$C(q)=\tfrac{1}{2}C_1q+\frac{C_3R}{q}. \qquad \ldots(12\cdot5)$$

It may be realized that some of the assumptions made are not satisfied in actual practice. For instance, it is seldom that a customer demand is known exactly and that production time is negligible.

Corollary 1. In the above model if the setup cost is C_3+bq instead of being fixed, where b is the setup cost per unit item produced; we can prove that there is no change in the optimum order quantity produced due to the changed setup cost.

Proof. The average cost per unit time, $C(q)=\tfrac{1}{2}\ C_1q+\dfrac{R}{q}$

$$(C_3+bq). \quad \left[\text{From equation } (12\cdot5)\right]$$

For the minimum cost $\dfrac{dC\ (q)}{dq}=0$ and $\dfrac{d^2C\ (q)}{dq^2}$ is positive.

i e., $\tfrac{1}{2}\ C_1-\dfrac{RC_3}{q^2}=0$ or $q=\sqrt{\dfrac{2RC_3}{C_1}}$,

and $\dfrac{d^2C\ (q)}{dq^2}=\dfrac{2\ RC_3}{q^3}$, which is necessarily positive for above value of q.

$$\therefore \quad q_0=\sqrt{\frac{2C_3R}{C_1}}, \text{ which is same as equation } (12\cdot3).$$

Hence there is no change in optimum order quantity produced as a result of change in the setup cost.

Corollary 2. In model 1 (a) discussed above, the lead time has been assumed to be zero. Most practical problems, however, have a positive lead time L from the time the order is placed until it is actually delivered. The ordering policy of the above model, therefore, must satisfy the reorder point.

If L is the lead time in days and R is the inventory consumption rate in units per day, the total inventory requirements during the lead time$=$LR. Thus we should place an order q as soon as the stock level becomes LR. This is called reorder point $p=$LR.

In practice, this is equivalent to continuously observing the level of inventory until the reorder point is obtained. That is why the economic lot size model is also called *continuous review model*. Figure 12·2 shows the reorder points.

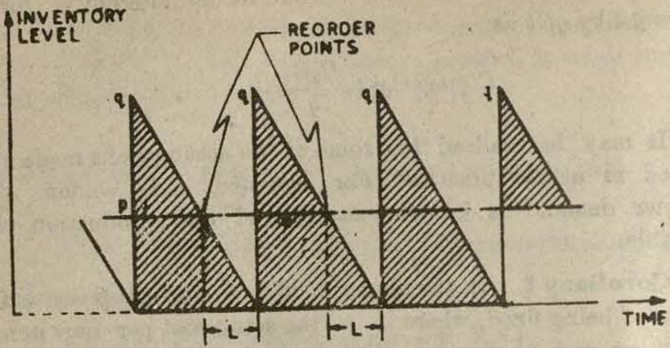


Fig. 12·2. Inventory situation with lead time.

If a buffer stock B is to be maintained, reorder point will be

$$p = B + LR \qquad \ldots(12{\cdot}7a)$$

Furthermore, if D days are required for reviewing the system,

$$p = B + LR + \frac{RD}{2}$$

$$= B + R\left(L + \frac{D}{2}\right). \qquad \ldots(12{\cdot}7b)$$

EXAMPLE 12.5-1

A manufacturer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and the shortage cost is assumed is to be infinite. The inventory holding cost is Rs. 0.20 per unit per month and the setup cost per run is Rs. 350. Determine

(*i*) the optimum run size q_0,
(*ii*) optimum scheduling period t_0,
(*iii*) minimum total expected yearly cost.

Solution

Supply rate $R = \dfrac{12{,}000}{12} = 1{,}000$ units/month,

$C_1 = $ Rs. 0·20 per unit per month,

$C_3 = $ Rs. 350 per run.

(*i*) $q_0 = \sqrt{\dfrac{2C_3R}{C_1}} = \sqrt{\dfrac{2 \times 350 \times 1\,000}{0\ 20}} = 1{,}870$ units/run.

(*ii*) $t_0 = \sqrt{\dfrac{2C_3}{C_1R}} = \sqrt{\dfrac{2 \times 350}{0{\cdot}20 \times 1{,}000}} = 1{\cdot}87$ months $= 8{\cdot}1$ weeks between runs.

(*iii*) $C_0 = \sqrt{2C_1C_3R} = \sqrt{2 \times 0{\cdot}20 \times 350 \times (1{,}000 \times 12)}$

$=$ Rs. 4, 490 per year.

EXAMPLE 12.5-2

A particular item has a demand of 9,000 units/year. The cost of one procurement is Rs. 100 and the holding cost per unit is Rs. 2·40 per year. The replacement is instantaneous and no shortages are allowed. Determine

(*i*) the economic lot size,

(*ii*) the number of orders per year,

(*iii*) the time between orders,

(*iv*) the total cost per year if the cost of one unit is Re. 1.

Solution

$R = 9{,}000$ units/year,

$C_3 =$ Rs. 100/procurement,

$C_1 =$ Rs. 2.40/year.

$$(i)\quad q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2\times 100\times 9{,}000}{2\cdot 40}} = 866 \text{ units/procurement.}$$

$$(ii)\quad N_0 = \frac{1}{t_0} = \sqrt{\frac{C_1R}{2C_3}} = \sqrt{\frac{2\cdot 40\times 9{,}000}{2\times 100}} = \sqrt{108}$$

$$= 10\cdot 4 \text{ order/year.}$$

$$(iii)\quad t_0 = \frac{1}{N_0} = \frac{1}{10\cdot 4} = 0\cdot 0962 \text{ years between procurement.}$$

$$(iv)\quad C_0 = 9{,}000 + \sqrt{2C_1C_3R}$$

$$= 9{,}000 + \sqrt{2\times 2\cdot 40\times 100\times 9{,}000}$$

$$= 9{,}000 + 2{,}080 = \text{Rs. } 11{,}080/\text{year.}$$

EXAMPLE 12.5-3

A stockist has to supply 400 units of a product every Monday to his customers. He gets the product at Rs. 50 per unit from the manufacturer. The cost of ordering and transportation from the manufacturer is Rs. 75 per order. The cost of carrying inventory is 7·5% per year of the cost of the product. Find

(*i*) the economic lot size,

(*ii*) the optimal cost.

Solution

$R = 400$ units/week,

$C_3 =$ Rs. 75/per order,

$C_1 = 7.5\%$ per year of the cost of the product

$$= \text{Rs.}\left(\frac{7.5}{100}\times \quad\right) \text{ per year}$$

$$=\text{Rs. }\left(\frac{7\cdot5}{100}\times\frac{50}{52}\right)\text{ per week}$$

$$=\text{Rs. }\frac{3.75}{52}\text{ per week.}$$

(*i*) $$q_0=\sqrt{\frac{2C_3R}{C_1}}=\sqrt{\frac{2\times75\times400\times52}{3\cdot75}}=288\text{ units/order.}$$

(*ii*) $$C_0=400\times50+\sqrt{2C_1C_3R}$$

$$=20{,}000+\sqrt{\frac{2\times3.75}{52}}\times75\times400$$

$$=20{,}000+65\cdot80=\text{Rs. }20{,}065\cdot80\text{ per week.}$$

EXAMPLE 12·5-4

A stockist purchases an item at the rate of Rs. 40 per piece from a manufacturer. 2,00(units of the item are required per year. What should be the order quantity per order if the cost per order is Rs. 15 and the inventory charges per year are 20 paise ?

Solution

$R=2{,}000$ units/year,

$C_3=$ Rs. 15/order,

$C_1=$ Rs. $0\cdot20$/rupee/year $=$ Rs. $0\cdot20\times40=$ Rs. 8/unit/year.

$$q_0=\sqrt{\frac{2C_3R}{C_1}}$$

$$=\sqrt{\frac{2\times15\times2{,}000}{8}}$$

$$=87\text{ units/order.}$$

EXAMPLE 12.5-5

The demand for a commodity is 100 units per day. Every time an order is placed, a fixed cost of Rs. 400 is incurred. Holding cost is Rs. 0.08 per unit per day. If the lead time is 13 days, determine the economic lot size and the reorder point.

Solution

$$q_0=\sqrt{\frac{2C_3R}{C_1}}=\sqrt{\frac{2\times400\times100}{0.08}}=1{,}000\text{ units.}$$

Length of cycle, $t_0=\frac{1{,}000}{100}=10$ days.

As the lead time is 13 days and cycle length is 10 days, reordering should occur when the level of inventory is sufficient to satisfy the demand for $13-10=3$ days.

$\therefore$ Reorder point $=100\times3=300$ units.

It may be noted that the 'effective' lead time is taken equal to 3 days rather than 13 days. It is because the lead time is longer than t_0.

12.5.2. Model 1. (b) (Demand Rate Non-uniform, Production Rate Infinite).

In this model all assumptions are same as in model 1 (*a*) with the exception that instead of uniform demand rate R, we are given some total demand D, to be satisfied during some long time period T. Thus demand rates are different in different production runs.

Let q be the fixed quantity produced, each time the production run is made.

Number of production runs $N=\dfrac{D}{q}$.

If t_1 is the time interval between run 1 and 2, t_2 the time interval between run 2 and 3 and so on, the total time T will be

$$=t_1+t_2+\dots+t_n. \qquad \dots(12.8)$$

This model is illustrated schematically in figure 12.3.

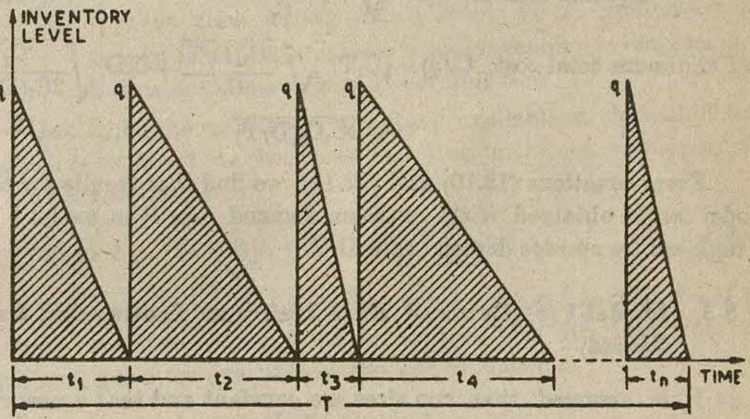


Figure 12.3. Inventory situation for different rates of demand in different cycles.

Holding costs for time period T will be

$$=(\tfrac{1}{2}qt_1)C_1+(\tfrac{1}{2}qt_2)C_1\dots+(\tfrac{1}{2}qt_n)C_1$$
$$=\tfrac{1}{2}qC_1\,(t_1+t_2+\quad+t_n)$$
$$=\tfrac{1}{2}qC_1T,$$

and the setup costs will be

$$=C_3.N$$

$$= C_3 . \frac{D}{q}, \text{ where } C_3 \text{ is the setup cost per cycle (run).}$$

$\therefore$ Total cost equation for fixed run size q will be

$$C(q) = \tfrac{1}{2} q C_1 T + C_3 \frac{D}{q}. \quad ...(12.9)$$

For minimum cost $\frac{d}{dq}[C(q)] = 0$ and $\frac{d^2}{dq^2}[C(q)]$ should be positive.

$\therefore$ Differentiating equation (8.9) w.r.t. q,

$$\frac{d}{dq}[C(q)] = \tfrac{1}{2} C_1 T - C_3 \frac{D}{q^2} = 0 \quad \therefore \quad q = \sqrt{\frac{2C_3 D}{C_1 T}} = \sqrt{\frac{2C_3 . D/T}{C_1}}$$

and $\frac{d^2}{dq^2}[C(q)] = \frac{2C_3 D}{q^3}$, which is positive for the value of q given above.

$\therefore$ Optimal lot size, $q_0 = \sqrt{\frac{2C_3 (D/T)}{C_1}}$, ...(12.10)

and minimum total cost, $C_0(q) = \tfrac{1}{2} C_1 T . \sqrt{\frac{2C_3 (D/T)}{C_1}} + C_3 D \sqrt{\frac{C_1}{2C_3 (D/T)}}$

$$= \sqrt{2C_1 C_3 (D/T)} . \quad ...(12.11)$$

From equations (12.10) and (12.11) we find that results for this model can be obtained if the uniform demand rate R in model 1 (*a*) is replaced by *average* demand rate D/T.

12.5-3. Model 1 (c) (Demand Rate Uniform, Production Rate Finite)

It is assumed that run sizes are constant and that a new run will be started whenever inventory is zero. Let

R = number of items required per unit time,

K = number of items produced per unit time,

C_1 = Cost of holding per item per unit time,

C_3 = cost of setting up a production run,

q = number of items produced per run, $q = Rt$,

t = interval between runs.

Figure 12·4 shows the variation of inventory with time.

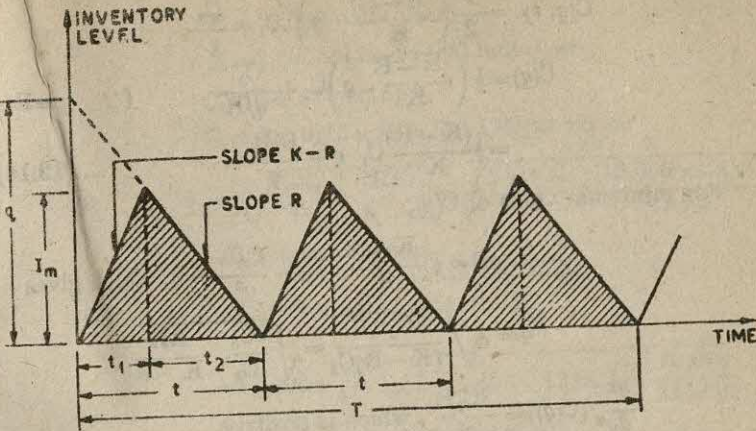


Figure 12.4. Inventory situation with finite rate of production.

Here, each production run of length t consists of two parts t_1 and t_2, where

(i) t_1 is the time during which the stock is building up at a constant rate of K−R units per unit time,

(ii) t_2 is the time during which there is no production (or supply or no replenishment) and inventory is decreasing at a constant demand rate R per unit time.

Let I_m be the maximum inventory available at the end of time t_1 which is expected to be consumed during the remaining period t_2 at the demand rate R.

Then $I_m=(K-R)t_1$ or $t_1=\frac{I_m}{K-R}$. ...(12.12)

Now the total quantity produced during time t_1 is q and the quantity consumed during the same period is Rt_1, therefore the remaining quantity available at the end of time t_1 is

$$I_m=q-Rt_1 \quad \text{...(12.13)}$$

$$=q-\frac{R.\ I_m}{K-R}.$$

$$\therefore\ I_m\left(1+\frac{R}{K-R}\right)=q \quad \text{or} \quad I_m=\frac{K-R}{K}q. \quad \text{...(12.14)}$$

Now holding cost per production run *i.e.*, for time period t

$$=\tfrac{1}{2}.\ I_m.t.C_1.$$

and setup cost per production run $=C_3$.

$\therefore$ Total average cost per unit time, $C\ (I_m.t)=\frac{1}{2}I_m\ C_1+C_3/t$ ...(12.15)

or $$C(q, t) = \frac{1}{2}\left(\frac{K-R)}{K} \cdot q\right) C_1 + \frac{C_3}{t}$$

or $$C(q) = \tfrac{1}{2}\left(\frac{K-R}{K} \cdot q\right) C_1 + \frac{C_3}{q/R} \qquad (\because\ q = Rt)$$

$$= \tfrac{1}{2}\frac{(K-R)}{K} C_1 \cdot q + \frac{C_3 R}{q}. \qquad \ldots(12.16)$$

For minimum value of $C(q)$,

$$\frac{d}{dq}[C(q)] = \tfrac{1}{2}\frac{K-R}{K} \cdot C_1 - \frac{C_3 R}{q^2} = 0, \text{ which gives}$$

$$q = \sqrt{\frac{2C_3 RK}{(K-R)C_1}} = \sqrt{\frac{2C_3}{C_1} \cdot \frac{RK}{K-R}},$$

and $$\frac{d^2}{dq^2}[C(q)] = \frac{2C_3 R}{q^3}, \text{ which is positive.}$$

$\therefore$ Optimum lot size, $q_0 = \sqrt{\dfrac{2C_3}{C_1} \cdot \dfrac{RK}{K-R}}$

$$= \sqrt{\frac{K}{K-R}} \cdot \sqrt{\frac{2C_3 R}{C_1}}. \qquad \ldots(12.17)$$

Optimum time interval, $t_0 = \dfrac{q_0}{R} = \sqrt{\dfrac{2C_3}{C_1} \cdot \dfrac{K}{R(K-R)}}$

$$= \sqrt{\frac{K}{K-R} \cdot \frac{2C_3}{C_1 R}}. \qquad \ldots(12.18)$$

Optimum average cost/unit time, C_0

$$= \tfrac{1}{2}\frac{(K-R)}{K} C_1 \cdot \sqrt{\frac{2C_3}{C_1} \cdot \frac{RK}{K-R}} + C_3 R \sqrt{\frac{C_1(K-R)}{2C_3 . RK}}$$

$$= \sqrt{2C_1 C_3 R . \frac{K-R}{K}} = \sqrt{\frac{K-R}{K}} \cdot \sqrt{2C_1 C_3 R}. \qquad \ldots(12.19)$$

Particular Cases : (*i*) If $K = R$, then $C_0 = 0$, which means that there will be no holding cost and no setup cost.

(*ii*) If $K = \infty$, *i.e.*, production rate is infinite, this model reduces to model 1 (*a*).

EXAMPLE 12.5-6

A company has a demand of 12,000 units/year for an item and it can produce 2,000 such items per month. The cost of one setup is Rs. 400 and the holding cost/unit/month is Rs. 0.15. Find the optimum lot size and the total cost per year, assuming the cost of 1 unit as Rs. 4. Also find the maximum inventory, manufacturing time and total time.

Solution

$R = 12{,}000$ units/year,

$K = 2{,}000 \times 12 = 24{,}000$ units/year,

$C_3 =$ Rs. 400/setup,

$C_1 =$ Rs. $0.15 \times 12 =$ Rs. 1.80/unit/year.

(*i*) $$q_0 = \sqrt{\frac{2C_3}{C_1} \cdot \frac{RK}{K-R}} = \sqrt{\frac{2 \times 400}{1.8} \times \frac{12{,}000 \times 24{,}000}{12{,}000}}$$

$= 3{,}264$ units/setup.

(*ii*) $$C_0 = 12{,}000 \times 4 + \sqrt{2C_1C_3R \cdot \frac{K-R}{K}}$$

$$= 48{,}000 + \sqrt{2 \times 1{\cdot}8 \times 400 \times 12{,}000 \times \frac{12{,}000}{24{,}000}}$$

$= 48{,}000 + 2{,}740 =$ Rs. 50,740/year.

(*iii*) Max. inventory $I_{m0} = \frac{K-R}{K} \cdot q_0$

$$= \frac{24{,}000 - 12{,}000}{24{,}000} \times 3{,}264 = 1{,}632 \text{ units.}$$

(*iv*) Manufacturing time $t_1 = \frac{I_{m0}}{K-R} = \frac{1{,}632}{12{,}000} = 0.136$ years.

(*v*) Total time $t_0 = \frac{q_0}{R} = \frac{3{,}264}{12{,}000} = 0.272$ year.

12.5-4. Model 2 (a) (Demand Rate Uniform, Production Rate Infinite, Shortages Allowed)

This model is just the extension of model 1 (*a*), allowing shortage. Let

R = number of items required per unit time *i.e.*, demand rate,

C_1 = cost of holding the item per unit time,

C_2 = shortage cost per item per unit time,

C_3 = cost of setting up a production run,

q = number of items produced per production run,

$q = Rt$,

t = interval between runs,

I_m = number of items that form inventory at the beginning of time interval t; it is a variable.

Lead time is assumed to be zero. Figure 12.5 shows the variation of inventory with time.

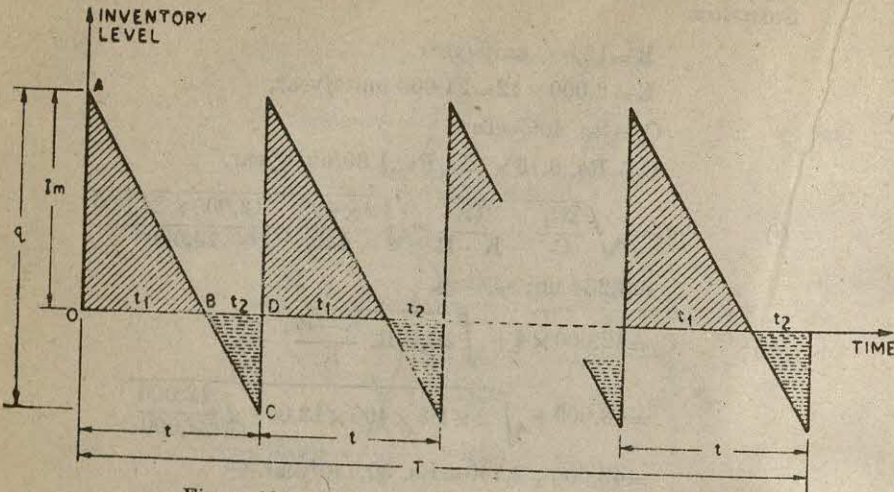


Figure 12.5. Inventory situation for model 2 (*a*).

Here, the total time period T is divided into n equal time intervals, each of value t. The time interval t is further divided into two parts t_1 and t_2.

$$i.e., \; t = t_1 + t_2$$

where t_1 is the time interval during which items are drawn from inventory and t_2 is the interval during which the items are being accumulated but not filled. Using the relationship of similar triangles,

$$\frac{t_1}{t} = \frac{I_m}{q} \qquad t_1 = \frac{I_m . t}{q},$$

and
$$\frac{t_2}{t} = \frac{q - I_m}{q} \qquad \therefore \quad t_2 = \frac{q - I_m}{q} . \, t.$$

Now total inventory during time t = area of $\triangle OAB = \frac{1}{2} I_m . t_1$.

$\therefore$ Inventory holding cost during time $t = \frac{1}{2} C_1 I_m . t_1$.

Similarly, total shortage during time t

$$= \text{area of} \triangle BCD = \tfrac{1}{2}(q - I_m) t_2.$$

$\therefore$ Shortage cost during time $t = \frac{1}{2} C_2 (q - I_m) t_2$,

and setup cost during time $t = C_3$.

$\therefore$ Total cost during time $t = \frac{1}{2} C_1 I_m t_1 + \frac{1}{2} C_2 (q - I_m) t_2 + C_3$

or total average cost per unit time,

$$C(I_m, t) = \frac{1}{t}\left[\tfrac{1}{2} C_1 I_m t_1 + \tfrac{1}{2} C_2 (q - I_m) \; t_2\right] + \frac{C_3}{t} \qquad \ldots(12.20)$$

$$=\frac{1}{t}\left[\tfrac{1}{2}C_1I_m\cdot\frac{I_m\cdot t}{q}+\tfrac{1}{2}C_2(q-I_m)\cdot\frac{q-I_m}{q}\cdot t\right]+\frac{C_3}{t}$$

$$\therefore\quad C(I_m,q)=\tfrac{1}{2}C_1\frac{I_m^2}{q}+\tfrac{1}{2}C_2\cdot\frac{(q-I_m)^2}{q}+\frac{C_3\cdot R}{q}.\qquad\ldots(12.21)$$

Total average cost per unit time $C(I_m\ q)$ being a function of two variables I_m and q, has to be partially differentiated w.r.t. I_m and q separately and then put equal to zero.

i.e., $\frac{[\partial C(I_m,q)]}{\partial I_m}=0$, which gives

$$\tfrac{1}{2}C_1\cdot\frac{2I_m}{q}+\tfrac{1}{2}C_2\cdot\frac{2(q-I_m)}{q}\cdot(-1)+0=0$$

or $$\frac{C_1}{q}I_m-\frac{C_2}{q}(q-I_m)=0$$

or $$\frac{C_1+C_2}{q}\cdot I_m=C_2$$

or $$I_m=\frac{C_2}{C_1+C_2}\cdot q.$$

and $$\frac{\partial^2}{\partial I_m^2}\left[C(I_m,q)\right]=\frac{C_1}{q}+\frac{C_2}{q}=\frac{C_1+C_2}{q}$$ which is positive.

$\therefore$ Optimum value of I_m,

$$I_{m0}=\frac{C_2}{C_1+C_2}q.\qquad\ldots(12\cdot22)$$

Similarly, $\frac{\partial}{\partial q}\left[C(I_m,q)\right]=0$, which gives

$$\tfrac{1}{2}C_1I_m^2\left(-\frac{1}{q^2}\right)+\tfrac{1}{2}C_2\cdot\frac{q.2(q+I_m)-(q-m)\cdot^2 1}{q^2}-\frac{C_3R}{q^2}=0$$

or $C_1I_m^2-C_2\left\{(q-I_m)\cdot(2q-q+m)\right\}+2C_3R=0$

or $C_1I_m^2-C_2\cdot(q^2-I_m^2)+2C_3R=0$

or $C_2q^2=(C_1+C_2)I_m^2+2C_3R$

or $$C_2q^2=(C_1+C_2)\cdot\frac{C_2^2}{(C_1+C_2)^2}\cdot q^2+2C_3R$$

$$=\frac{C_2^2}{C_1+C_2}\cdot q^2+2C_3R$$

or $$\left(C_2-\frac{C_2^2}{C_1+C_2}\right)q^2=2C_3R$$

or $$q^2=\frac{C_1+C_2}{C_1C_2}\cdot 2C_3R$$

or $$q=\sqrt{\frac{C_1+C_2}{C_1C_2}}\cdot\sqrt{2C_3R},$$

and it can be proved that

$$\frac{\partial^2}{\partial q^2}\left[\ C\ (I_m,\ q)\ \right] \text{ is positive.}$$

$\therefore$ Optimal value of q,

$$q_0=\sqrt{\frac{C_1+C_2}{C_1C_2}}\cdot\sqrt{2C_3R}=\sqrt{\frac{C_1+C_2}{C_2}}\cdot\sqrt{\frac{2C_3R}{C_1}}. \quad ...(12\cdot23)$$

$\therefore$ For equation (12·22),

$$I_{m0}=\sqrt{\frac{C_2}{C_1(C_1+C_2)}}\cdot\sqrt{2C_3R}$$

$$=\sqrt{\frac{C_2}{C_1+C_2}}\cdot\sqrt{\frac{2C_3R}{C_1}}. \quad ...(12\cdot24)$$

Substituting values of I_{m0} and q_0 in equation (12·21), we get the minimum average cost per unit time as

$$C_0(I_m,\ q)=\frac{C_1}{2}\left\{\frac{C_2}{C_1+C_2}\right\}^2\cdot q+\frac{C_2}{2}\cdot\frac{\left\{q-\frac{C_2}{C_1+C_2}q\right\}^2}{q}+C_3\frac{R}{q}$$

$$=\frac{C_1}{2}\cdot\frac{C_2^2}{(C_1+C_2)^2}\cdot q+\frac{C_2}{2}\cdot\frac{C_1^2}{(C_1+C_2)^2}\cdot q+C_3\frac{R}{q}$$

$$=\frac{C_1C_2}{2.(C_1+C_2)^2}\cdot q\cdot(C_1+C_2)+C_3.\frac{R}{q}$$

$$=\frac{C_1C_2}{2\cdot(C_1+C_2)}\cdot\sqrt{\frac{C_1+C_2}{C_1C_2}}\cdot\sqrt{2C_3R}$$

$$+C_3R.\frac{\sqrt{C_1C_2}}{\sqrt{C_1+C_2}}\cdot\frac{1}{\sqrt{2C_3R}}$$

$$=\frac{1}{\sqrt{2}}\sqrt{\frac{C_1C_2}{C_1+C_2}}\cdot\sqrt{C_3R}+\frac{1}{\sqrt{2}}\sqrt{\frac{C_1C_2}{C_1+C_2}}\cdot\frac{1}{\sqrt{C_3R}}.$$

$$\therefore\ C_0(I_m,\ q)=\sqrt{\frac{C_1C_2}{C_1+C_2}}\cdot\sqrt{2C_3R}$$

$$=\sqrt{\frac{C_2}{C_1+C_2}}\ \sqrt{2C_1C_3R}. \quad ...(12\cdot25)$$

Optimal time interval t between runs is given by

$$t_0=\frac{q_0}{R}=\sqrt{\frac{C_1+C_2}{C_1C_2}}\cdot\sqrt{\frac{2C_3}{R}}$$

$$=\sqrt{\frac{C_1+C_2}{C_2}}\ \sqrt{\frac{2C_3}{C_1R}}. \quad ...(12\cdot26)$$

Particular cases (*i*) If shortages are prohibited *i.e.*, $C_2=\infty$, equations (12·23), (12·25) and (12·26) reduce to equations (12·3), (12·4) and (12·2) for model 1 (*a*), which must be true since model 1(*a*) is a special case of model 2(*a*).

(*ii*) If $C_2 \neq \infty$, $\sqrt{\dfrac{C_2}{C_1+C_2}}\cdot\sqrt{2C_1C_3R}$ from equation (12·25) $<\sqrt{2C_1C_3R}$ from equation (12·4).

$\therefore$ Total expected costs associated with policy of model 2(*a*) are $\dfrac{C_2}{C_1+C_2}$ of the costs associated with policy of model 1(*a*).

12.5-5. Model 2(b) (Demand Rate Uniform, Production Rate Infinite, Shortages Allowed, Time Interval Fixed).

Time interval t is fixed which means that inventory is to be replenished (produced) after every fixed time t. All other assumptions of model 2(*a*) hold good.

Here $$t_1=\frac{I_m.t}{q},$$

and $$t_2=\frac{q-I_m}{q}.t.$$

Total inventory during time $t=\frac{1}{2}I_m t_1$.

$\therefore$ Total inventory holding cost during time $t=\frac{1}{2}C_1 I_m t_1$.

Similarly, total shortage during time $t=\frac{1}{2}.(q-I_m)t_2$.

$\therefore$ Total shortage cost during time $t=\frac{1}{2}C_2.(q-I_m)t_2$.

Setup cost C_3 and time interval t are both constant, hence the average setup cost per unit time $\dfrac{C_3}{t}$ is also constant and hence is not to be considered.

$\therefore$ Total average cost per unit time,

$$C(I_m)=\frac{1}{t}\left[\frac{1}{2}C_1I_m t_1+\frac{1}{2}C_2(q-I_m)t_2\right]$$

$$=\frac{1}{t}\left[\frac{1}{2}C_1I_m.\frac{I_m t}{q}+\frac{1}{2}C_2(q-I_m).\frac{q-I_m}{q}.t\right].$$

$$\therefore\quad C(I_m)=\left[\frac{C_1}{2q}.I_m^2+\frac{C_2}{2q}(q-I_m)^2\right] \qquad ...(12\cdot27)$$

Now $C(I_m)$ will be optimal if $\dfrac{d}{dI_m}[C(I_m)]=0$ and $\dfrac{d^2}{dI_m^2}[C(I_m)]$ is positive.

$$\frac{d}{dI_m}[C(I_m)]=0 \text{ gives}$$

$$\frac{C_1}{2q}\cdot 2I_m+\frac{C_2}{2q}\cdot 2(q-I_m)(-1)=0$$

or
$$C_1I_m-C_2(q-I_m)=0$$

or
$$I_m=\frac{C_2}{C_1+C_2}\cdot q,$$

and $\frac{d^2}{dI_m^2}[C(I_m)]=\frac{C_1+C_2}{q}$, which is positive.

$\therefore$ Optimum order quantity is given by

$$\left.\begin{aligned} I_{m0}&=\frac{C_2}{C_1+C_2}\cdot q\\ &=\frac{C_2}{C_1+C_2}Rt.\end{aligned}\right\} \quad ...(12\cdot 28)$$

The minimum average cost per unit time from equation (12·27) is given by

$$\begin{aligned} C_0(I_m)&=\frac{C_1}{2q}\cdot\left(\frac{C_2}{C_1+C_2}\right)^2\cdot q^2+\frac{C_2}{2q}\cdot\left(q-\frac{C_2}{C_1+C_2}\cdot q\right)^2\\ &=\frac{1}{2}C_1q\cdot\left(\frac{C_2}{C_1+C_2}\right)^2+\frac{1}{2}C_2q\cdot\left(\frac{C_1}{C_1+C_2}\right)^2\\ &=\frac{1}{2}\cdot\frac{C_1C_2}{(C_1+C_2)^2}\cdot q(C_1+C_2)\end{aligned}$$

$$\left.\begin{aligned} &=\frac{1}{2}\cdot\frac{C_1C_2}{C_1+C_2}\cdot q\\ &=\frac{1}{2}\cdot\frac{C_1C_2}{C_1+C_2}Rt.\end{aligned}\right\} \quad ...(12\cdot 29)$$

From equation (12·28) we observe that unless C_1 is zero, optimum order level I_m is less than the demand q during the time interval t. Therefore, it is advantageous to plan for shortages.

12.5·6. Model 2(c) (Demand Rate Uniform, Production Rate Finite, Shortages Allowed).

This model has the same assumptions as in model 2(a) except that production rate is finite. Figure 12·6 shows the variation of inventory with time.

Referring to Figure 12·6 we find that inventory is zero in the beginning. It increases at constant rate (K—R) for time t_1 until it reaches a level I_m. There is no replenishment during time t_2, inventory decreases at constant rate R till it becomes zero. Shortage starts piling up at constant rate R during time t_3 until this backlog

reaches a level s. Lastly, production starts and backlog is filled at a constant rate K—R during time t_4 till the backlog becomes zero. This completes one cycle ; the total time taken during this cycle is

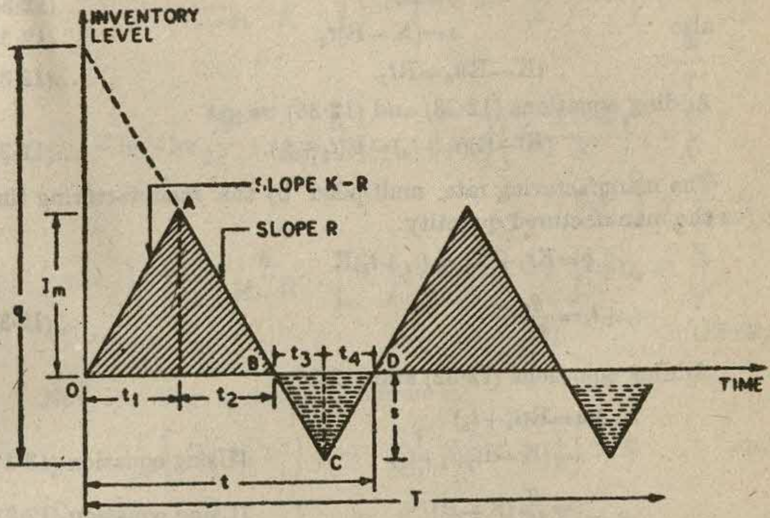


Fig. 12·6. Inventory situation for model 2(c).

$$t = t_1 + t_2 + t_3 + t_4.$$

This cycle repeats itself over and over again.

Now holding cost during time interval t

$$= C_1.\text{area of } \triangle OAB$$

$$= C_1.\tfrac{1}{2}.I_m(t_1 + t_2),$$

and shortage cost during time interval t

$$= C_2.\text{ area of } \triangle BCD$$

$$= C_2.\tfrac{1}{2}.s.(t_3 + t_4).$$

Also setup cost during time interval $t = C_3$.

Total average cost per unit time.

$$C = \frac{\tfrac{1}{2}.C_1 I_m(t_1 + t_2) + \tfrac{1}{2}C_2 s.(t_3 + t_4) + C_3}{t_1 + t_2 + t_3 + t_4}. \qquad ...(12\cdot30)$$

Now C is a function of six variables I_m, s, t_1, t_2, t_3, and t_4 but we can derive relationships which determine the values of I_m, t_1, t_2, t_3 and t_4 in terms of only two variables q and s. An inventory policy is given when we know how much to produce *i.e.*, q and when to start production, which can be found if s is known.

Now $$I_m=(K-R)t_1, \quad ...(12\cdot31)$$
also $$I_m=Rt_2. \quad ...(12\cdot32)$$
$$\therefore \quad (K-R)t_1=Rt_2. \quad ...(12\cdot33)$$
Further, $$s=Rt_3, \quad ...(12\cdot34)$$
also $$s=(K-R)t_4. \quad ...(12\cdot35)$$
$$\therefore \quad (K-R)t_4=Rt_3. \quad ...(12\cdot36)$$
Adding equations (12·33) and (12·36) we get
$$(K-R)(t_1+t_4)=R(t_2+t_3). \quad ...(12\cdot37)$$

The manufacturing rate multiplied by the manufacturing time gives the manufactured quantity.
$$\therefore \quad q=Kt_1+Kt_4=(t_1+t_4)K$$
$$\therefore \quad t_1+t_4=\frac{q}{K}. \quad ...(12\cdot38)$$

Adding equations (12·32) and (12·34),
$$I_m+s=R(t_2+t_3)$$
$$=(K-R)(t_1+t_4) \quad \text{[Using equation (12·37)]}$$
$$=\frac{q}{K}(K-R) \quad \text{[Using equation (12·38)]}$$
$$\therefore \quad I_m=q\left(1-\frac{R}{K}\right)-s. \quad ...(12\cdot39)$$

From equations (12·31) and (12·32),
$$t_1+t_2=\frac{I_m}{K-R}+\frac{I_m}{R}$$
$$=\left\{q\left(1-\frac{R}{K}\right)-s\right\}\left\{\frac{1}{K-R}+\frac{1}{R}\right\}.$$
[Using equation (12·39)] ...(12·40)

Similarly, $$t_3+t_4=\frac{s}{K-R}+\frac{s}{R}$$
$$=s\left(\frac{1}{K-R}+\frac{1}{R}\right), \quad ...(12\cdot41)$$

and $$t_1+t_2+t_3+t_4=\left(\frac{1}{K-R}+\frac{1}{R}\right)\left\{q\cdot\frac{K-R}{K}\right\}$$
$$=\frac{q}{R}. \quad ...(12\cdot42)$$

Substituting values of I_m, t_1+t_2, t_3+t_4 and $t_1+t_2+t_3+t_4$ in equation (12·30), we get
$$C(q, s)=\frac{R}{q}\left[\frac{1}{2}\cdot C_1\cdot\left\{q\left(1-\frac{R}{K}\right)-s\right\}\cdot\left\{q\left(1-\frac{R}{K}\right)-s\right\}\right.$$

$$\left\{\frac{1}{K-R}+\frac{1}{R}\right\}+\frac{1}{2}C_2\cdot\ s\left(\frac{1}{K-R}+\frac{1}{R}\right)\cdot s\ \Big]+\frac{R}{q}C_3.$$

$$=\frac{R}{q}\left[\frac{1}{2}\cdot\left(\frac{1}{K-R}+\frac{1}{R}\right)\right]\left[C_1\cdot\left\{q\left(1-\frac{R}{K}\right)-s\right\}^2+C_2 s^2\right]+\frac{R}{q}\cdot C_3.$$

$$\therefore\quad C(q,s)=\frac{R}{2q}\cdot\frac{K}{R(K-R)}\cdot\left[C_1\left\{\left(q\cdot\frac{K-R}{K}\right)-s^2\right\}^2+C_2 s^2\right]+\frac{R}{q}C_3$$

$$\text{or } C\left(q,s\right)=\frac{1}{2q}\cdot\frac{K}{K-R}\cdot\left[C_1\left\{q\cdot\frac{K-R}{K}-s\right\}^2+C_2 s^2\right]+\frac{R}{q}C_3. \qquad ...(12\cdot43)$$

Now cost $C(q, s)$ will be minimum if

$$\frac{\partial}{\partial q}\left[C\left(q,s\right)\right]=0,\ \frac{\partial^2}{\partial q^2}\left[C\left(q,s\right)\right]>0 \quad \text{and}$$

$$\frac{\partial}{\partial s}\left[C\left(q,s\right)\right]=0,\ \frac{\partial^2}{\partial s^2}\left[C\left(q,s\right)\right]>0.$$

Differentiating equation (12·43) partially w.r.t. s,

$$\frac{\partial}{\partial s}\left[C\left(q,s\right)\right]=\frac{1}{2q}\cdot\frac{K}{K-R}\cdot\left[2C_1\left(q\cdot\frac{K-R}{K}-s\right)\left(-1\right)+2C_2 s\right]=0$$

or
$$2C_1\cdot\left(q\cdot\frac{K-R}{K}\right)=2\left(C_1+C_2\right)\cdot s$$

or
$$s=q\cdot\frac{K-R}{K}\cdot\frac{C_1}{C_1+C_2}.$$

$$\frac{\partial^2}{\partial s^2}\left[C\left(q-s\right)\right]=\frac{1}{2q}\cdot\frac{K}{K-R}\left(2C_1+2C_2\right),$$

which is positive.

$$\therefore\quad s_0=q\cdot\frac{K-R}{K}\cdot\frac{C_1}{C_1+C_2}. \qquad ...(12\cdot44)$$

Differentiating equation (12·43) partially w.r.t. q,

$$\frac{\partial}{\partial q}\left[C\left(q,s\right)\right]=-\frac{1}{2q^2}\cdot\frac{K}{K-R}\cdot\left[C_1\left(q\cdot\frac{K-R}{K}-s\right)^2+C_2 s^2\right]$$
$$+\frac{1}{2q}\cdot\frac{K}{K-R}\left[2C_1\cdot\left(q\cdot\frac{K-R}{K}-s\right)\cdot\frac{K-R}{K}\right]-\frac{RC_3}{q^2}=0,$$

which on simplification gives,

$$q=\sqrt{2C_3 \cdot \frac{C_1+C_2}{C_1 C_2}} \cdot \sqrt{\frac{KR}{K-R}}.$$

It can be proved that $\frac{\partial^2}{\partial q^2}\left[C\left(q, s\right)\right]$ is positive, so that

$$q_0=\sqrt{2C_3 \cdot \frac{(C_1+C_2)}{C_1C_2}} \cdot \sqrt{\frac{K \cdot R}{K-R}}$$

$$=\sqrt{\frac{C_1+C_2}{C_2}} \cdot \sqrt{\frac{K}{K-R}} \cdot \sqrt{\frac{2C_3R}{C_1}} \qquad ...(12 \cdot 45)$$

$\therefore$ From equation (12·44),

$$s_0=\sqrt{2C_3 \cdot \frac{C_1}{(C_1+C_2)C_2}} \cdot \sqrt{\frac{R(K-R)}{K}}. \qquad ...(12 \cdot 46)$$

Substituting values of q_0 and s_0 in equation (12·30), and simplifying we get

$$C_0(q, s)=\sqrt{\frac{2C_1C_2}{C_1+C_2} \cdot C_3} \sqrt{\frac{R(K-R)}{K}}$$

$$=\sqrt{\frac{C_2}{C_1+C_2}} \cdot \sqrt{\frac{K-R}{K}} \cdot \sqrt{2C_1C_3R}. \qquad ...(12 \cdot 47)$$

Optimum time interval t_0 is given by

$$t_0=\frac{q_0}{R}=\sqrt{2C_3 \cdot \frac{(C_1+C_2)}{C_1C_2} \cdot \frac{K}{R(K-R)}}$$

$$=\sqrt{\frac{C_1+C_2}{C_2}} \cdot \sqrt{\frac{K}{K-R}} \cdot \sqrt{\frac{2C_3}{C_1R}}, \qquad ...(12 \cdot 48)$$

and
$$I_{m0}=q_0\left(1-\frac{R}{K}\right)-s_0 \qquad \text{[Equation (12·39)]}$$

$$=\sqrt{\frac{2C_2 \cdot C_3}{C_1(C_1+C_2)}} \sqrt{\frac{R(K-R)}{K}}$$

$$=\sqrt{\frac{C_2}{C_1+C_2}} \cdot \sqrt{\frac{K-R}{K}} \cdot \sqrt{\frac{2C_3R}{C_1}}. \qquad ...(12 \cdot 49)$$

Particular cases

(*i*) If $K=\infty$, *i.e.*, production rate is infinity, equations (12·45), (12·49) and (12·47) giving q_0, I_{m0} and C_0 reduce to equations (12·23), (12·24) and (12·25) for model 2 (*a*).

(*ii*) If $C_2=\infty$, *i.e.*, no shortages are allowed, equations (12·45), (12·48) and (12·47) reduce to (12·17), (12·18) and (12·19) for model 1 (*c*).

(*iii*) If $K=\infty$, $C_2=\infty$, this model becomes model 1 (*a*) and equations (12·45), (12·47), and (12·48) reduce to equations (12·3), (12·4) and (12·2) respectively.

EXAMPLE 12.5-7

Find the results of example 12·5-2 if in addition to the data given in this problem the cost of shortage is also given as Rs. 5 per unit per year.

Solution. $R=9{,}000$ units/year,

$C_3=$ Rs. 100/procurement,

$C_1=$ Rs. 2·40/year,

$C_2=$ Rs. 5 per unit/year.

(*i*) From equation (12·23)

$$q_0=\sqrt{\frac{C_1+C_2}{C_2}}\cdot\sqrt{\frac{2C_3R}{C_1}}$$

$$=\sqrt{\frac{2{\cdot}40+5}{5}}\cdot\sqrt{\frac{2\times100\times9{,}000}{2{\cdot}4}}$$

$$=\sqrt{11{,}10{,}000}=1{,}053 \text{ units/run.}$$

(*ii*) From equation (12·25),

$$C_0(I_m, q)=9{,}000\times1+\sqrt{\frac{C_2}{C_1+C_2}}\ \sqrt{2C_1C_3R}=9{,}000$$

$$+\sqrt{\frac{5}{2{\cdot}4+5}}\cdot\ \sqrt{2\times2{\cdot}4\times100\times9{,}000}$$

$$=\text{Rs. }(9{,}000+1{,}710)$$

$$=\text{Rs. }10{,}710\text{/year.}$$

(*iii*) Number of orders/year,

$$N_0=\frac{9{,}000}{1{,}053}=8{\cdot}55.$$

(*iv*) Time between orders,

$$t_0=\frac{1}{N_0}=\frac{1}{8{\cdot}55}=0{\cdot}117 \text{ year.}$$

EXAMPLE 12·5-8

The data for this example is same as that of example 12·5-6 except that the shortage cost of one unit is Rs. 20 per year. Find the various results.

Solution. $R=12{,}000$ units/year,

$K=2{,}000\times12=24{,}000$ units/year,

$C_3=$ Rs. 400/setup,

$C_1=$ Rs. $0{\cdot}15\times12=1{\cdot}80$ per unit/year,

$C_2=$ Rs. 20 per year.

(*i*) Using equation (12·45),

$$q_0=\sqrt{\frac{2C_3R}{C_1}}\cdot\sqrt{\frac{C_1+C_2}{C_2}}\cdot\sqrt{\frac{K}{K-R}}$$

$$=\sqrt{\frac{2\times 400\times 12{,}000}{1\cdot 8}}\cdot\sqrt{\frac{1\cdot 8+20}{20}}\cdot\sqrt{\frac{24{,}000}{24{,}000-12{,}000}}$$

$$=\sqrt{\frac{2\times 400\times 12{,}000}{3}}\cdot\sqrt{\frac{10\cdot 9}{10}}\cdot\sqrt{2}$$

$=3{,}413$ units/year.

(*ii*) From equation (12·47),

$$C_0\,(q,\,s)=12{,}000\times 4+\sqrt{2C_1C_3R}\,.\sqrt{\frac{C_2}{C_1+C_2}}\cdot\sqrt{\frac{K-R}{K}}$$

$$=48{,}000+\sqrt{2\times 1.8\times 400\times 12{,}000}.\sqrt{\frac{20}{20+1\cdot 8}}$$

$$\times\sqrt{\frac{24{,}000-12{,}000}{24{,}000}}$$

$$=48{,}000+\sqrt{2\times 1.8\times 400\times 12{,}000}\,\sqrt{\frac{10}{10.9}}\cdot\frac{1}{\sqrt{2}}$$

$=$ Rs. 51,336/year.

(*iii*) Using equation (12·49),

$$I_{m0}=\sqrt{\frac{2C_3R}{C_1}}\cdot\sqrt{\frac{C_2}{C_1+C_2}}\cdot\sqrt{\frac{K-R}{K}}$$

$$=\sqrt{\frac{2\times 400\times 12{,}000}{1\cdot 80}}\times\sqrt{\frac{20}{1\cdot 80+20}}\cdot\sqrt{\frac{24{,}000-12{,}000}{24{,}000}}$$

$$=\sqrt{\frac{2\times 400\times 12{,}000}{1\cdot 80}\times\frac{10}{10\cdot 9}\times\frac{1}{2}}$$

$=1{,}564$ units/production run.

(*iv*) **Manufacturing** time interval t_1+t_4

$$=\frac{q_0}{K}\ \text{[equation (12·38)]}$$

$$=\frac{3{,}413}{24{,}000}=0\cdot 1422\ \text{years.}$$

(*v*) **Total** time interval, t_0

$$=\frac{q_0}{R}\ \text{[equation (12·48)]}$$

$$=\frac{3{,}413}{12{,}000}$$

$=0\cdot 2844$ year.

EXAMPLE 12·5·9

A contractor supplies diesel engines to a truck manufacturer at the rate of 20 per day. He has to pay a penalty of Rs. 10 per engine per day for missing the scheduled delivery date. Holding cost of a complete engine is Rs. 12 per month. The manufacturing of engines starts with the beginning of the month and is completed at the end of the month. What should be the inventory level at the beginning of each month ?

Solution. $R = 20$ engines/day,

$C_2 =$ Rs. 10 per engine per day,

$C_1 =$ Rs. 12 per month $= \frac{12}{30} =$ Rs. 0·40/day,

$t = 1$ month $= 30$ days.

Using equation (12·28).

$$I_{m_0} = \frac{C_2}{C_1 + C_2} \cdot q$$

$$= \frac{C_2}{C_1 + C_2} \cdot Rt$$

$$= \frac{10}{0{\cdot}40 + 10} \times 20 \times 30$$

$$= 577 \text{ engines/month.}$$

12·6. Inventory Models with Probabilistic Demand

The models discussed in the previous sections are only artificial since in practical situations, demand is hardly known precisely. In most situations demand is probabilistic since only *prabability distribution* of future demand, rather than the exact value of demand itself, is known. The probability distribution of future demand is usually determined from the data collected from past experience. In such situations we choose policies that minimize the *expected* costs rather than the actual costs. Expected costs are obtained by multiplying the actual costs for a particular situation with the probability of occurrence of that situation and then either summing or integrating according as the probability distribution is discrete or continuous.

12·6·1. Model 3 (a) (Instantaneous Demand, Setup Cost Zero, Stock Levels Discrete and Lead Time Zero).

Let $R =$ discrete demand rate with probability p_R,

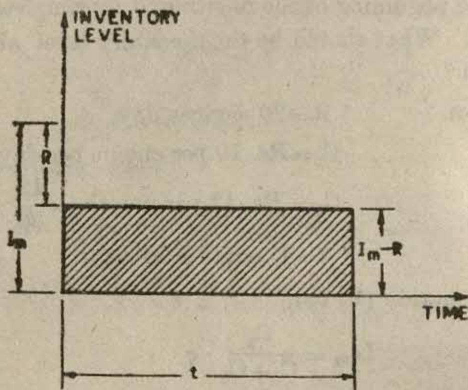
$I_m =$ discrete stock level for time interval t,

$t =$ constant interval between orders,

$C_1 =$ holding cost per item per unit time,

$C_2 =$ shortage cost per item per unit time.

Production is assumed to be instantaneous and lead time negligibly small. The problem is to determine the optimal inventory level I_m, where $R \leqslant I_m$ or $R > I_m$ at the beginning of each time interval. The variation of inventory with time for these two cases is shown in Fig. 12·7 (*a*) and (*b*).



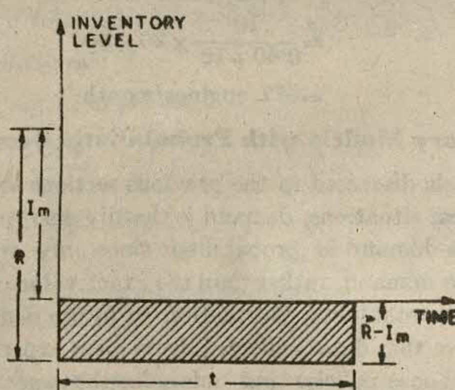


Fig. 12·7. Inventory situation for model 3 (*a*), (*a*) $R \leqslant I_m$ (*b*) $R > I_m$.

When $R \leqslant I_m$, as shown in figure 12·7 (*a*), there are no shortages; when $R > I_m$, as shown in figure 12·7 (*b*), shortages occur. The inventory units may be either discrete or continuous. In this model we shall consider inventory comprising of discrete units only. The cost equation for this model may be derived as follows :

1. When number of units used is $\leqslant$ number of units in stock *i.e.*, $R \leqslant I_m$, the cost of *oversupplying* due to the units *lying surplus* in stock $= C_1(I_m - R)$.

2. When number of units required is > number of units in stock *i.e.*, $R > I_m$, the cost of *undersupplying* due to shortage of units in stock $= C_2.(R - I_m)$.

$\therefore$ Expected cost per unit time associated with any particular value of R is either $C_1 (I_m - R) \cdot p_R$, where $R \leqslant I_m$

or $\qquad C_2 (R - I_m)\, p_R$, where $R > I_m$.

If $\qquad R = I_m$, expected cost is zero.

The total expected cost is the sum of all the expected costs *i.e.*, the costs associated with each possible value of R.

$\therefore$ Total expected cost per unit time

$$C(I_m) = C_1 \sum_{R=0}^{I_m} (I_m - R) \cdot p_R + C_2 \sum_{R=I_m+1}^{\infty} (R - I_m) \cdot p_R. \qquad \ldots(12\cdot50)$$

Now we have to determine the value of I_m that minimizes the total expected cost.

First we substitute $(I_m + 1)$ for I_m in equation (12·50), obtaining

$$C(I_m+1) = C_1 \sum_{R=0}^{I_m+1} (I_m + 1 - R)\, p_R + C_2 \sum_{R=I_m+2}^{\infty} (R - \overline{I_m+1})\, p_R$$

$$= C_1 \sum_{R=0}^{I_m} (I_m + 1 - R)\, p_R + C_1 \left[(I_m+1) - (I_m+1) \right] p_{I_m+1}$$

$$+ C_2 \sum_{R=I_m+1}^{\infty} (R - I_m - 1)\, p_R - C_2 \left[(I_m+1) - (I_m+1) \right] p_{I_m+1}$$

$$= C_1 \sum_{R=0}^{I_m} (I_m + 1 - R) \cdot p_R + C_2 \sum_{R=I_m+1}^{\infty} (R - I_m - 1) \cdot p_R$$

$$= C_1 \sum_{R=0}^{I_m} (I_m - R)\, p_R + C_1 \sum_{R=0}^{I_m} p_R$$

$$+ C_2 \sum_{R=I_m+1}^{\infty} (R - I_m)\, p_R - C_2 \sum_{R=I_m+1}^{\infty} p_R.$$

$$\text{Now} \quad \sum_{R=I_m+1}^{\infty} p_R = \sum_{R=0}^{\infty} p_R - \sum_{R=0}^{I_m} p_R$$

$$=1-\sum_{R=0}^{I_m} p_R.$$

$$\therefore\quad C(I_m+1)=\left[C_1\sum_{R=0}^{I_m}(I_m-R)p_R+C_2\sum_{R=I_m+1}^{\infty}(R-I_m)p_R\right]$$

$$+C_1\sum_{R=0}^{I_m}p_R-C_2\left\{1-\sum_{R=0}^{I_m}p_R\right\}$$

$$=C(I_m)+(C_1+C_2)\sum_{R=0}^{I_m}p_R-C_2.$$

$$\therefore\quad C(I_m+1)=C(I_m)+(C_1+C_2)\cdot p_{R\leq I_m}-C_2. \qquad \text{...(12·51)}$$

Similarly, $C(I_m-1)=C(I_m)-(C_1+C_2)\cdot p_{R\leq I_m-1}+C_2$....(12·52)

Now consider I_{m0} such that

$$\left.\begin{aligned}&(C_1+C_2)\cdot p_{R\leq I_{m_0}}-C_2>0,\\ \text{and}\quad &-(C_1+C_2)\,p_{R\leq I_{m_0}-1}+C_2>0.\end{aligned}\right\} \qquad \text{...(12·53)}$$

For any integer I'_m greater than I_{m_0} and for any integer I''_m lesser than I_{m_0}, inequations (12·53) would hold since $p_{R\leq I_{m_0}}$ is non-decreasing for increasing I_{m_0}. Hence if inequations (12·53) hold, then

$$C(I'_m)>C(I_{m_0}),\ \text{for } I'_m>I_{m_0};$$

and

$$C(I''_m)>C(I_{m_0}),\ \text{for } I''_m<I_{m_0}.$$

Thus I_{m_0} is the optimum value of I_m which minimizes the total expected cost and it satisfies inequations (12·53). These inequalities may be rearranged to give

$$P_{R\leq I_{m_0}-1}<\frac{C_2}{C_1+C_2}<p_{R\leq I_{m_0}} \qquad \text{...(12·54)}$$

$\therefore$ If the oversupply cost C_1 and the shortage cost C_2 are known, the optimum quantity I_{m_0} is determined when the value of

cumulative probability distribution exceeds the ratio

$$\frac{C_2}{C_1+C_2},$$

i.e., I_{m_0} is determined by comparing a cost ratio with probability figures.

Note that if I_{m_0} is such that

$$p_{R \leq I_{m_0}-1} < \frac{C_2}{C_1+C_2} = p_{R \leq I_{m_0}}, \text{ then equation (12·51) gives}$$

$$C(I_{m_0}+1) = C(I_{m_0}).$$

In this case optimum value of I_m is either I_{m_0} or $I_{m_0}+1$.

Similarly, if I_{m_0} is such that

$$p_{R \leq I_{m_0}-1} = \frac{C_2}{C_1+C_2} < p_{R \leq I_{m_0}}, \text{ the equation (12·52) gives}$$

$$C(I_{m_0}-1) = C(I_{m_0}).$$

In this case optimum value of I_m is either $I_{m_0}-1$ or I_{m_0}.

EXAMPLE 12·6-1

A newspaper boy buys papers for 5 paise each and sells them for 6 paise each. He cannot return unsold newspapers. Daily demand R for newspapers follows the distribution :

R :	10	11	12	13	14	15	16
p_R :	0·05	0·15	0·40	0·20	0·10	0·05	0·05

If each day's demand is independent of the previous day's how many papers should be ordered each day ?

Solution. Let I_m be the number of newspapers ordered per day and R be the demand for it *i.e.*, the number that are actually sold per day.

Now C_1 = Rs. 0·05,

C_2 = Rs. (0·06−0·05) = Rs. 0·01.

The probabilities for demand are

R :	10	11	12	13	14	15	16
R_R :	0·05	0·15	0·40	0·20	0·10	0·05	0·05

$$\sum_{R=0}^{I_m} p_R : 0{\cdot}05 \quad 0{\cdot}20 \quad 0{\cdot}60 \quad 0{\cdot}80 \quad 0{\cdot}90 \quad 0{\cdot}95 \quad 1{\cdot}00$$

The desired optimum value for I_m is determined by double inequality,

$$P_{R \leq I_m - 1} < \frac{C_2}{C_1+C_2} < p_{R < I_m}$$

Now $\dfrac{C_2}{C_1+C_2} = \dfrac{0{\cdot}01}{0{\cdot}01+0{\cdot}5} = \dfrac{1}{6} = 0{\cdot}167.$

This suggests that I_{m_0} must lie between 10 and 11 because

$$0{\cdot}05 < 0{\cdot}167 < 0{\cdot}20.$$

$\therefore \quad I_{m_0} = 11.$

EXAMPLE 12·6·2

Some of the spare parts of a ship cost Rs. 50,000 each. These spare parts can only be ordered together with the ship. If not ordered at the time the ship is constructed, these parts cannot be available on need. Suppose that a loss of Rs. 4,500,000 is suffered for each spare that is needed when none is available in the stock. Further suppose that the probabilities that the spares will be needed as replacement during the life term of the class of ship discussed are

Spares required	*Probability*
0	0·900
1	0·040
2	0·025
3	0·020
4	0·010
5	0·005
6 or more	0·000
Total	1·000

How many spare parts should be procured ?

Solution. $C_1 = \text{Rs. } 50{,}000,$

$C_2 = \text{Rs. } 4{,}500{,}000.$

$$\frac{C_2}{C_1+C_2} = \frac{4{,}500{,}000}{4{,}550{,}000} = 0{\cdot}988.$$

$$\text{Now} \sum_{R=0}^{I_m} p_R > 0.988.$$

$\therefore$ Cumulative frequency distribution is

Spares required :	0	1	2	3	4	5	6 or more
$\sum_{0}^{I_m} p_R$:	0.900	0.940	0.965	0.985	0.995	1.000	1.000

$$\therefore \qquad I_{m_0} = 4.$$

EXAMPLE 12·6-3

(*a*) A firm is to order a new lathe. Its power unit is an expensive part and can be ordered only with the lathe. Each of these units is uniquely built for a particular lathe and cannot be used on any other. The firm wants to know how many spare units should be incorporated in the order for each lathe. Cost of the unit when ordered with the lathe is Rs. 700. If a spare unit is needed (because of its failure during service) and is not available, the whole lathe becomes useless. The cost of the unit made to order and the down time cost of lathe is Rs. 9,300. The analysis of 100 similar units on similar lathes yields the following information given in table 12·1.

(*b*) If in the above problem the shortage cost of the part is unknown and the firm wants to maintain stock level of 4 parts, find the shortage cost.

Table 12·1

No. of spare units required	*No. of lathes requiring indicated number of spare units*	*Estimated probability of occurrence of indicated number of failures.*
0	87	0.87
1	5	0.05
2	3	0·03
3	2	0.02
4	1	0.01
5	1	0.01
6	1	0.01
7 or more	0	0.00

Solution (*a*) The range of optimum value of stock level, I_m is given by

$$P_{R \leq I_m - 1} < \frac{C_2}{C_1 + C_2} < P_{R \leq I_m} \qquad \text{[Equation (12·54)]}$$

Table 12.2 gives the data of table 12.1 after reformulation.

Table 12.2

I_m	R	P_R	$P_R \leq I_m \left(\sum_{R=0}^{I_m} p_R \right)$
0	0	0.87	0.87
1	1	0.05	0.92
2	2	0.03	0.95
3	3	0.02	0.97
4	4	0.01	0.98
5	5	0.01	0.99
6	6	0.01	1.00
7 or more		0.00	1.00
		1.00	

Now $\dfrac{C_2}{C_1+C_2} = \dfrac{9{,}300}{700+9{,}300} = 0.93.$

$\therefore \; P_{R \leq I_m - 1} < 0.95 < P_{R \leq I_m}.$

$\therefore$ Optimum value of $I_m = 2$, since $P_{R \leq 1} < 0.93 < P_{R \leq 2}$, *i.e.*, $0.92 < 0.93 < 0.95$.

(*b*) Here $I_m = 4$.

$$\therefore \qquad P_{R \leq 3} < \frac{C_2}{700 + C_2} < P_{R \leq 4}$$

or
$$0.97 < \frac{C_2}{700+C_2} < 0.98.$$

$\therefore$ The least value of C_2 is given by

$$\frac{C_2}{700+C_2} = 0.97 \quad \text{or} \quad C_2 = \frac{700 \times 0.97}{0.03} = \text{Rs. } 22,633.33,$$

and the greatest value of C_2 is given by

$$\frac{C_2}{700+C_2} = 0.98 \quad \text{or} \quad C_2 = \frac{700 \times 0.98}{0.02} = \text{Rs. } 34,300.$$

$\therefore$ The value of shortage cost ranges from Rs. 22,633.33 to Rs. 34,300.

12.6-2 Model 3 (b) (Instantaneous Demand, No Setup Cost, Stock Levels Continuous, Lead Time Zero).

In this model, all conditions are same as in model 3 (*a*) except that the stock levels are continuous (rather than discrete). Therefore, probability $f(x)\,dx$ will be used instead of p_R, where $f(x)$ is the probability density function.

The cost equation for this model is similar to the one derived for model 3 (*a*) with the difference that p_R is replaced by $f(R).\,dR$ and the summation is replaced by an integral. Let I_m be the quantity is stock, C_1 be the holding (penalty) cost per unit of oversupply ($R < I_m$) and C_2 the penalty cost per unit of undersupply ($R > I_m$), where R is the demand rate with probability $f(R)$.

Expected size of oversupply

$$= \int_0^{I_m} (I_m - R).\, f(R).\, dR, \text{ and}$$

expected size of undersupply

$$= \int_{I_m}^{\infty} (R - I_m) f(R).\, dR.$$

$\therefore$ The total expected cost,

$$C(I_m) = C_1 \int_0^{I_m} (I_m - R) f(R).\, dR + C_2 \int_{I_m}^{\infty} (R - I_m) f(R).\, dR \quad \ldots(12.55)$$

C (I_m) will be minimum if

$$\frac{d}{d\,I_m}[C(I_m)] = 0 \text{ and } \frac{d^2}{dI_m^2}[C(I_m)] > 0.$$

Now we know that if

$$C(I_m) = \int_{a(I_m)}^{b(I_m)} f(R, I_m).\, dx, \quad \text{then}$$

$$\frac{d}{dI_m}\Big[\, C(I_m) \,\Big] = \int_{a(I_m)}^{b(I_m)} \frac{\partial}{\partial I_m}.\, f(R, I_m).\, dx + \Big[\, f(R, I_m).\, \frac{dR}{d\,I_m} \Big]_{a(I_m)}^{(b)I_m}.$$

$\therefore$ Differentiating equation (12.55) w.r.t. I_m we get

$$\frac{d}{dI_m}\Big[\, C\,(I_m) \,\Big] = C_1 \int_{R=0}^{I_m} (1 - 0).\, f(R).\, dR + C_1 \Big[(I_m - R).f(R).\, \frac{dR}{dI_m} \Big]_{R=0}^{I_m}$$

$$+C_2\int_{I_m}^{\infty}(0-1).\,f(R).\,dR+C_2\Big[(R-I_m).\,f(R).\,\frac{dR}{dI_m}\Big]_{R=I_m}^{\infty}$$

$$=C_1\int_{R=0}^{I_m} f(R).\,dR+0-C_2\int_{I_m}^{\infty} f(R).\,dR+0$$

$$=C_1\int_{R=0}^{I_m} f(R).\,dR-C_2\int_{I_m}^{\infty} f(R).\,dR$$

$$=C_1\int_{R=0}^{I_m} f(R).\,dR-C_2\left[\int_{0}^{\infty} f(R).\,dR-\int_{0}^{I_m} f(R).\,dR\right]$$

$$=(C_1+C_2)\int_{R=0}^{I_m} f(R).\,dR-C_2\int_{R=0}^{\infty} f(R).\,dR.$$

$$\therefore\quad \frac{d}{dI_m}\Big[\,C(I_m)\,\Big]=(C_1+C_2)\int_{R=0}^{I_m} f(R)\,dR-C_2.$$

$$\because\quad \frac{d}{d\,I_m}\Big[\,C(I_m)\,\Big]=0 \text{ for optimality,}$$

$$(C_1+C_2)\int_{R=0}^{I_m} f(R).\,dR-C_2=0$$

or
$$\int_{R=0}^{I_m} f(R).\,dR=\frac{C_2}{C_1+C_2}. \qquad \ldots(12\cdot56)$$

Furthermore $\dfrac{d^2}{d\,I_m^2}\Big[\,C(I_m)\,\Big]=(C_1+C_2)\Big[f(R).\,\dfrac{dR}{d\,I_m}\Big]_0^{I_m}$

$$=(C_1+C_2).\,f(I_m).$$

$\because$ C_1, C_2 are positive constants and $f(I_m)$ is >0,

$$\frac{d^2}{dI_m^2}\Big[\,C(I_m)\,\Big]>0.$$

$\therefore$ The optimum value of I_m is the one that satisfies equation (12.56).

EXAMPLE 12.6-4

A baking company sells one of its types of cakes by weight. It makes a profit of 95 paise a pound on every pound of cake sold on the day it is baked. It disposes of all cakes not sold on the day they are baked at a loss of 15 paise a pound. If demand is known to be rectangular between 3,000 and 4,000 pounds, determine the optimum amount to be baked.

[*Delhi M.Sc.* (*Math.*) *1973*]

Solution. Penalty cost/unit of oversupply, $C_1 = \text{Rs. } 0.15$,

Penalty cost/unit of undersupply, $C_2 = \text{Rs. } 0.95$,

$R_1 = 3{,}000$ pounds,

$R_2 = 4{,}000$ pounds.

$$f(R) = \frac{1}{R_2 - R_1} = \frac{1}{1{,}000}.$$

Optimum value of I_m is given by equation (12·36), which is

$$\int_0^{I_m} f(R).\, dR = \frac{C_2}{C_1 + C_2}.$$

$$\therefore \int_{3{,}000}^{I_m} \frac{1}{1{,}000}\,.\,dR = \frac{0.95}{0.15 + 0.95} = \frac{0.95}{1{\cdot}10}.$$

$$\therefore \frac{1}{1{,}000}\Big[\,I_m - 3{,}000\,\Big] = \frac{0.95}{1{\cdot}1}$$

or $\quad I_m = \dfrac{950}{1{\cdot}1} + 3{,}000 = 3{,}864$ units.

EXAMPLE 12.6-5

A baking company sells one of its types of cake by weight. It makes a profit of 95 paise a pound on every pound of cake sold on the day it is baked. It disposes of all cakes not sold on the day they are baked at a loss of 15 paise a pound. If demand is known to be triangular with probability density function

$$f(R) = 0.03 - 0.0003R,$$

find the optimum amount of cake the company should bake daily.

Solution. Using the relation

$$\int_0^{I_m} f(R).\, dR = \frac{C_2}{C_1 + C_2}, \text{ we get}$$

$$\int_0^{I_m} (0.03-0.0003R)\,dR = \frac{0.95}{0.15+0.95} = \frac{0.95}{1.1}$$

$$\therefore\ 0.03\,I_m - \frac{0.0003\,I_m^2}{2} = \frac{0.95}{1.1}$$

or $0.03\,I_m - 0.00015\,I_m^2 = 0.8636$

or $3{,}000\,I_m - 15I_m^2 = 0.86360$

or $200\,I_m - I_m^2 = 5{,}757$

or $I_m^2 - 200\,I_m + 5{,}757 = 0$

or $$I_m = \frac{200 \pm \sqrt{(200)^2 - 4\times 5{,}757}}{2}$$

$$= \frac{200 \pm 130.3}{2} = 165.15 \text{ or } 34.85 \text{ pounds.}$$

$I_m = 165.15$ pounds is not feasible since the given probability distribution of R is not applicable above 100 pounds.

$\therefore$ Optimum value of $I_m = 34.85$ pounds/day.

12.6-3. Model 4 (a) (Continuous Demand, Setup Cost Zero, Stock Levels Discrete, Lead Time Zero)

This model is similar to model 3(*a*) with the difference that demand is continuous rather than instantaneous *i.e.*, withdrawals from stock are continuous rather than instantaneous. Also the rate of withdrawals is assumed to be constant. Let

R = demand rate with probability p_R,

I_m = discrete stock level for time interval t,

t = prescribed constant time interval between orders,

C_1 = holding cost per item per unit time,

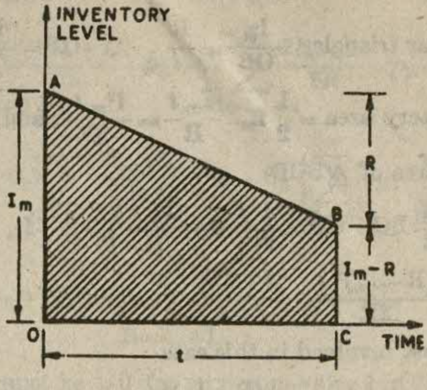
C_2 = shortage cost per item per unit time.

The reorder time is assumed to be fixed and known; hence setup cost is not included in calculations. Production is assumed to be instantaneous and lead time negligibly small.

The problem is to determine the optimal order level I_m where $R \leqslant I_m$ or $R > I_m$ at the beginning of each time period. The variation of inventory with time for these two cases is shown in Fig. 12.8(*a*) and (*b*).

When $R \leqslant I_m$, as shown in Fig. 12.8 (*a*), there are no shortages; when $R > I_m$, as shown in figure 12.8 (*b*), shortages occur. The inventory units may be either discrete or continuous ; in this model we

shall consider inventory comprising of discrete units only. The cost equation for this model may be derived as follows :



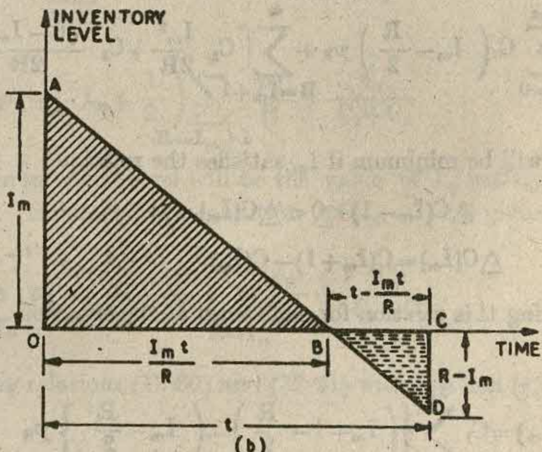


Fig. 12.8. Inventory situation for model 4 (*d*) ; (*a*) $R \leqslant I_m$ (*b*) $R > I_m$.

1. When $R \leqslant I_m$, the only cost involved is the holding cost, which is

$$= C_1 . \text{ (Inventory area OABC)}$$

$$= C_1 \frac{I_m + (I_m - R)}{2} . t$$

$$= C_1 \left(I_m - \frac{R}{2} \right) t.$$

2. When $R > I_m$, the shortage cost is also involved.

Now inventory area = area of $\triangle$OAB

$$=\frac{1}{2}\ \text{OA.OB}=\frac{1}{2}\ I_m.\text{OB}.$$

From similar triangles, $\dfrac{I_m}{\text{OB}}=\dfrac{R}{t}$ $\quad\therefore\quad$ $\text{OB}=\dfrac{I_m\cdot t}{R}$.

$\therefore$ Inventory area $=\dfrac{1}{2}I_m\cdot\dfrac{I_m t}{R}=\dfrac{I_m^2 t}{2R}$, and

shortage area = area of $\triangle$BCD

$$=\frac{1}{2}\,\text{BC}\,.\,\text{CD}=\frac{1}{2}\left(t-\frac{I_m t}{R}\right)\left(R-I_m\right)$$

$$=\frac{(R-I_m)^2\,t}{2R}.$$

$\therefore$ The cost involved in this case

$$=C_1\cdot\frac{I_m^2\,t}{2R}+C_2\,\frac{(R-I_m)^2\,t}{2R}.$$

$\therefore$ The total expected cost/unit time is given by

$$C(I_m)=\sum_{R=0}^{I_m}C_1\left(I_m-\frac{R}{2}\right)p_R+\sum_{R=I_m+1}^{\infty}\left[C_1\,\frac{I_m^2}{2R}+C_2\,\frac{(R-I_m)^2}{2R}\right]\cdot p_R. \qquad \ldots(12\cdot57)$$

$C(I_m)$ will be minimum if I_m satisfies the relation

$$\triangle C(I_m-1)<0<\triangle C(I_m), \qquad \ldots(12\cdot58)$$

where $$\triangle C(I_m)=C(I_m+1)-C(I_m). \qquad \ldots(12\cdot59)$$

Applying this relation for each term of the equation (12·57), we have

$$\triangle C(I_m)=C_1\sum_{R=0}^{I_m}\left\{\left(I_m+1-\frac{R}{2}\right)-\left(I_m-\frac{R}{2}\right)\right\}p_R$$

$$+C_1\sum_{R=I_m+1}^{\infty}\left\{\frac{(I_m+1)^2}{2R}-\frac{I_m^2}{2R}\right\}p_R$$

$$+C_2\sum_{R=I_m+1}^{\infty}\left\{\frac{(R-I_m-1)^2}{2R}-\frac{(R-I_m)^2}{2R}\right\}\cdot p_R$$

$$=C_1\sum_{R=0}^{I_m}p_R+C_1\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R+C_2\sum_{R=I_m+1}^{\infty}\frac{(2R-2I_m-1)(-1)}{2R}\cdot p_R$$

$$=C_1\sum_{R=0}^{I_m} p_R+C_1\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R-C_2\sum_{R=I_m+1}^{\infty}\frac{2R-2I_m-1}{2R}\cdot p_R$$

$$=C_1\sum_{R=0}^{I_m} p_R+C_1\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R+C_2\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R-C_2\sum_{R=I_m+1}^{\infty} p_R$$

$$=C_1\sum_{R=0}^{I_m} p_R+(C_1+C_2)\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R-C_2\left\{\sum_{R=0}^{\infty} p_R-\sum_{R=0}^{I_m} p_R\right\}$$

$$=\left(C_1+C_2\right)\sum_{R=0}^{I_m} p_R+(C_1+C_2)\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R-C_2.$$

But $\triangle C(I_m)$ must be >0 for minimum value of $C(I_m)$.

$$\therefore\quad \sum_{R=0}^{I_m} p_R+\sum_{R=I_m+1}^{\infty}\frac{2I_m+1}{2R}\cdot p_R>\frac{C_2}{C_1+C_2}$$

or
$$\sum_{R=0}^{I_m} p_R+\left(I_m+\frac{1}{2}\right)\sum_{R=I_m+1}^{\infty}\frac{p_R}{R}>\frac{C_2}{C_1+C_2} \qquad \text{...(12·60)}$$

The optimum stock level will be the value of I_m satisfying the above relation. Similarly, the condition $\triangle C(I_m-1)<0$ gives

$$\sum_{R=0}^{I_m-1} p_R+\left(I_m-\frac{1}{2}\right)\sum_{R=I_m}^{\infty}\frac{p_R}{R}<\frac{C_2}{C_1+C_2}. \qquad \text{...(12·61)}$$

Combining relations (12·60) and (12·61) with relation (12·58), we get

$$\sum_{R=0}^{I_m-1} p_R+\left(I_m-\frac{1}{2}\right)\sum_{R=I_m}^{\infty}\frac{p_R}{R}<\frac{C_2}{C_1+C_2}<\sum_{R=0}^{I_m} p_R+\left(I_m+\frac{1}{2}\right)\sum_{R=I_m+1}^{\infty}\frac{p_R}{R}. \qquad \text{...(12·62)}$$

Relation (12·62) may also be written as

$$L(I_m-1)<\frac{C_2}{C_1+C_2}<L(I_m). \qquad \text{...(12·63)}$$

Relation (12·62) gives the range of optimum value of I_m.

12·6·4. Model 4(b) (Continuous Demand, Setup Cost Zero, Continuous Stock Levels, Lead Time Zero)

The cost equation for this model is similar to that of model 4(*a*) with the difference that p_R is replaced by $f(R) \cdot dR$ and summation (Σ) is replaced by integration ($\int$). Thus the expected cost equation is

$$C(I_m)=\int_0^{I_m} C_1\left(I_m-\frac{R}{2}\right) f(R) \cdot dR+\int_{I_m}^{\infty}\left\{\frac{C_1 I_m^2}{2R}+\frac{C_2(R-I_m)^2}{2R}\right\} f(R) \cdot dR. \quad ...(12\cdot 64)$$

Differentiating equation (12·64) w.r.t. I_m,

$$\frac{dC}{dI_m}[(I_m)]=C_1\int_0^{I_m}(1-0)\, f(R) \cdot dR+C_1\left[\left(I_m-\frac{R}{2}\right)\cdot f(R)\cdot\frac{dR}{dI_m}\right]_0^{I_m}$$

$$+\int_{I_m}^{\infty}\left\{\frac{C_1}{2R}\cdot 2I_m+\frac{C_2}{2R}\cdot 2\cdot(R-I_m)(-1)\right\}\cdot f(R)\cdot dR$$

$$+\left[\left\{C_1\frac{I_m^2}{2R}+C_2\frac{(R-I_m)^2}{2R}\right\}\cdot f(R)\cdot\frac{dR}{dI_m}\right]_{I_m}^{\infty}$$

$$=C_1\int_0^{I_m} f(R)\, dR+C_1\left[\left(I_m-\frac{I_m}{2}\right)\cdot f(I_m)\cdot 1\right]$$

$$+\int_{I_m}^{\infty}\left(\frac{C_1+C_2}{R}\cdot I_m-C_2\right) f(R)\cdot dR+(-1)\left[\left\{C_1\frac{I_m}{2}+C_2(0)\right\} f(I_m)\cdot 1\right]$$

$$=C_1\int_0^{I_m} f(R)\, dR+\frac{C_1}{2} I_m f(I_m)+\int_{I_m}^{\infty}(C_1+C_2)\cdot\frac{I_m}{R} f(R)\cdot dR$$

$$-C_2\int_{I_m}^{\infty} f(R)\cdot dR-\frac{C_1}{2} I_m\cdot f(I_m)$$

$$=C_1\int_0^{I_m} f(R)\, dR+(C_1+C_2)\int_{I_m}^{\infty} I_m\frac{f(R)}{R}\cdot dR-C_2\int_{I_m}^{\infty} f(R)\, dR$$

$$=C_1\int_0^{I_m} f(R)dR+(C_1+C_2)\int_{I_m}^{\infty} I_m\,\frac{f(R)}{R}\,dR-C_2\left[\int_0^{\infty} f(R)\,dR-\int_0^{I_m} f(R).dR\right]$$

$$\therefore \frac{d\,[C(I_m)]}{d\,I_m}=(C_1+C_2)\int_0^{I_m} f(R)\,.\,dR+(C_1+C_2)\int_{I_m}^{\infty} I_m\,.\,\frac{f(R)}{R}\,.\,dR-C_2. \quad ...(12\cdot65)$$

For $C(I_m)$ to be minimum, $\frac{d}{dI_m}\left[\,C(I_m)\,\right]=0.$

$$\therefore \quad \int_0^{I_m} f(R)\,.\,dR+\int_{I_m}^{\infty} I_m\,.\,\frac{f(R)}{R}\,.\,dR=\frac{C_2}{C_1+C_2}, \qquad ...(12\cdot66).$$

and

$$\frac{d^2}{dI_m^2}\left[\,C(I_m)\,\right]=(C_1+C_2)\Big[\,f(R)\,\Big]_0^{I_m}+(C_1+C_2)\int_{I_m}^{\infty}\frac{f(R)}{R}\,.\,dR+(C_1+C_2)$$

$$\left[\,I_m\,.\,\frac{f(R)}{R}\,.\,\frac{dR}{dI_m}\right]_{I_m}^{\infty}-0$$

$$=(C_1+C_2)\,f(I_m)+(C_1+C_2)\int_{I_m}^{\infty}\frac{f(R)}{R}\,.\,dR+(C_1+C_2)$$

$$\left[0-I_m\,\frac{f(I_m)}{I_m}\,.\,1\right]$$

$$=(C_1+C_2)\int_{I_m}^{\infty}\frac{f(R)}{R}\,.\,dR$$

$$=+ve.$$

Equation (12·66) gives the optimum value of I_m for minimum total expected cost per unit time.

EXAMPLE 12.6-6

Let the probability density of demand of a certain item during a week be

$$f(x)=\begin{cases}0\cdot1, & 0\leqslant x\leqslant 10;\\ 0, & \text{otherwise.}\end{cases}$$

This demand is assumed to occur with a uniform pattern over the week. Let the unit carrying cost of the item in inventory be Rs. 2 per week and unit shortage cost be Rs. 8 per week. How will you determine the optimal order level of the inventory ?

[*Agra M. Stat. 1973*]

Solution. Here $f(x)=0{\cdot}1,\ 0\leqslant x\leqslant 10,$

$C_1=$ Rs. 2/week,

$C_2=$ Rs. 8/week.

As the demand is uniform over the week, the optimum order level of the inventory, I_m is given by

$$\int_0^{I_m} f(x)\,.\,dx+I_m\int_{I_m}^{\infty}\frac{f(x)}{x}\,.\,dx=\frac{C_2}{C_1+C_2}$$

or
$$\int_0^{I_m} 0{\cdot}1\,dx+I_m\int_{I_m}^{10}\frac{0{\cdot}1}{x}\,.\,dx=\frac{8}{2+8}$$

or
$$0{\cdot}1\,I_m+0{\cdot}1\,I_m\,(\log 10-\log I_m)=0{\cdot}8$$

or
$$I_m+2{\cdot}3\,I_m-I_m\log I_m=8$$

or
$$3{\cdot}3\,I_m-I_m\log I_m=8.$$

On solving the equation by trial and error method, we get $I_m=4{\cdot}5$.

EXAMPLE 12·6-7

The probability distribution of monthly sales of a certain item is as follows :

Monthly sales :	0	1	2	3	4	5	6	7	8
Probability :	0·1	0·04	0·25	0·30	0·23	0·08	0·05	0·03	0·01

The cost of holding inventory is Rs. 8 per unit per month. A stock of 5 items is maintained at the start of each month. If the shortage cost is proportional to both time and quantity short, find the imputed cost of shortage of unit item for unit time.

Solution. As the problem is stated in discrete units, the answer will consist of a range of values for the imputed cost. Here,

optimum inventory, $I_m=5$ units,

holding cost, $C_1=$ Rs. 8 per unit per month.

Range of monthly sales $=0$ to 8.

Probability p_R for sale R in each month is

p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
0·01	0·04	0·25	0·30	0·23	0·08	0·05	0·03	0·01

Range of optimum value of I_m given by

$$\sum_{R=0}^{I_m-1} p_R + \left(I_m - \frac{1}{2}\right)\sum_{R=I_m}^{\infty} \frac{p_R}{R} < \frac{C_2}{C_1+C_2} < \sum_{R=0}^{I_m} p_R$$

$$+\left(I_m+\frac{1}{2}\right)\sum_{R=I_m+1}^{\infty} \frac{p_R}{R}. \quad \left[\text{ Equation } (12\cdot62)\right]$$

$\therefore$ The least value of C_2 is given by

$$\sum_{R=0}^{I_m-1} p_R + \left(I_m - \frac{1}{2}\right)\sum_{R=I_m}^{\infty} \frac{p_R}{R} = \frac{C_2}{C_1+C_2}$$

or
$$\sum_{R=0}^{4} p_R + \left(5 - \frac{1}{2}\right)\sum_{R=5}^{8} \frac{p_R}{R} = \frac{C_2}{8+C_2}$$

or
$$(p_0+p_1+p_2+p_3+p_4)+\frac{9}{2}\left(\frac{p_5}{5}+\frac{p_6}{6}+\frac{p_7}{7}+\frac{p_8}{8}\right)=\frac{C_2}{8+C_2}$$

or
$$(0\cdot01+0\cdot04+0\cdot25+0\cdot30+0\cdot23)+\frac{9}{2}\left(\frac{0\cdot08}{5}+\frac{0\cdot05}{6}+\frac{0\cdot03}{7}+\frac{10\cdot0}{8}\right)=\frac{C_2}{8+C_2}$$

or
$$0\cdot83+4\cdot5(0\cdot016+0\cdot0083+0\cdot0043+0\cdot00125)=\frac{C_2}{8+C_2}$$

or
$$0\cdot83+4\cdot5\times 0.02985=\frac{C_2}{8+C_2}$$

or
$$0\cdot9643=\frac{C_2}{8+C_2} \quad \text{or} \quad C_2=\frac{0\cdot9643\times 8}{0\cdot0357}=\text{Rs } 216.$$

Similarly, the greatest value of C_2 is given by

$$\sum_{R=0}^{I_m} p_R + \left(I_m + \frac{1}{2}\right)\sum_{R=I_m+1}^{\infty} \frac{p_R}{R} = \frac{C_2}{C_1+C_2}$$

or
$$\sum_{0}^{5} p_R + \left(5 + \frac{1}{2}\right)\sum_{R=6}^{8} \frac{p_R}{R} = \frac{C_1}{8+C_2}$$

or
$$(p_0+p_1+p_2+p_3+p_4+p_5)+\frac{11}{2}\left(\frac{p_6}{6}+\frac{p_7}{7}+\frac{p_8}{8}\right)=\frac{C_2}{8+C_2}$$

or
$$(0\cdot01+0\cdot04+0\cdot25+0\cdot30+0\cdot23+0.08)+\frac{11}{2}\left(\frac{0\cdot05}{6}+\frac{0\cdot03}{7}+\frac{0\cdot01}{8}\right)=\frac{C_2}{8+C_2}$$

or
$$0\cdot91+\frac{11}{2}\,(0\cdot01385)=\frac{C_2}{8+C_2}$$

or $\quad 0{\cdot}91+0{\cdot}076165=\frac{C_2}{8+C_2}$

or $\quad C_2=\text{Rs. } 576{\cdot}62.$

$\therefore$ Range of values for the imputed cost C_2 is Rs. $216<C_2$ $<$Rs. 576·62.

EXAMPLE 12·6-8

A manufacturer wants to determine the optimum stock level of a certain part. The part is used in filling orders which come in at a constant rate. The delivery of these parts to him is almost instantaneous. He places his orders for these parts at the start of every month. The requirements per month are associated with probabilities shown in table 12·3. Holding cost is Re. 1 per part per month and shortage cost is Rs. 19 per part per month. Also find the expected cost associated with the optimum stock level.

Table 12·3

No. of parts required/month	*Probability*
0	0.10
1	0.15
2	0.25
3	0.30
4	0.15
5	0.05
6 or more	0.00

Solution. We write equation (12·62) as

$$\sum_{R=0}^{I_m-1} p_R+(I_m-\tfrac{1}{2})\sum_{R=I_m}^{\infty}\frac{p_R}{R}<\frac{C_2}{C_1+C_2}<\sum_{R=0}^{I_m} p_R+(I_m+\tfrac{1}{2})\sum_{R=I_m+1}^{\infty}\frac{p_R}{R}.$$

In this example, $\frac{C_2}{C_1+C_2}=\frac{19}{1+19}=0.95.$

The data given in table 12.3 is reformulated and presented in table 12.4.

Table 12·4

I_m	R	p_R	$\sum_{R=0}^{I_m} p_R$	$\frac{p_R}{R}$	$\sum_{R=I_m+1}^{\infty} \frac{p_R}{R}$	$\left(I_m+\frac{1}{2}\right)\sum_{R=I_m+1}^{\infty} \frac{p_R}{R}$	$\sum_{R=0}^{I_m} p_R+(I_m+\frac{1}{2}) \sum_{R=I_m+1}^{\infty} \frac{p_R}{R}$
0	0	0.10	0.10	∞	0.4225	0.21125	0.31125
1	1	0.15	0.25	0.1500	0.2725	0.40875	0.65875
2	2	0.25	0.50	0.1250	0.1475	0.36875	0.86875
3	3	0.30	0.80	0.1000	0.0475	0.16625	0.96625
4	4	0.15	0.95	0.0375	0.0100	0.04500	0.99500
5	5	0.05	1.00	0.0100	0.0000	0.00000	1.00000
6 or more	>5	0.00	1.00	0.0000	0.0000	0.00000	1.00000

From table 12·4, we now select that value of I_m which satisfies relation (12.62) written above. We find that $I_m=3$ satisfies this relation, since

$$0.86875 < 0.95 < 0.96625.$$

The total expected cost associated with stock level of 3 units (parts) is given by the equation

$$C(I_m)=\sum_{R=0}^{I_m} C_1\left(I_m-\frac{R}{2}\right)p_R+\sum_{R=I_m+1}^{\infty}$$

$$\left[C_1\cdot\frac{I_m^2}{2R}+C_2\cdot\frac{(R-I_m)^2}{2R}\right]p_R \qquad \text{[Equation (12.57)]}$$

$$\text{or } C(I_m)=C_1\sum_{R=0}^{I_m}\left(I_m-\frac{R}{2}\right)p_R+C_1\sum_{R=I_m+1}^{\infty}\frac{I_m^2}{2R}\cdot p_R$$

$$+C_2\sum_{R=I_m+1}^{\infty}\frac{(R-I_m)^2}{2R}\cdot p_R$$

$$=\text{Rs.}\left[1.\sum_{R=0}^{3}\left(3-\frac{R}{2}\right)p_R+1.\sum_{R=4}^{\infty}\frac{3^2}{2}\cdot\frac{p_R}{R}\right.$$

$$\left.+19\sum_{R=4}^{\infty}\frac{(R-3)^2}{2R}\cdot p_R\right]$$

$$=\text{Rs.}\left[\left\{(3-0)\ (0.10)+\left(3-\frac{1}{2}\right)(0.15)+(3-1)\ (0.25)\right.\right.$$

$$\left.+\left(3-\frac{3}{2}\right)(0.30)\right\}+\left\{\frac{9}{2}\left(\frac{0.15}{4}+\frac{0.05}{5}+0\right)\right\}$$

$$\left.+19\left\{\frac{(4-3)^2}{2\times4}\cdot\ (0.15)+\frac{(5-3)^2}{2\times5}(0.05)+0\right\}\right]$$

$$=\text{Rs. }[(0.30+0.375+0.50+0.45)+0.21375+0.73625]$$

$$=\text{Rs. } 2.58.$$

12.6-5. Model 5 (a) (Continuous Demand, Zero Setup Cost, Stock Levels Discrete with Lead Time)

This model is similar to model 4 (*a*) with the difference that lead time is not zero. Lead time is the time interval between placing an order for inventory and arrival of goods for inventory. The discrete case is discussed here while the continuous case will be dealt with in the next section.

Let

I_m = discrete stock level,

t_p = the prescribed interval between orders,

C_1 = holding cost per item per unit time,

C_2 = shortage cost per item per unit time,

C_3 = setup cost per production run,

x = demand during prescribed time t_p with probability p_x,

y = demand during lead time L with probability p_y.

Since setup cost C_3 and prescribed time interval t_p are both constant, the average setup (replenishment) cost per unit time $\frac{C_3}{t_p}$ is also constant and hence is not to be considered. It may be noted that quantity in stock at any time depends upon the inventory order level I_m, demand x during time t_p and demand y during lead time L. There are three possible inventory situations depending upon the relative magnitudes of I_m, x and y. They are represented graphically in figures 12.9 (a), (b) and (c).

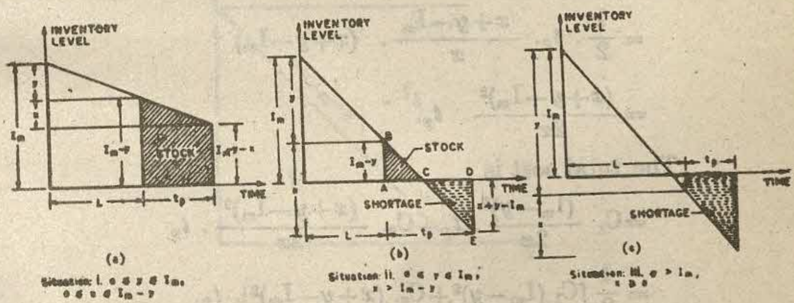


Figure 12.9 (a), (b) and (c). **Inventory situations for model 4 (a).**

The period t_p starts after the lead time L has elapsed. Decision taken in the beginning *i.e.*, before the elapse of lead time L, affects the situation during time t_p. Objective is to minimize the expected cost during t_p.

Situation I : The prescribed time t_p starts with an inventory $I_m - y$ and ends with inventory $I_m - y - x$, so that the total (actual) cost is

$$= C_1 \times \text{hatched area}$$

$$= C_1 \cdot \frac{(I_m - y) + (I_m - y - x)}{2} \cdot t_p$$

$$= C_1 \cdot \left(I_m - y - \frac{x}{2}\right) t_p.$$

Situation II. The prescribed time t_p starts with an inventory $I_m - y$ and ends with a shortage $x+y-I_m$. The area representing inventory cost is the area of hatched triangle ABC, which is

$$=\frac{1}{2}(I_m - y).\ \text{side AC}$$

$$=\frac{1}{2}(I_m - y).\ \frac{I_m - y}{x} t_p$$

$$\left(\because \text{ from similar triangles, } \frac{AC}{t_p} = \frac{I_m - y}{x}\right)$$

$$=\frac{(I_m - y)^2}{2x}.\ t_p.$$

Also, the area representing shortage cost is

$$=\text{area of } \triangle CDE$$

$$=\frac{1}{2}.\ CD.\ DE$$

$$=\frac{1}{2}.\ (t_p - AC).\ (x+y-I_m)$$

$$=\frac{1}{2}\left[\ t_p - \frac{I_m - y}{x}.\ t_p\right]\ (x+y-I_m)$$

$$=\frac{1}{2}.\ t_p.\ \frac{x+y-I_m}{x}.\ (x+y-I_m)$$

$$=\frac{(x+y-I_m)^2}{2x}.\ t_p.$$

$\therefore$ The total cost is

$$=C_1\ \frac{(I_m - y)^2}{2x}.\ t_p + C_2.\ \frac{(x+y-I_m)^2}{2x}.\ t_p$$

$$=\frac{1}{2x}[C_1\ (I_m - y)^2 + C_2.\ (x+y-I_m)^2].\ t_p.$$

Situation III. t_p starts with a shortage $y - I_m$ and ends with a shortage $x+y-I_m$ so that the cost is

$$=C_2.\ \frac{(y-I_m)+(x+y-I_m)}{2}.\ t_p$$

$$=C_2.\ \left(\frac{x}{2} + y - I_m\right) t_p.$$

Let us assume the random variables x and y to be independent so that the joint probability density function is $p_x.\ p_y$. Let C (I_m) represent the total expected cost per unit time. To obtain this cost we multiply the cost associated with each of the three situations by the joint probability of demands for y items during the lead time L and x items during the prescribed time t_p (*i.e.*, we multiply by $p_x p_y$) and sum over the appropriate range of x and y.

$\therefore$ The expected cost per unit time,

$$C(I_m)=\sum_{y=0}^{I_m}\sum_{x=0}^{I_m-y} C_1\left(I_m-y-\frac{x}{2}\right)p_xp_y$$

$$+\sum_{y=0}^{I_m}\sum_{x=I_m-y+1}^{\infty}\frac{1}{2x}\left[C_1\cdot(I_m-y)^2+C_2(x+y-I_m)^2\right]p_xp_y$$

$$+\sum_{y=I_m+1}^{\infty}\sum_{x=0}^{\infty} C_2\cdot\left(\frac{x}{2}+y-I_m\right)\cdot p_xp_y. \quad \ldots(12.67)$$

Now $C(I_m)$ will be minimum if $\triangle C(I_m)>0>\triangle C(I_m-1)$, where $\triangle C\,(I_m)=C(I_m+1)-C(I_m)$.

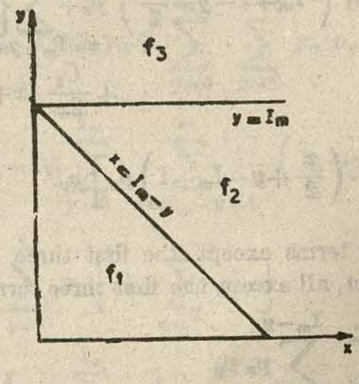


Figure 12.10.

To find $\triangle C(I_m)$ we proceed as in section B. 3-4.2 of appendix B.

The summation is over the first quadrant of xy-plane divided into parts as shown in figures 12.10. Here,

$$f_1(x,\,y;\,z)=C_1\left(I_m-y-\frac{x}{2}\right)p_xp_y,$$

$$f_2(x,\,y;\,z)=\frac{1}{2x}\left[C_1(I_m-y)^2+C_2(x+y-I_m)^2\right]\cdot p_xp_y,$$

$$f_3(x\,,y;\,z)=C_2\cdot\left(\frac{x}{2}+y-I_m\right)p_xp_y,$$

$$b(z)=I_m,$$

$$c(y,\,z)=I_m-y,$$

$$d\,(y,\,z)=+\infty.$$

$$\therefore \triangle C(I_m) = \sum_{y=0}^{I_m} \sum_{x=0}^{I_m-y} C_1 p_x p_y$$

$$+\sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{\infty} \frac{1}{x}\Big[(C_1+C_2)\,(I_m-y+\tfrac{1}{2})-C_2 x\Big] p_x p_y$$

$$-\sum_{y=I_m+1}^{\infty} \sum_{x=0}^{\infty} C_2 p_x p_y$$

$$+\sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{I_m-y+1} \Big[C_1\Big(I_m+1-y-\frac{x}{2}\Big)-\frac{C_1}{2x}\Big(I_m+1-y\Big)^2$$

$$+\frac{C_2}{2x}(x+y-I_m-1)^2\Big] p_x p_y$$

$$+\sum_{y=I_m+1}^{I_m+1} \Big[\sum_{x=0}^{I_m-y+1} C_1\Big(I_m+1-y-\frac{x}{2}\Big) p_x + \sum_{x=I_m-y+2}^{\infty}\Big\{\frac{C_1}{2x}(I_m+1-y)^2$$

$$+\frac{C_2}{2x}(x+y-I_m-1)^2\Big\} p_x$$

$$-\sum_{x=0}^{\infty} C_2 \cdot \Big(\frac{x}{2}+y-I_m-1\Big) p_x\Big] p_y. \qquad \ldots(12\cdot 68)$$

We find that all terms except the first three reduce to single summation, and in fact, all except the first three terms vanish. Thus

$$\triangle C\,(I_m) = C_1 \sum_{y=0}^{I_m} \sum_{x=0}^{I_m-y} p_x\, p_y$$

$$+\sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{\infty} \frac{1}{x}\Big[\Big(C_1+C_2\Big)\Big(I_m-y+\frac{1}{2}\Big)-C_2 x\Big] p_x\, p_y$$

$$-C_2 \sum_{y=I_m+1}^{\infty} \sum_{x=0}^{\infty} p_x\, p_y \qquad \ldots(12\cdot 69)$$

or $$\triangle C(I_m) = C_1 \sum_{y=0}^{I_m} \sum_{x=0}^{I_m-y} p_x\, p_y$$

$$+(C_1+C_2) \sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{\infty} \frac{1}{x}\Big(I_m-y+\frac{1}{2}\Big) p_x\, p_y$$

$$-C_2 \sum_{y=0}^{I_m} p_y \left[\sum_{x=0}^{\infty} p_x - \sum_{x=0}^{I_m-y} p_x\right]$$

$$-C_2 \left[\sum_{y=0}^{\infty} \sum_{x=0}^{\infty} p_x\, p_y - \sum_{y=0}^{I_m} \sum_{y=0}^{\infty} p_x\, p_y\right].$$

The above result can be considerably simplified by using relations

$$\sum_{x=0}^{\infty} p_x = 1 \text{ and}$$

$$\sum_{y=0}^{\infty} p_y = 1.$$

$\therefore$ **We have** $\triangle C(I_m) = C_1 \sum_{y=0}^{I_m} \sum_{x=0}^{I_m-y} p_x\, p_y$

$$+(C_1+C_2) \sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{\infty} \frac{1}{x}\left(I_m - y + \frac{1}{2}\right) p_x\, p_y$$

$$-C_2 \sum_{y=0}^{I_m} p_y \left[1 - \sum_{x=0}^{I_m-y} p_x\right]$$

$$-C_2 \left[1 - \sum_{y=0}^{I_m} p_y\right]$$

or $$\triangle C(I_m) = (C_1+C_2)\left[\sum_{y=0}^{I_m} \sum_{x=0}^{I_m-y} p_x\, p_y + \sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{\infty} \frac{1}{x}\left(I_m - y + \frac{1}{2}\right) p_x\, p_y\right] - C_2. \quad ...(12\cdot70)$$

Now $C(I_m)$ will be minimum if $\triangle C(I_m) \geqslant 0$

or if $$\sum_{y=0}^{I_m} \sum_{x=0}^{I_m-y} p_x\, p_y + \sum_{y=0}^{I_m} \sum_{x=I_m-y+1}^{\infty}$$

$$\frac{1}{x}\left(\mathrm{I}_m-y+\frac{1}{2}\right) p_x\, p_y \geqslant \frac{\mathrm{C}_2}{\mathrm{C}_1+\mathrm{C}_2}. \qquad \ldots(12\cdot71)$$

Equation (12·71) determines the optimum order level.

12.6-6 Model 5(b) (Continuous Demand, Zero Setup Cost, Stock Levels Continuous, Significant Lead Time)

The cost equation for this model is similar to that of model 5(*a*) with the difference that p_x and p_y are replaced by $f(x)\,dx$ and $g(y)dy$ and summations ($\Sigma\Sigma$) are replaced by double integrals ($\int\int$). Thus the expected cost equation is

$$\mathrm{C}(\mathrm{I}_m)=\int_0^{\mathrm{I}_m} g(y)\,.\,dy \int_0^{x-y} \mathrm{C}_1\left(\mathrm{I}_m-y-\frac{x}{2}\right) f(x)\,dx$$

$$+\int_0^{\mathrm{I}_m} g(y)\;dy \int_{\mathrm{I}_m-y}^{\infty}\left[\mathrm{C}_1\,\frac{(\mathrm{I}_m-y)^2}{2x}+\mathrm{C}_2\,\frac{(x+y-\mathrm{I}_m)^2}{2x}\right] f(x)\,dx$$

$$+\int_0^{\mathrm{I}_m} g(y)\;dy \int_0^{\infty}\mathrm{C}_2\left(y-\mathrm{I}_m+\frac{x}{2}\right) f(x)\,dx. \qquad \ldots(12\cdot72)$$

Differentiating equation (12.72) by using the result of case 1 of section C-2 of appendix C (at the end of the book) and simplifying, we get

$$\frac{d}{d\,\mathrm{I}_m}[\mathrm{C}(\mathrm{I}_m)]=(\mathrm{C}_1+\mathrm{C}_2)\int_0^{\mathrm{I}_m} g(y)\,dy\left[\int_0^{\mathrm{I}_m-y}\left(1+\frac{\mathrm{I}_m-y}{x}\right) f(x)\,dx\right]$$

$$-\mathrm{C}_2=0, \qquad \ldots(12\cdot73$$

which is analogous to equation (12·70).

Differentiating once again, we get

$$\frac{d^2}{d\,\mathrm{I}^2_m}[\mathrm{C}(\mathrm{I}_m)]=(\mathrm{C}_1+\mathrm{C}_2)\int_0^{\mathrm{I}_m} g(y)\,dy\left[f(\mathrm{I}_m-y)+\int_0^{\mathrm{I}_m-y}\frac{f(x)}{x}\,.\,dx\right]>0.$$

$$\ldots(12\cdot74)$$

∴ Optimum order level is determined by the equation

$$\int_0^{\mathrm{I}_m} g(y)\,.\,dy\int_0^{\mathrm{I}_m-y}\left[1+\frac{\mathrm{I}_m-y}{x}\right] f(x)\,dx=\frac{\mathrm{C}_2}{\mathrm{C}_1+\mathrm{C}_2} \qquad \ldots(12\cdot75)$$

12·7 Inventory Models with Price Breaks

In the inventory models discussed so far the production or purchase cost per unit was assumed to be constant. It was not considered during their formulation since it did not affect the level of inventory. In this section we shall consider a class of inventory problems in which this cost is variable and depends upon the quantity manufactured or purchased. This usually happens when *discounts* are offered for the purchase of large quantities. These discounts take the form of *price breaks*. For example, the price breaks may be given as

Rs. 1 per item for purchase of items upto 500,

Rs. 0·95 per item for purchase of items upto 1,000,

Rs. 0·90 per item for purchase of items 1,001 or more.

Clearly, the purchase cost $C(q)$ is a variable and is given by the expression

$$C(q)=\begin{cases} 1\cdot q & 0 \leqslant q \leqslant 500, \\ 1\times 500+0\cdot 95(q-500) & 501 \leqslant q \leqslant 1{,}000 \\ 1\times 500+0\cdot 95(1{,}000-500)+0\cdot 90\ (q-1{,}000 & q \geqslant 1{,}001. \end{cases}$$

Such a variable cost must be considered in the inventory model. Further, as this variable production or purchase cost per unit is most appropriate for purchased parts (because of quantity discounts), we shall, hereafter, refer only to purchased parts and the problem, then, is to determine

(*i*) how often the parts be purchased,

(*ii*) how many units should be purchased at any one time.

12·7·1 Basic Cost Equations (Purchase-Inventory Model)

Let R = number of units purchased or manufactured per unit time, *i.e.*, demand rate,

t = interval between placing orders,

q = number of items purchased per order,

K_1 = purchasing cost of one unit,

P = holding cost/month expressed as a fraction of the value of the unit,

C_3 = setup cost per production run or setup cost associated with procurement of the purchased items.

It is assumed that

(*i*) demand rate R is constant,

(*ii*) demand is fixed and known,

(*iii*) no shortages are permitted, *i.e.*, shortage cost $C_2 = \infty$,

(*iv*) production or supply of items is instantaneous.

The problem is to determine

(*a*) how often units be purchased,

(*b*) how many units be purchased at a time.

This model when represented graphically for any *one* value of the unit purchase cost K_1, has the same shape as that of model 1(*a*). As in that model, $q = Rt$ or $t = \frac{q}{R}$.

$\therefore$ For each run or procurement, the number of part-month inventories

$$= \tfrac{1}{2}qt = \tfrac{1}{2}q\left(\frac{q}{R}\right) = \frac{q^2}{2R},$$

while the number of lot-month inventories

$$= \frac{\frac{1}{2}qt}{q} = \frac{1}{2}t = \frac{q}{2R}.$$

The component costs associated with each run of size q are

C_3 = setup cost per procurement,

qK_1 = purchasing cost of q items,

$C_3\left(\frac{q}{2R}\right)P$ = holding cost, associated with the setup, of inventory for time period t,

$qK_1\left(\frac{q}{2R}\right)P$ = holding cost, associated with the purchase, of inventory for time period t.

Total expected cost for time period t is

$$C_3 + qK_1 + C_3\left(\frac{q}{2R}\right)P + qK_1\left(\frac{q}{2R}\right)P.$$

$\therefore$ Total expected cost per unit time is

$$C(q) = \frac{1}{t}\left[C_3 + qK_1 + C_3\left(\frac{q}{2R}\right)P + qK_1\left(\frac{q}{2R}\right)P\right]$$

$$= \frac{R}{q}\left[C_3 + qK_1 + C_3\left(\frac{q}{2R}\right)p + qK_1\left(\frac{q}{2R}\right)P\right]$$

$$= C_3\frac{R}{q} + K_1R + \tfrac{1}{2}C_3P + \tfrac{1}{2}K_1Pq. \qquad \text{...(12·76)}$$

$C(q)$ will be minimum if $\frac{d}{dq}\left[C(q)\right] = 0$ and $\frac{d^2}{dq^2}\left[C(q)\right] > 0$.

$$\therefore \quad \frac{d}{dq}[C(q)] = -\frac{C_3R}{q^2} + \tfrac{1}{2}K_1P = 0$$

or $$\frac{C_3R}{q^2}=K_1P$$

or $$q=\sqrt{\frac{2C_3R}{K_1P}}.$$

Also $\frac{d^2}{dq^2}\left[\ C(q)\ \right]=\frac{2C_3R}{q^3}$, which is positive.

$\therefore$ Optimal order quantity, $q_0=\sqrt{\frac{2C_3R}{K_1P}}$. ...(12·77)

Substituting this value of q in equation (12 76), we get

$$C_0(q)=C_3R\sqrt{\frac{K_1P}{2C_3R}}+K_1R+\tfrac{1}{2}C_3P+\tfrac{1}{2}K_1P\sqrt{\frac{2C_3R}{K_1P}}$$

$$=\sqrt{\frac{C_3K_1PR}{2}}+K_1R+\tfrac{1}{2}C_3P+\sqrt{\frac{C_3K_1PR}{2}}$$

$$=\sqrt{2C_3K_1PR}+K_1R+\tfrac{1}{2}C_3P. \qquad ...(12.78)$$

12·7·2. Purchase-Inventory Model With One Price Break

When there is one price break (one quantity discount), the situation may be represented as follows :

Unit purchasing cost	*Range of quantity*
K_{11}	$0<q<b$,
K_{12}	$b\leqslant q<\infty$ (or $q\geqslant b$),

where q is a (constant) quantity at and beyond which the discount applies and $K_{12}<K_{11}$.

Total cost per unit time for $0<q<b$ is

$$C_1(q)=\frac{C_3R}{q}+\overbrace{K_{11}R+\tfrac{1}{2}C_3P}+\tfrac{1}{2}K_{11}Pq. \qquad ...(12.79)$$

For $q\geqslant b$, this cost is

$$C_2(q)=\frac{C_3R}{q}+\overbrace{K_{12}R+\tfrac{1}{2}C_3P}+\tfrac{1}{2}K_{12}Pq. \qquad ...(12·80)$$

Equations (12.79) and (12.80) will now be represented graphically. However, while doing so the terms $K_{11}R+\frac{1}{2}C_3P$ in equation (12.79) and $K_{12}R+\frac{1}{2}C_3P$ in equation (12.80) being independent of the value of q, will be ignored. The graphical representation of the above two functions is shown in figure 12.11.

The solid portions in curves $C_1(q)$ and $C_2(q)$ represent the *actual* cost curves while the dotted portions represent the costs that would be obtained without the price breaks. Let q_1 and q_2 be the quantities at which minimum values of $C_1(q)$ and $C_2(q)$ occur.

The solid curve $C(q)$ in figure 12.11 represents the total cost function for the entire range of q. This curve shows a discontinuity

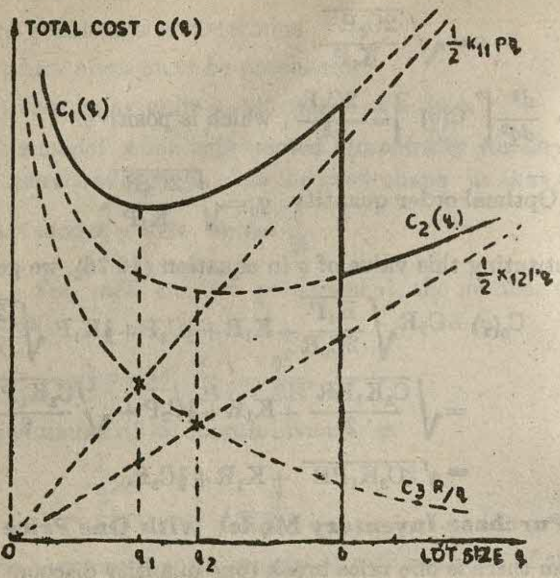
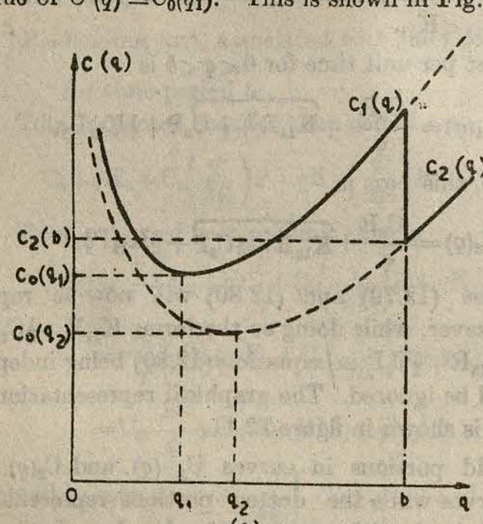


Figure 12.11

at $q=b$ and the minimum value $C(q)$ will depend upon the respective positions of q_1, q_2 and b. The various situations that can exist are shown in figures 12.12 (*a*), (*b*), (*c*) and (*d*).

1. If $b>q_2$ and $C_2(b)>C_0(q_1)$, the optimal lot size is q_1, and minimum value of $C(q)=C_0(q_1)$. This is shown in Fig. 12.11 (*a*).



Figures 12.11 (*a*)

2. If $b > q_2$ and $C_2(b) < C_0(q)$, the optimal lot size is b and min. $C(q) = C_2(b)$. This is shown in figure 12.11 (b).

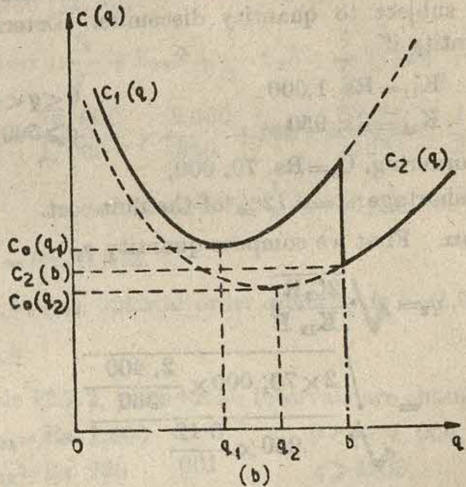
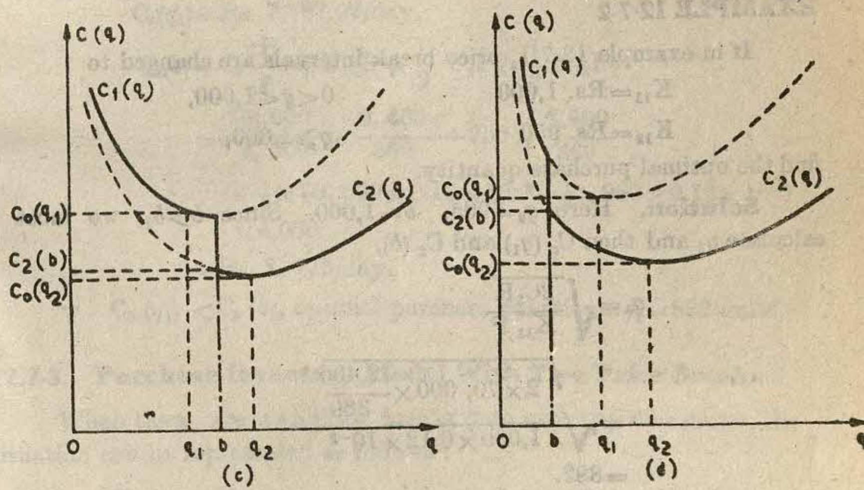


Figure 12.11 (b)

3. If $b < q_2$, the optimal lot size is q_2 and min. $C(q) = C_0(q_2)$. This is shown in figures 12.11 (c) and (d).



Figures 12.11 (c) and (d).

Thus the first quantity to be calculated is q_2 which will then be compared with b. Next we calculate, according to the situation, $C_0(q_1)$, $C_2(b)$ and $C_0(q_2)$.

EXAMPLE 12.7-1

An automobile manufacturer purchases 2,400 castings over a period of 360 days. This requirement is fixed and known. These castings are subject to quantity discounts. Determine the optimal purchase quantity if

$$K_{11} = \text{Rs. } 1{,}000 \qquad 0 < q < 500,$$
$$K_{12} = \text{Rs. } 950 \qquad q \geqslant 500,$$

cost of ordering, $C_3 = \text{Rs. } 70,000$,

cost of shortage $= 0.12\%$ of the unit cost.

Solution. First we compute quantity q_2.

$$q_2 = \sqrt{\frac{2C_3R}{K_{12}P}}$$
$$= \sqrt{\frac{2 \times 70,000 \times \frac{2,400}{360}}{950 \times \frac{0.12}{100}}}$$
$$= \sqrt{\frac{2 \times 70,000 \times 2,400 \times 100}{950 \times 0.12 \times 360}}$$
$$= 905.$$

$\because \; b = 500, \; b < q_2. \qquad \therefore$ Optimal lot size $= q_2 = 905$ units.

EXAMPLE 12·7-2

If in example 12.7-1, price break intervals are changed to

$$K_{11} = \text{Rs. } 1{,}000 \qquad 0 < q < 1{,}000,$$
$$K_{12} = \text{Rs. } 950 \qquad q \geqslant 1{,}000,$$

find the optimal purchase quantity.

Solution. Here $q_2 = 905$, $b = 1{,}000$. Since $b > b_2$, we must calculate q_1 and then $C_0(q_1)$ and $C_2(b)$.

$$q_1 = \sqrt{\frac{2C_3R}{K_{11}P}}$$
$$= \sqrt{\frac{2 \times 70,000 \times \frac{2,400}{360}}{1,000 \times 0.12 \times 10^{-2}}}$$
$$= 882.$$

From equation (12.78),

$$C_0(q_1) = \sqrt{2C_3K_{11}PR} + K_{11}R + \tfrac{1}{2}C_3P$$
$$= \sqrt{2 \times 70,000 \times 1,000 \times 0.12 \times 10^{-2} \times \frac{2,400}{360}}$$

$$+1,000\times\frac{2,400}{360}+\frac{1}{2}\times 70,000\times 0.12\times 10^{-2}$$
$$=\text{Rs. } 7,767.07.$$

From equation (12.76),

$$C_2(b)=C_3\frac{R}{q}+K_{12}R+\frac{1}{2}C_3P+\frac{1}{2}K_{12}Pq$$
$$=\frac{70,000}{1,000}\times\frac{2,400}{360}+950\times\frac{2,400}{360}+\frac{1}{2}\times 70,000$$
$$\times 0.12\times 10^{-2}+\frac{1}{2}\times 950\times 1,000\times 0.12\times 10^{-2}$$
$$=\text{Rs. } 7,412.$$

$\because$ $C_2(b)<C_0(q_1)$, optimal order quantity is $b=1{,}000$ units.

EXAMPLE 12.7-3

If in example 12.7-2, price break intervals are changed to

$K_{11}=\text{Rs. } 1{,}000$ $\quad 0<q<4,000,$

$K_{12}=\text{Rs. } 950$ $\quad q\geqslant 4000,$

find the optimal purchase quantity.

Solution. Here $q_2=905$, $b=4{,}000$. Since $b>q_2$, we must calculate q_1 and then $C_0(q_1)$ and $C_2(b)$.

Now $\quad q_1=882.$

$$C_0(q_1)=\text{Rs. } 7,767.07/\text{day}.$$
$$C_2(b)=C_3\frac{R}{q}+K_{12}R+\frac{1}{2}C_3P+\frac{1}{2}K_{12}Pq$$
$$=\frac{70,000}{4,000}\times\frac{2,400}{360}+950\times\frac{2,400}{360}$$
$$+\tfrac{1}{2}\times 70,000\times 0.12\times 10^{-2}+\tfrac{1}{2}\times 950\times 0.12\times 10^{-2}$$
$$\times 4,000$$
$$=\text{Rs. } 8,772/\text{day}.$$

$\because$ $C_0(q_1)<C_2(b)$, optimal purchase quantity $=q_1=882$ units.

12.7-3. Purchase-Inventory Model With Two Price Breaks

When there are two price breaks (two quantity discounts), the situation can be represented as follows :

Unit purchasing cost	*Range of quantity*
K_{11}	$0<q<b_1,$
K_{12}	$b_1\leqslant q<b_2,$
K_{13}	$b_2\leqslant q,$

where b_1 and b_2 are the quantities which determine the price breaks. The function $C(q)$ will have the shape shown in figure 12.12, where there are obviously two discontinuities.

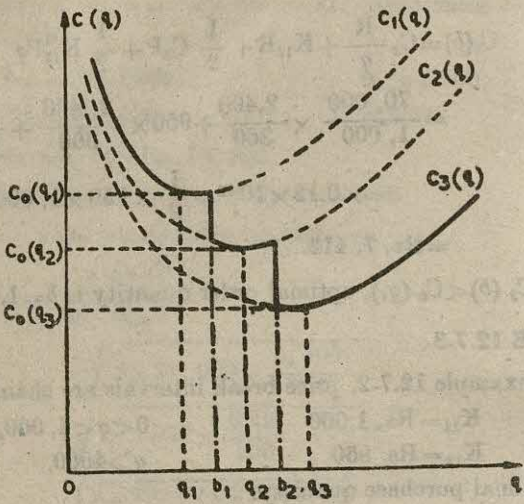


Figure 12.12.

The optimal purchase quantity is determined in the following way :

1. Calculate q_3. If $q_3 > b_2$, optimal purchase quantity is q_3.

2. If $q_3 < b_2$, calculate q_2. Since $q_3 < b_2$, then necessarily $q_2 < b_2$. As a consequence we have $q_2 < b_1$ or $q_2 > b_1$.

3. If $q_3 < b_2$ and $b_1 < q_2 < b_2$, compare $C_0(q_2)$ with $C_3\,(b_2)$. The smaller of these quantities will be the optimal purchase quantity.

4. If $q_3 < b_2$ and $q_2 < b_1$, calculate $C_3(q_1)$, which will necessarily satisfy the inequality $q_1 < b_1$. In this case compare $C_0\,(q_1)$, $C_2(b_1)$ and $C_3(b_2)$ to determine the optimum purchase quantity.

EXAMPLE 12.7-4

Find the optimal order quantity for a product for which the price breaks are as follows :

Unit Cost (Rs.)	*Quantity*
Rs. 1, 000	$0 < q < 500$,
Rs. 925	$500 \leqslant q < 4{,}000$,
Rs. 875	$4000 \leqslant q$.

Ordering cost C_3=Rs. 35,000, demand D=2,400, time period T=360 days and cost of shortage=0.06% of the unit cost.

Solution. First, we calculate q_3.

$$q_3=\sqrt{\frac{2C_3R}{K_{13}P}}$$

$$=\sqrt{\frac{2\times 35,000}{875\times \frac{0.06}{100}}\times\frac{2,400}{360}}$$

=956 units.

b_2=4,000 units.

Since $q_3<b_2$, we next calculate q_2.

$$q_2=\sqrt{\frac{2c_3R}{K_{12}P}}$$

$$=\sqrt{\frac{2\times 35,000\times 2,400}{925\times 0\cdot 06\times 10^{-2}\times 360}}$$

=917 units.

b_1=500 units.

$\therefore$ $b_1<q_2<b_3$.

$\therefore$ We shall determine C_0 (q_2) and C_3 (b_2).

Now $C_0\,(q_2)=\sqrt{2C_3K_{12}PR}+K_{12}R+\frac{1}{2}C_3P$

$$=\text{Rs.}\left[\sqrt{2\times 35,000\times 925\times 0\cdot 06\times 10^{-2}\times\frac{2,400}{360}}+925\times\frac{2,400}{360}+\frac{1}{2}\times 35,000\times 0\cdot 06\times 10^{-2}\right]$$

$$=\text{Rs. }[509+6,167+10\cdot 50]$$

$$=\text{Rs. }6,686\cdot 50,$$

and $C_3\,(b_2)=C_3\frac{R}{q}+K_{13}R+\frac{1}{2}C_3P+\frac{1}{2}K_{13}Pq$

$$=\text{Rs.}\left[\frac{35,000}{4,000}\times\frac{2,400}{360}+875\times\frac{2,400}{360}+\frac{1}{2}\times 35,000\times 0\cdot 06\times 10^{-2}+\frac{1}{2}\times 8\cdot 75\times 0\cdot 06\times 10^{-2}\times 4,000\right]$$

$$=\text{Rs. }6,952\cdot 17.$$

Since $C_0(q_2) < C_3(b_2)$, the optimal order quantity is $q_2 = 917$ units. This situation is represented in figure 12·13.

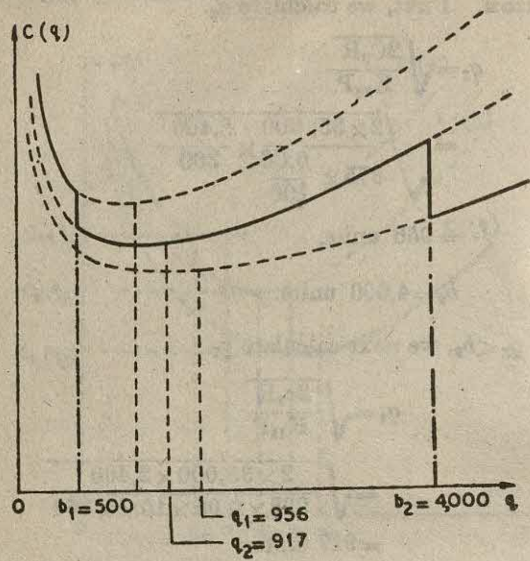


Fig. 12·13

EXAMPLE 12·7·5

Find the optimal order quantity for a product for which the price breaks are as follows :

Quantity	*Unit Cost (Rs.)*
$0 < q < 500$	Rs. 10,
$500 \leqslant q < 750$	Rs. 9·25,
$750 \leqslant q$	Rs. 8·75.

The monthly demand for the product is 200 units, shortage cost is 2% of the unit cost and cost of ordering is Rs. 100.

[*Delhi M.Sc. (Math.) 1976*]

Solution. Here, we calculate

$$q_3 = \sqrt{\frac{2C_3R}{K_{13}P}}$$

$$= \sqrt{\frac{2 \times 100 \times 200}{8{\cdot}75 \times 0{\cdot}02}}$$

$$= 478 \text{ units.}$$

$$b_3 = 750.$$

$\because \quad q_3 < b_2$, we next calculate q_2.

$$q_2 = \sqrt{\frac{2C_3R}{K_{12}P}}$$

$$= \sqrt{\frac{2 \times 100 \times 200}{9{\cdot}25 \times 0{\cdot}02}}$$

$$= 465 \text{ units.}$$

$$b_1 = 500 \text{ units.}$$

$\because \quad q_2 < b_1$, we next compute q_1.

$$q_1 = \sqrt{\frac{2C_3R}{K_{11}P}}$$

$$= \sqrt{\frac{2 \times 100 \times 200}{10 \times 0{\cdot}02}}$$

$$= 447 \text{ units.}$$

Next we compute

$$C_0(q_1) = \sqrt{2C_3K_{11}PR} + K_{11}R + \frac{1}{2}C_3P$$

$$= \text{Rs.} \left[\sqrt{2 \times 100 \times 10 \times 0{\cdot}02 \times 200} + 10 \times 200 + \frac{1}{2} \times 100 \times 0{\cdot}02\right]$$

$$= \text{Rs. } 2{,}090{\cdot}42.$$

$$C_2(b_1) = C_3\frac{R}{q} + K_{12}R + \frac{1}{2}C_3P + \frac{1}{2}K_{12}Pq$$

$$= \text{Rs.} \left[100 \times \frac{200}{500} + 9{\cdot}25 \times 200 + \frac{1}{2} \times 100 \times 0{\cdot}02 + \frac{1}{2} \times 9{\cdot}25 \times 0{\cdot}02 \times 500\right]$$

$$= \text{Rs. } 1{,}937{\cdot}25.$$

$$C_3(b_2) = C_3\frac{R}{q} + K_{13}R + \frac{1}{2}C_3P + \frac{1}{2}K_{13}Pq$$

$$= \text{Rs.} \left[100 \times \frac{200}{750} + 8{\cdot}75 \times 200 + \frac{1}{2} \times 100 \times 0{\cdot}02 + \frac{1}{2} \times 8{\cdot}75 \times 0{\cdot}02 \times 750\right]$$

$$= \text{Rs. } 1{,}843{\cdot}29.$$

Since $C_3(b_2) < C_2(b_1) < C_0(q_1)$, the optimal order quantity is $b_2 = 750$ units. This situation is represented in Fig. 12·14.

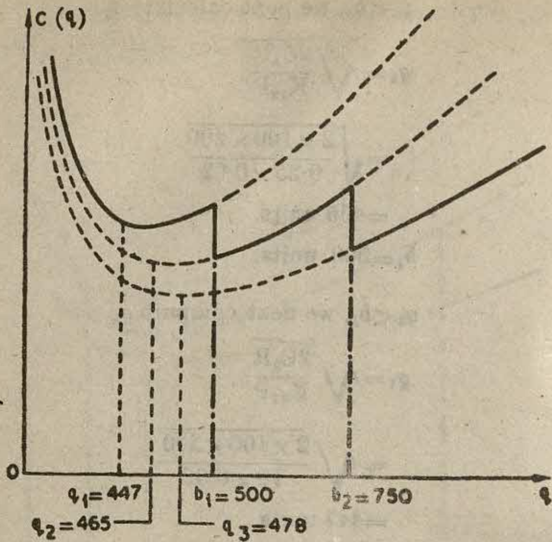


Fig. 12·14

12·7·4. Purchase Inventory Model With n Price Breaks

When there are n price breaks (n quantity discounts), the situation can be represented as follows :

Unit purchasing cost	*Range of quantity*
K_{11}	$0<q<b_1$,
K_{12}	$b_1 \leqslant q < b_2$,
K_{13}	$b_2 \leqslant q < b_3$,
.	.
.	.
.	.
K_{1n}	$b_{n-1} \leqslant q$,

where $b_1, b_2, b_3, \ldots\ldots, b_{n-1}$ are the quantities which determine the price breaks.

The method for finding the optimal purchase quantity in this case consists of the following steps :

1. Calculate q_n. If $q_n \geqslant b_{n-1}$, the optimum purchase quantity is q_n.

2. If $q_n < b_{n-1}$, calculate q_{n-1}. If $q_{n-1} \geqslant b_{n-2}$, proceed as in the case of one price break; *i.e.*, compare $C_0(q_{n-1})$ with $C\ (b_{n-1})$ to determine the optimum purchase quantity.

3. If $q_{n-1} < b_{n-2}$, compute q_{n-2}. If $q_{n-2} \geqslant b_{n-3}$, proceed as in

the case of two price breaks, *i.e.*, compare $C_0(q_{n-2})$ with $C(b_{n-2})$ and $C(b_{n-1})$ to determine the optimal purchase quantity.

4. If $q_{n-2} < b_{n-3}$, compute q_{n-3}. If $q_{n-3} \geqslant b_{n-4}$, compare $C_0(q_{n-3})$ with $C(b_{n-3})$, $C(b_{n-2})$ and $C(b_{n-1})$.

5. Continue in this manner until $q_{n-j} \geqslant b_{n-(j+1)}$, $(0 \leqslant j \leqslant n-1)$ and then compare $C_0(q_{n-j})$ with $C(b_{n-j})$, $C(b_{n-j+1})$, $C(b_{n-j+2}), \ldots, C(b_{n-1})$. The above procedure involves only a finite number of steps; at the most n, where n is the number of price ranges.

12·8. Multi-Item Deterministic Model

This model considers the inventory system consisting of several items. There are limitations on production facilities or storage capacity or time or money. This limitation results in an interaction between the different items so that it is not possible to consider each item separately. However, simple cases can be handled by using the technique of Langrange multiplier.

Consider an inventory consisting of n items. For simplicity let us assume that production is instantaneous, there is no quantity discount and that no shortages are permitted. Further let us assume that the demand is known and uniform at a rate of R_i per unit time for the ith item. Let C_{1i} be the inventory holding cost per unit per unit time and C_{3i} be the setup cost per production run for the ith quantity.

Total cost of ith item per production run

$$= C_{1i}(\tfrac{1}{2} q_i t) + C_{3i}.$$

$\therefore$ Total cost of ith item per unit time

$$= \tfrac{1}{2} C_{1i} q_i + \frac{C_{3i}}{t}$$

or

$$C(q_i) = \tfrac{1}{2} C_{1i} q_i + C_{3i} \frac{R_i}{q_i} \qquad \left(t = \frac{q_i}{R_i} \right)$$

$$= \tfrac{1}{2} C_{1i} q_i + \frac{C_{3i} R_i}{q_i},$$

where q_i is the order quantity for ith item.

$\therefore$ Total cost per unit time,

$$C(q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} \left(\tfrac{1}{2} C_{1i} q_i + \frac{C_{3i} R_i}{q_i} \right), \; i = 1, 2, \ldots, n. \qquad \ldots(12\cdot81)$$

To minimize this cost, we must differentiate it w.r.t. q_i and put it equal to zero.

i.e., $\dfrac{\partial [C(q_1, q_2, \ldots, q_n)]}{\partial q_i} = \tfrac{1}{2} C_{1i} - \dfrac{C_{3i} R_i}{q_i^2} = 0$, which gives optimal

value of q_i as

$$q_i^0 = \sqrt{\frac{2C_{3i}\,R_i}{C_{1i}}},\ i=1,\ 2,\ldots,n. \qquad \ldots(12.82)$$

Limitation on Inventories

If, now, there is a limitation on inventories that restricts the average number of all types of stocked item, to I_m, the cost $C(q_1, q_2, \ldots, q_n)$ must be minimized subject to the restriction that

$$\frac{1}{2}\sum_{i=1}^{n} q_i \leqslant I_m. \qquad \ldots(12.83)$$

If $\frac{1}{2}\sum_{i=1}^{n} q_i \leqslant I_m$, there is no problem; the optimum values given by equation (12.82) are the required values.

If $\frac{1}{2}\sum_{i=1}^{n} q_i > I_m$, *equality* condition must be obtained by reducing one or more of q_i^0'*s*. We define a quantity λ such that

$$\lambda < 0 \text{ when } \sum_{i=1}^{n} q_i - 2I_m = 0,$$

$$\lambda = 0 \text{ when } \sum_{i=1}^{n} q_i - 2I_m < 0,$$

where λ is Langragian multiplier.

Then quantity $\lambda\left(\sum_{i=1}^{n} q_i - 2I_m\right)$ is identically equal to zero and it may be added to the cost equation (12.81) without changing the value of $C(q_1, q_2, \ldots, q_n)$.

$$\therefore \quad C(\lambda;\ q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n}\left(\tfrac{1}{2}\ C_{1i}\,q_i + \frac{C_{3i}\,R_i}{q_i}\right)$$

$$+\lambda\left(\sum_{i=1}^{n} q_i - 2I_m\right),\quad i=1, 2, \ldots, n. \qquad \ldots(12.84)$$

For optimum values of q_i, we put $\dfrac{\partial[C(\lambda;\ q_1, q_2, \ldots, q_n)]}{\delta q_i}$

$$=\frac{1}{2}\,C_{1i}-\frac{C_{3i}\,R_i}{q_1^2}+\lambda=0,$$

and
$$\frac{\partial\,[C(\lambda;\,q_1,\,q_2,\ldots,\,q_n)]}{\partial\lambda}=\sum_{i=1}^{n} q_i-2I_m=0.$$

The second equation implies that q_i^0 must satisfy the inventory constraint in equality sense.

From the first equation, $q_i^0=\sqrt{\dfrac{2C_{3i}\,R_i}{C_{1i}+2\lambda^0}}$ ...(12·85)

and from the second, $\sum_{i=1}^{n} q_i^0=2I_m$ ...(12·86)

Notice that q_i^0 is dependent on λ^0, the optimal value of λ. Also for $\lambda^\circ=0$, q_i° gives the solution of the unconstrained case. The value of λ° can be found by systematic trial and error. By definition, $\lambda<0$ for the above minimization case. Thus if we try successive negative values of λ, its optimum value λ° should result in simultaneous values of $q^0{}_i$, which satisfy the given constraint in equality sense. Thus determination of λ° automatically yields q_i^0.

Limitation on Storage Area

Let A be the maximum storage area available for n items and a_i be the storage area required by one unit of ith item. If q_i is the order quantity for ith item, the storage requirements constraint becomes

$$\sum_{i=1}^{n} a_i\,q_i \leqslant A.$$

The Langrangian function for this problem is

$$C(\lambda\,;\,q_1\,,\,q_2\,,\ldots,\,q_n)=\sum_{i=1}^{n}\left(\frac{1}{2}\,C_{1i}\,q_i+\frac{C_{3i}R_i}{q_i}\right)$$
$$+\lambda\left(\sum_{i=1}^{n} a_i\,q_i-A\right),\ i=1,2,\ldots,n \quad\ldots(12{\cdot}87)$$

For optimum values of q_i, we put

$$\frac{\partial}{\partial q_i}\Big[C(\lambda;\,q_1\,,\,q_2,\ldots,\,q_n)\Big]=\frac{1}{2}\,C_{1i}-\frac{C_{3i}\,R_i}{q_i^2}+\lambda\,a_i=0,$$

and $$\frac{\partial[C(\lambda;\, q_1, q_2, \ldots, q_n)]}{\partial \lambda} = \sum_{i=1}^{n} a_i\, q_i - A = 0,$$

which give $$q_i^o = \sqrt{\frac{2C_{3i}\, R_i}{C_{1i} + 2\lambda^o a_i}},\ i = 1, 2, \ldots, n, \quad \text{.. (12.88)}$$

and $$\sum_{i=1}^{n} a_i\, q_i = A. \quad \text{...(12.89)}$$

The value of λ^o can be found by systematic trial and error.

EXAMPLE 12.8-1

A company producing three items has limited storage space of averagely 750 items of all types. Determine the optimal production quantities for each item separately, when the following information is given :

Product	1	2	3
Holding cost, C_1 (Rs.)	0.05	0.02	0.03
Setup cost, C_3 (Rs.)	50	40	60
Demand rate	100	120	75

Solution : Using economic lot-size formula (12.82), we get for

$$\text{item } 1 : q_1^0 = \sqrt{\frac{2C_{31}\, R_1}{C_{11}}} = \sqrt{\frac{2 \times 50 \times 100}{0.05}} = 447,$$

$$\text{item } 2 : q_2^0 = \sqrt{\frac{2 \times 40 \times 120}{0.02}} = 693,$$

$$\text{item } 3 : q_3^0 = \sqrt{\frac{2 \times 60 \times 75}{0.04}} = 464.$$

$\therefore$ Average inventory at any time $= \frac{1}{2}$ [447 + 693 + 464] = 802, which is greater than the limiting value of 750 items. Therefore, we use equation (12.85), which includes the parameter λ. Let $\lambda = 0.005$. Then using equation (12.85) we get for

$$\text{item } 1 : q_1^0 = \sqrt{\frac{2C_{31}\, R_1}{C_{11} + 2\lambda^o}} = \sqrt{\frac{2 \times 50 \times 100}{0.05 + 2 \times 0.005}} = 409,$$

$$\text{item } 2 : q_2^0 = \sqrt{\frac{2 \times 40 \times 120}{0.02 + 2 \times 0.005}} = 556,$$

$$\text{item } 3 : q_3^0 = \sqrt{\frac{2 \times 60 \times 75}{0.04 + 2 \times 0.005}} = 424.$$

$\therefore$ Average inventory at any time $= \frac{1}{2}(409 + 566 = 424) \simeq 700$, which is less than the limiting value of 750 items.

$\therefore$ Smaller value of λ should be used in computations.

The best way of finding it is by linear interpolation.

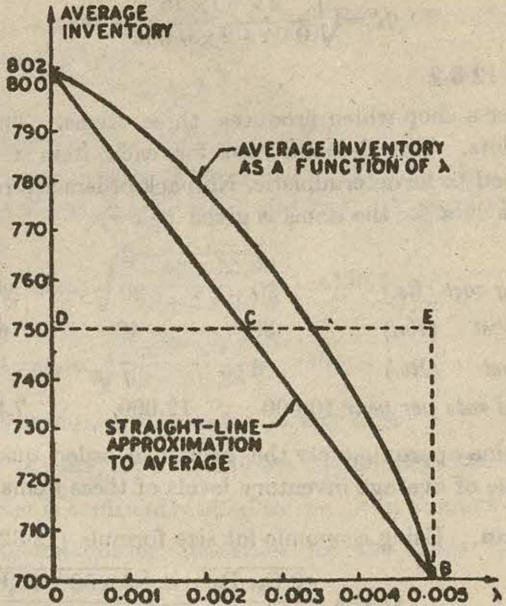


Fig. 12·15

Figure 12·15 shows the variation of average inventory as a function of λ. Points A and B have coordinates (0, 802) and (0·005, 700) respectively and between these points the unknown curve can be approximated to a straight line. Point C, which corresponds to average inventory of 750 can be computed as follows :

From similar triangles ACD and BCE,

$$\frac{DC}{AD}=\frac{CE}{BE}=\frac{DE-DC}{BE}$$

or $$\frac{DC}{802-750}=\frac{0{\cdot}005-DC}{50}$$

or $$50\ DC=52\times 0{\cdot}005-52DC$$

or $$DC=\frac{52\times 0{\cdot}005}{102}\simeq 0{\cdot}0025$$

or $$\lambda^{\circ}\simeq 0{\cdot}0025.$$

Now $$q^{\circ}_i=\sqrt{\frac{2C_{3i}\ R_i}{C_{1i}+2\lambda^{\circ}}}$$

$\therefore$ $$q^{\circ}_1=\sqrt{\frac{2\times 50\times 100}{0{\cdot}05+2+0.0025}}=428.$$

$$q_2^0 = \sqrt{\frac{2 \times 40 \times 120}{0.02 + 2 \times 0.0025}} = 628,$$

and
$$q_3^0 = \sqrt{\frac{2 \times 60 \times 75}{0.04 + 2 \times 0.005}} = 444.$$

EXAMPLE 12·8-2

Consider a shop which produces three items. The items are produced in lots. The demand rate for each item is constant and can be assumed to be deterministic. No back orders are to be allowed. The pertinent data for the items is given below.

Item	1	2	3
Holding cost (*Rs.*)	20	20	20
Setup cost (*Rs.*)	50	40	60
Unit cost (*Rs.*)	6	7	5
Demand rate per year	10,000	12,000	7,500

Determine approximately the economic order quantities when the total value of average inventory levels of these items is Rs. 1,000.

Solution. Using economic lot size formula (12·82), we get for

item 1 : $$q_1^0 = \sqrt{\frac{2\,C_{31}\,R_1}{C_{11}}} = \sqrt{\frac{2 \times 50 \times 10{,}000}{20}} = 223,$$

item 2 : $$q_2^0 = \sqrt{\frac{2 \times 40 \times 12{,}000}{20}} = 216,$$

item 3 : $$q_3^0 = \sqrt{\frac{2 \times 60 \times 7{,}500}{20}} = 210.$$

$\therefore$ Average inventory value at any time

$$= \tfrac{1}{2}(223 \times 6 + 216 \times 7 + 210 \times 5)$$
$$= \text{Rs. } 1{,}950,$$

which is greater than the limiting value of Rs. 1,000. Therefore, we use equation (12·85) which includes the parameter λ.

Let $\lambda = 30$.

Then using this equation we get for

item 1 : $$q_1^0 = \sqrt{\frac{2\,C_{31}\,R_1}{C_{11} + 2\lambda^\circ}} = \sqrt{\frac{2 \times 50 \times 10{,}000}{20 + 2 \times 30}} = 111,$$

item 2 : $$q_2^0 = \sqrt{\frac{2 \times 40 \times 12{,}000}{20 + 2 \times 30}} = 108,$$

item 3 : $$q_3^0 = \sqrt{\frac{2 \times 60 \times 7{,}500}{20 + 2 \times 30}} = 105.$$

$\therefore$ Average inventory value at any time

$$=\tfrac{1}{2}[111\times 6+108\times 7+105\times 5]$$

$$\text{Rs. } 973{\cdot}50,$$

which is less than the limiting value of Rs. 1,000. Hence, smaller value of λ should be used.

Let $\lambda = 28{\cdot}5.$

Then using equation (12·85) we get for

$$\text{item 1}: q^0_1 = \sqrt{\frac{2C_{31}\,R_1}{C_{11}+2\lambda^0}} = \sqrt{\frac{2\times 50\times 10{,}000}{20+2\times 28{\cdot}5}} = 114,$$

$$\text{item 2}: q^0_2 = \sqrt{\frac{2\times 40\times 12{,}000}{20+2\times 28{\cdot}5}} = 110{\cdot}5,$$

$$\text{item 3}: q^0_3 = \sqrt{\frac{2\times 60\times 7{,}500}{20+2\times 28{\cdot}5}} = 109.$$

$\therefore$ Average inventory value at any time

$$=\tfrac{1}{2}[114\times 6+110{\cdot}5\times 7+109\times 5] = \text{Rs. } 1{,}001{\cdot}50.$$

This value is sufficiently close to the given amount of Rs. 1,000.

$\therefore$ Economic order quantities for the three items are 114, 110·5 and 109 units respectively.

12·9. Buffer (Safety) Stock

In section 12·5·1, figure 12·2 represents an inventory model in which demand rate as well as the lead time is constant and known.

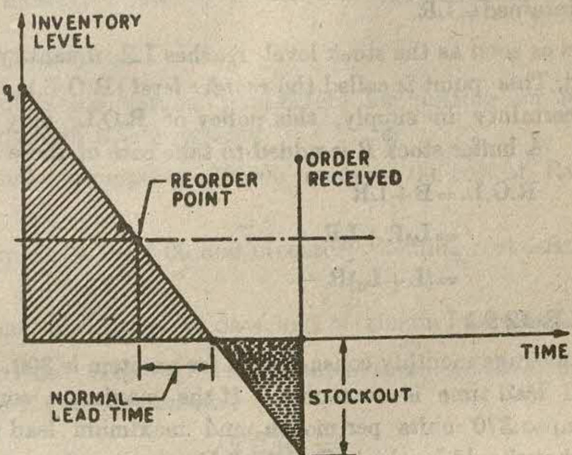


Fig. 12·16.

However, these assumptions are not applicable to all inventory situations. Demand can be more or less than anticipated due to internal

and external factors such as power failures or changes in weather. Similarly, the lead time can also vary due to supply and/or transportation facilities.

If, due to internal or external factors, inventory is not available when required, a stock-out will result. Such a situation will result in reduced profits and possibly losses. Figure 12·16 shows the problem of stock out. It may be noted that inventory level does not return to its original point as back orders are to be filled. Reorder point is defined as the condition that indicates that a purchase order should be placed to replenish the inventory stock. To avoid stockout, extra inventory is held as a buffer, resulting in *buffer stock* or *safety stock*. When buffer stock is low, the inventory holding cost is low but the shortage cost is high. On the other hand for a large buffer stock, the inventory holding cost is high. but since the shortages are reduced, the shortage cost will be low. Thus it is necessary to strike a balance between the shortage cost and inventory holding cost in order to determine an optimum value of buffer stock.

Let B = buffer stock,

L = normal lead time,

L_d = difference between the maximum lead time and normal lead time,

R = demand rate.

Then $B = L_d . R$.

Total inventory consumption during lead time, if buffer stock is not maintained $= LR$.

Thus as soon as the stock level reaches LR, quantity q should be ordered. This point is called the *reorder level* (R.O.L.). However, due to uncertainty in supply, this policy of R.O.L. may result in shortages. A buffer stock B is added to take care of these shortages.

Thus
$$\begin{aligned} \text{R.O.L.} &= B + LR \\ &= L_d R + LR \\ &= (L + L_d) R. \end{aligned}$$

EXAMPLE 12·9·1

The average monthly consumption for an item is 300 units and the normal lead time is one month. If the maximum consumption has been upto 370 units per month and maximum lead time is $1\frac{1}{2}$ months, what should be the buffer stock for the item ?

Solution. Optimum buffer stock = (Maximum lead time − Normal lead time) × monthly consumption

$$=(1\tfrac{1}{2}-1)\times 370=185 \text{ units.}$$

EXAMPLE 12·9-2

A firm uses every year 12,000 units of a raw material costing Rs. 1·25 per unit. Ordering cost is Rs. 15·00 per order and the holding cost is 5% per year of average inventory.

(*i*) Find the economic order quantity.

(*ii*) The firm follows E.O.Q. purchasing policy. It operates for 300 days per year. Procurement time is 14 days and safety stock is 400 units. Find the re-order point, the maximum inventory and the average inventory. [*Meerut M. Com. 1975*]

Solution

(*i*) E.O.Q. $= q_0 = \sqrt{\dfrac{2C_3R}{C_1}} = \sqrt{\dfrac{2\times 15\times 12{,}000}{5/100\times 1{\cdot}25}}$

$$=\sqrt{\frac{2\times 15\times 12{,}000\times 100}{5\times 1{\cdot}25}}=2{,}400 \text{ units.}$$

(*ii*) Re-order level = buffer stock + consumption during normal lead time

$$=400+14\times\frac{12{,}000}{300}=960 \text{ units.}$$

(*iii*) Maximum inventory $=q_0+B=2{,}400+400=2{,}800$ units.

(*iv*) Minimum inventory $=B=400$ units.

$\therefore$ Average inventory $=\dfrac{2{,}800+400}{2}=1{,}600$ units.

EXAMPLE 12·9-3

Calculate the various parameters for putting an item with following data on E.O.Q. system :

Annual consumption = 12,000 units at the cost of Rs. 7·50 per unit.

Setup cost = Rs. 6·00 and inventory holding cost = Rs. 0·12 per unit.

Normal lead time = 15 days and maximum lead time = 20 days.

Solution. The various parameters to be calculated for putting an item on E.O.Q. system are :

(*i*) E.O.Q. $=q_0=\sqrt{\dfrac{2\,C_3R}{C_1}}$

$$=\sqrt{\frac{2\times 6\times 12{,}000}{0{\cdot}12}}=1{,}096 \text{ units.}$$

(*ii*) Optimum buffer stock, $B=\frac{(20-15)}{30}\times\frac{12,000}{12}$

$$=\frac{5\times 12,000}{360}=\frac{500}{3}=167 \text{ units}.$$

(*iii*) R.O.L. = B + consumption during normal lead time

$$=167+\frac{15}{30\times 12}\times 12,000=167+500=667 \text{ units}.$$

12·10. Application of Models

Inventory control models discussed in previous sections face certain difficulties when applied in actual practice. Some of these difficulties are

(*i*) demand is not always Poissonly distributed,

(*ii*) stock-out cost is unknown,

(*iii*) costs are difficult to obtain.

We shall briefly discuss each of them now.

Distribution of demand

Demand is often more variable than would be expected with a Poisson distribution. This is particularly so when there are also sub-stores below the stores level under consideration. In such cases total demand may be regarded as the sum of individual demands, where each one of them has a Poisson distribution.

In place of Poisson distribution, hyper-exponential distribution can be used to fit the distribution $f(x)$ of demands in a fixed period. This distribution is simple and is a better fit than the Poisson distribution when standard deviations of demand are greater than the means. It is expressed as

$$f(x)=a\lambda_1 \, . \, e^{-\lambda_1 x}+(1-a)\lambda_2 e^{-\lambda_2 x}, \qquad \ldots(12\cdot 90)$$

where x is demand in a fixed interval and a, λ_1 and λ_2 are parameters.

Unknown cost of stock-out

Stock-out costs do exist but are difficult to assess. The effect of stock-out in any problem can be partially incorporated by finding the effect of different stock levels on the frequency of stock-outs. This can be done with the help of a graph shown below.

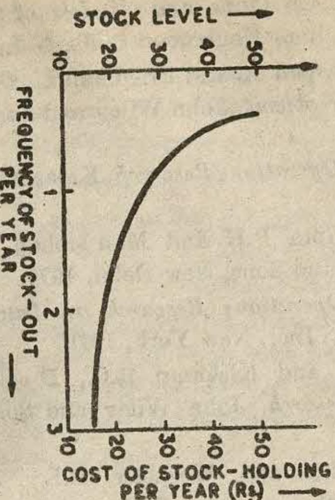


Fig. 12·17

Management, when using this graph, has to base its decision on intuition or has to put a rational estimate of stock-out cost. It is easy to do so since all the available information has been expressed in a le form.

Costs difficult to obtain

In addition to the difficulty in assessing stock-out costs, ordering costs and inventory holding costs may not directly depend upon the numbers of orders placed and inventory value respectively. However, these costs may be expressed as functions of some variables and these functions can be introduced into the appropriate models. If this rational technique results in large savings, then the apparent difficulties in finding the exact ordering and holding costs will not appear to be so serious.

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EXERCISES

Section 12.5-1

1. In each of the following cases, stock is replenished instantaneously and no shortage is allowed. Find the economic lot size, the associated total cost and length of time between two orders.

(*a*) C_3=Rs. 100, C_1=Rs. 0·05 and R=30 units/year.

(*b*) C_3=Rs. 50, C_1=Rs. 0·05 and R=30 units/year.

(*c*) C_3=Rs. 100, C_1=Rs. 0·01 and R=40 units/year.

(*d*) C_3=Rs. 100, C_1=Rs. 0·04 and R=20 units/year.

(*Ans.* (*a*) $q_0=346{\cdot}4$, $t_0=11{\cdot}55$, $C_0(q)$=Rs. 17.30)

2. In each case of exercise no. 1, determine the reorder level if lead time is 14 units.

3. A company purchases 10,000 items per year for use in its production shop. The unit cost is Rs. 10 per year, holding cost is Rs. 0·80 per month and cost of making a purchase is Rs. 200. Determine the following if no shortages are allowed :

(*i*) the cptimum order quantity,

(*ii*) the optimum total year cost,

(*iii*) the number of orders per year,

(*iv*) the time between orders.

4. A certain item costs Rs. 235 per ton. The monthly requirement is 5 tons and each time the stock is replenished, there is a setup cost of Rs. 1,000. The cost of carrying of inventory has been estimated at 10% of the value of the stock per year. What is the optimal order quantity ? [*Delhi M.Sc. (Math.) 1973*]

5. The XYZ manufacturing company has determined from an analysis of its accounting and production data for part number 625, that its cost to purchase is Rs. 36 per order and Rs. 2 per part. Its inventory carrying charge is 18% of the average inventory. The demand for this part is 10,000 units per annum. Find

(*a*) what should the economic order quantity be ?

(*b*) what is the optimal no. of days' supply per optimum order ? [*Delhi M.B.A. 1975*]

(*Ans.* 1,144 units, 0·1414 year)

6. An aircraft company uses rivets at an approximate customer rate of 2,500 kg per year. The rivets cost Rs. 30 per kg and the company personnel estimate that it costs Rs. 130 to place an order and the inventory carrying cost is 10% per year. How frequently should orders for rivets be placed and what quantities should be ordered ?

[*Meerut M.Sc.* (*Stat.*) *1976*]

(*Ans.* 466 kg, 5.3/year)

7. Consider the inventory system with the following data in usual notations: $R=1{,}000$ units/year, $I=0.30$, $P=$Rs. 0.50 per unit, $C_3=$Rs. 10, $L=2$ years (lead time) and $C_1=IP$. Determine

(*i*) optimal order quantity,

(*ii*) re-order point,

(*iii*) minimum average cost.

[*Delhi 1968*]

(*Ans.* 365 units, 2,000 units, Rs. 54.80)

Section 12.5-3

8. A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is 2 paise, and the setup cost of a production run is Rs. 18. What is the optimum lot size and how frequently should production run be made ?

[*Meerut M.Sc.* (*Math.*) *1969*; *Delhi M. Com. 1975*]

(*Ans.* 1,05,000 ; 10.5 days)

9. A product is produced at the rate of 50 items per day. The demand occurs at the rate of 30 items per day. Given that setup cost per order=Rs. 100 and holding cost per unit per unit time =Rs. 0.05, find the economic lot size and the associated total cost per cycle assuming that no shortage is allowed.

(*Ans.* 225, Rs. 11)

10. The manager of a company manufacturing car parts has entered into a contract of supplying 1,000 nos. per day of a particular part to a car manufacturer. He finds that his plant has a capacity of producing 2,000 nos. per day of the part. The cost of the part is Rs. 50, cost of holding stock is 12% per annum and setup cost per production run is Rs. 100. What should be run size for each production run and total optimum cost/month ? How frequently should production runs be made ? Shortage is not permissible.

[*Baroda Univ. B E. 1973*]

Section 12.5-4

11. A contractor undertakes to supply diesel engines to a truck manufacturer at the rate of 25 per day. He finds that the cost of holding a completed engine in stock is Rs. 16 per month, and there is a clause in the contract penalising him Rs. 10 per engine per day late for missing the scheduled delivery date. Production of engines is in batches, and each time a new batch is started there are setup costs of Rs. 10,000. How frequently should batches be started, and what should be the initial inventory level at the time each batch is completed? [*Baroda B.Sc.* (*Math.*) *1978*]

(*Ans.* 40 days, 943 engines)

12. A manufacturer receives an order for 6,890 items to be delivered over a period of a year as follows:

at the end of the 1st week : 5 items,

at the end of 2nd week : 10 items,

at the end of 3rd week : 15 items, etc.

The cost of carrying inventory is Rs. 2.60 per item per year and the cost of a setup is Rs. 450 per production run.

Compute costs for the following policies:

(1) Make all 6,890 at start of year.

(2) Make 3,445 now and 3445 in 6 months.

(3) Make 1/12th of the order each month.

(4) Make 1/52 of the order each week.

(*Ans.* Rs. 12,000, Rs. 8,000, Rs. 9,000, Rs. 27,000 (approx.))

13. A manufacturer received an annual contract for supplying 4,000 gears to be delivered over a period of one year. Deliveries re to be affected as under:

First quarter —3,500, second quarter —4,500, third quarter

—2000, fourth quarter —4,000. The manufacturer wants to plan out his production on his vital machine which costs Rs. 590 for setting up. The cost of gear is Rs. 60 and inventory carrying cost comes to 10% per year. Calculate the annual cost for producing this quantity in a number of equal lots. What will be the minimum cost over the year ? Explain and derive the formula used.
[*Baroda Univ. 1975*)

14. (*a*) Discuss in detail the inventory costs which are to be considered for determination of economic order quantity.

(*b*) The demand for a product is 25 units per month, and the items are withdrawn uniformly. The setup cost each time a production run is made is Rs. 15. The inventory holding cost is Rs. 0.30 per item per month. Assuming that shortages are not allowed, determine how often to make a production run and what size should it be. Prove the formula used. If shortages cost Rs. 1.50 per item/month, determine how often to make production run and what size should it be. (*Baroda Univ. B.E. 1973*)

Section 12.5-6

15. A product is produced at the rate of 50 items per day. The demand occurs at the rate of 30 items per day. Given that

setup cost per order =Rs. 100,

holding cost per item per unit time =Rs. 0.05,

and shortages being allowed, what is the shortage cost per unit under optimal conditions if the lot size is of 600 items.
(*Ans.* Rs. 1/220)

16. A newspaper boy buys papers for 3 paise and sells them for 7 paise each. He cannot return unsold newspapers. Daily demand has the following distribution :

No. of customers:	23	24	25	26	27	28	29	30	31	32
Probability:	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05

If each day's demand is independent of the previous day's, how many papers should he order each day ?
[*Meerut M.Sc.* (*Stat.*) *1971, 74*; *Bombay B.Sc.* (*Stat.*) *1975*]
(*Ans.* 28)

17. The probability distribution of monthly sales of a certain item is as follows:

Monthly sales:	0	1	2	3	4	5	6
Probability:	0.01	0.06	0.25	0.35	0.20	0.03	0.10

The cost of carrying inventory is Rs. 30 per unit per month and

the cost of unit shortage is Rs. 70 per month. Determine the optimum stock level which minimizes the total expected cost.

[*Delhi M.Sc.* (*Math.*) *1976*]

(*Ans.* 3)

18. An electric company is about to order a new generator for its plant. One of the essential parts of the generator is expensive and complicated. Its failure cannot be foreseen and its failure leads to the breakdown of the generator. Each of these parts is uniquely built for a particular generator and may not be used on any other. The cost of part when ordered with generator is Rs. 4,000. If a spare part is required and is not available, the cost of having the part to order plus the cost of down time is Rs. 80,000. The planned life of the generator is 20 years and records of similar parts in similar generators give the following information:

No. of spare parts required in 20 years	*No. of generators requiring indicated no. of spares*	*Estimated probability of indicated no. of failures*
0	90	0.90
1	5	0.05
2	2	0.02
3	1	0.01
4	1	0.01
5	1	0.01
6 or more	0	0.00

How many spare parts should be incorporated in the order for each generator?

(*Ans.* 2)

19. An ice-cream company sells one of its types of ice-cream by weight. If the product is not sold on the day it is prepared, it can be sold at a loss of 50 paise per pound. There is, however, an unlimited market for one day old ice-cream. On the other hand the company makes a profit of Rs. 3.20 on every pound of ice-cream sold on the day it is prepared. Past daily orders form a distribution with

$$f(x)=0.02-0.0002\,x,\ 0\leqslant x\leqslant 100.$$

How many pounds of ice-cream should the company prepare every day ?

[*Baroda B.Sc.* (*Math*). *1978*; *Agra M.Sc.* (*Stat.*) *1974*]

(*Ans.* 63.3 pounds)

20. A baking company sells cake by the pound. It makes a

profit of 50 paise a pound on every pound sold on the day it is baked. It disposes of all cakes not sold on the date it is baked at a loss of 12 paise a pound. If the demand is known to be rectangular between 2,000 and 3,000 pounds, determine the optimum daily amount baked ?

[*Meerut M. Sc. (Math.) 1969 Kuru. M.Sc. (Math). 1977*]
(*Ans.* 2,807 pounds)

21. Show that when considering the optimum level of inventory S_0, which minimizes the total expected cost in case of continuous (non-discrete) quantities, the condition to be satisfied is

$$F(S_0)=\frac{C_2}{C_1+C_2},$$

where $$F(S_0)=\int_0^{S_0} f(r).dr.$$

Here, $f(r)$ = the probability density function of requirement of r quantity,

C_2 = the shortage cost,

C_1 = the holding cost per unit of quantity per unit of time.

[*Gujarat Univ. B. E. 1976*)

Section 12·6·3

22. A shopkeeper has to decide how much quantity of bread he should stock every week. The quantity of bread demand in any week is assumed to be a continuous random variable with a given probability function $f(x)$. Let 'a' be the unit cost of purchasing bread and 'd' be the unit penalty cost. Find the optimum quantity of bread to be stocked.

[*I.S.I. 1962*]

Section 12·6·4

23. The probability distribution of monthly sales of a certain item is as follows :

Monthly sales	0	1	2	3	4	5	6
Probability	0·02	0·05	0·03	0·27	0·20	0·10	0·06

The cost of carrying inventory is Rs. 10 per unit per month. The current policy is to maintain a stock of 4 items at the beginning of each month. Assuming that the cost of shortage is proportional to both time and quantity short, obtain the imputed cost of shortage of one item for one time unit.

[*Meerut M.Sc. (Stat) 1970*]

Section 12·6.5

22. An airline runs a school for air hostesses each month; it takes two months to assemble a group of girls and to train them. Past records of turnover in hostesses show that the probability of requiring x new trained hostesses in any one month is $g(x)$ $[x=0, 1, 2, ...]$, and the probability of requiring y new hostesses in any two months period is $h(y)$. In the event that a trained hostess is not required for flying duties, the air line still has to pay her salary at the rate of C_1 per month. If insufficient hostesses are available, there is a cost of C_2 per girl short per month. Show how to determine decision rules for the size of classes.

[**Hint.** See model 5 (*a*).] (*Meerut 1971, 75*)

25. Two products are stocked by a company. The company has limited space and cannot store more than 40 units. The demand distributions for the products are as follows :

First Product		*Second Product*	
Demand	*Probability of demand*	*Demand*	*Probability of demand*
0	0·10	0	0·05
10	0·20	10	0·20
20	0·35	20	0·40
30	0·25	30	0·20
40	0·10	40	0·15

The inventory carrying costs are Rs. 5 and Rs. 10 per unit of the ending inventories for the first and second products, respectively. The shortage costs are Rs. 20 and Rs. 50 per unit of the ending shortages for the first and second products respectively. Find out the economic order quantities for both the products.

(*Delhi M.B.A. 1976*)

26. The uniform annual demands for two bulky items are 90 units and 160 units respectively. The carrying costs are Rs. 250 and Rs. 200 per ton per year, and setup costs are Rs. 50 and Rs. 40 per production respectively. No shortages are allowed. Space considerations restrict the average amount inventory of both items to 4,000 c. ft. A ton of the first item occupies 1, 000 c. ft., and a ton of the second item 500 c. ft. Find the optimal lot size.

[*I.S.I. 1971*]

(*Ans.* 2 tons, 12 tons)

Section 12·7.2

27. A manufacturer of engines is required to purchase 2,400n

castings per year. This requirement is assumed to be fixed and known. The manufacturer is given a lower price for quantity purchased within certain ranges. The problem is to determine the optimal purchase quantity.

The following data is given :

Time period $T = 12$ months,

Total demand $R = 2{,}400$ units,

$P = 2\%$.

Setup cost per procurement, $C_3 = \text{Rs. } 350$,

$K_{11} = \text{Rs. } 10 \qquad 0 < q < 500$,

$K_{12} = \text{Rs. } 9{\cdot}25 \qquad q \geqslant 500$.

(*Ans.* 870)

28. In exercise 27, if the setup cost per procurement is Rs. 100 only, find the optimal order quantity. (*Ans.* 500)

29. If in exercise 28, the price break does not occur until $q = 3{,}000$ find the optimal lot size. (*Ans.* 447)

30. The annual demand of a product is 10,000 units. Each unit costs Rs. 100 if orders are placed in quantities below 200 units but for orders of 200 or above the price is Rs. 95. The annual inventory holding cost is 10% of the value of the item and the ordering cost is Rs. 5 per order. Find the economic lot size.

(*Meerut M. Com. 1970*)

Section 12·7·3

31. The demand for a product is 2,400 units over 360 days. The storage cost is 0·06% of the unit cost of the product and the ordering cost is Rs. 35,000. Find the optimal order quantity if the price breaks are as follows :

Quantity range	*Purchasing cost (Rs.)*
$0 < q < 1{,}000$	1,000,
$1{,}000 \leqslant q < 4{,}000$	925,
$4{,}000 \leqslant q$	850.

32. A manufacturer's requirement for an item is 2,000 units per year. Ordering costs are Rs. 100 per order and inventory costs are 16% per year per unit of average inventory. Calculate the economic order quantity. If the price quoted is Rs. 10 each for quantities below 1,000 units, Rs. 9·50 for quantities between 1,000 and below 2,000 and Rs. 9·30 for lots of Rs. 2,000 or more, compute total ordering cost when ordering in lots of (*i*) 500 (*ii*) 1,000 and (*iii*) 2,000 units. (*Gawalior Univ. 1975*)

33. Find the optimal order quantity for a product for which the following data is given :

$R = 100$ units/week,

$C_s =$ Rs. 300,

$P = 10\%$/unit/week,

$K_1 =$ Rs. 1 if $q < 500$,

Rs. 0·95 if $500 \leqslant q < 1{,}000$,

Rs. 0·90 if $1{,}000 \leqslant q$. (*Ans.* 1,000 units)

34. Find the optimal order quantity for a product where the annual demand for the product is 500 units, cost of storage per unit item per year is 10% of the unit cost and ordering cost per order is Rs. 180. The unit costs are given below.

Quantity	*Unit cost in Rs.*
$0 < q < 500$	25·00,
$500 \leqslant q < 1{,}500$	24·80,
$1{,}500 \leqslant q \leqslant 3{,}000$	24·60,
$3{,}000 \leqslant q$	24·40.

[*Delhi M. Sc. (Math.) 1975*]

35. Determine an optimal ordering rule for the following case:

$R = 5{,}000$ units/month,

$C_3 =$ Rs. 50 per order,

$P = 0\ 50$ per unit per month.

Unit costs are

$K_{11} =$ Rs. 1·50	$0 \leqslant q < 100$,
$K_{12} =$ Rs. 1·40	$100 \leqslant q < 250$,
$K_{13} =$ Rs. 1·30	$250 \leqslant q < 500$,
$K_{14} =$ Rs. 1·00	$500 \leqslant q < 1{,}000$,
$K_{15} =$ Rs. 0·90	$1{,}000 \leqslant q < 5{,}000$,
$K_{16} =$ Rs. 0·85	$5{,}000 \leqslant q$.

Section 12·8

36. For an inventory problem with three items, the parameters of the problem are given below :

Item (i)	C_{3i} (*Rs.*)	R_i	C_{1i} (*Rs.*)	a_i (m^2)
1	10	2	0·30	1
2	5	4	0·10	1
3	15	4	0·20	1

If the total available storage area is 25 m^2, find the economic order quantities.

(*Ans.* 6·7, 7·6 and 10·6)

37. Four different items are kept in a store. The demand rates are constant for the four items. Production rate is instantaneous and no shortages are permitted. The data for the problem is given below.

Item i	*Setup cost* C_{3i} (*Rs.*)	*Demand rate* R_i	*Holding cost* C_{1i} (*Rs.*)	*Annual demand* D_i
1	50	20	0.2	5,000
2	100	10	0·1	10,000
3	20	10	0·1	5,000
4	90	5	0·2	7,500

Find the economic lot sizes if the total number of orders per year for the four items is limited to 200 only.

38. A small shop produces three machines parts 1, 2 and 3 in lots. The shop has only 650 sq. ft. of storage space. The appropriate data for three items are presented in the following table :

Item	1	2	3
Demand rate (*units/year*)	5,000	2,000	10,000
Procurement cost (*Rs.*)	100	200	75
Cost per unit (*Rs.*)	10	15	5
Floor space required (*sq. fit./unit*)	0·70	0·80	0·40

The carrying charges on each item are 20% of average inventory evaluation per annum. If no stock-outs are allowed, determine the optimal lot size for each item. [*Delhi M.B.A. 1976*]

Section 12·9

39. For a fixed order quantity system find out the various parameters for a problem with the following data :

Annual consumption (R) = 10,000 units,

Cost of one unit = Rs. 1·00,

C_3 = Rs. 12·00 per production run,

C_1 = Rs. 0·24 per unit,

past lead times : 15 days, 25 days, 13 days, 14 days, 30 days, 17 days.

(*Ans.* E.O.Q. = 1,000, B = 416·66 ≃ 417, R.O.L. = 834, $q_{average}$ = 917.)

40. A factory uses Rs. 32,000 worth of a new material per year. The ordering cost per order is Rs. 50 and the carrying cost is 20% per year of the average inventory. If the company follows the E.O.Q. purchasing policy, calculate the reorder point, the maximum inventory, the minimum inventory and the average inventory; it being given that the factory works for 360 days a year, the replacement time is 9 days and the safety stock is worth Rs. 300.

[*Meerut M. Com. 1975*]

(*Ans.* 1,100, 4,3(0, 300 and 2,300 units)

41. Based on the following data, is a lot of 1,000, 3,000, 6,000, or 12,000 units the most economical to manufacture ?

setup cost :	Rs. 3·00,
value :	Rs. 0.04 per unit,
carrying cost :	20% of value of average inventory,
storage cost :	Rs. 0·03 per unit per year,
consumption :	12,000 units per year,
minimum safety stock on hand at any time. :	1,000 units.

[*Bombay D.I.M. 1975*]

13

Simulation

The technique of simulation has long been used by the designers and analysts in physical sciences and it promises to become an important tool for tackling the complicated problems of managerial decision making. Scale models of machines have been used to simulate the plant layouts and models of aircrafts have been tested in wind tunnels to determine their aerodynamic characteristics. Simulation, which can appropriately be called management laboratory, determines the effect of a number of alternate policies without disturbing the real system. It helps in selecting the best policy with the prior assurances that its implementation will be beneficial.

Probably the first important application of simulation was made by John Von Neumann and Stanislaw Ulam for studying the tedius behaviour of neutrons in a nuclear shielding problem which was too complex for mathematical analysis. With the remarkable success of the technique on neutron problem, it became popular and found many applications in business and industry. Development of digital computer in early 1950s is further responsible for the rapid progress made by the simulation techniques. The range of simulation application varies from simple queuing models to models of large integrated systems of production.

In this chapter we will limit our discussion to the Monte Carlo method of simulation. A few simplified applications are discussed to illustrate the Monte Carlo method as well as the general simulation technique. Though the use of computer is essential in all simulation problems for obtaining reliable results, yet the illustrative examples have been solved manually to demonstrate the approach.

13·1. Examples on the Applications of Simulation Technique

EXAMPLE 13·1·1

At a small store of ready made garments, there is one clerk at the counter who is to check the bills, receive payments and place the packed garments into fancy bags, etc. The customers' arrivel at the check counter is a random phenomenon and the time between the arrivals varies from one minute to five minutes, the frequency distribution for which is given in table 13·1. The service time (time taken by the counter clerk) varies from one minute to three minutes and the frequency distribution for it is given in table 13·2. The Manager of the store feels that the counter clerk is not sufficiently loaded with work and wants to assign to him some additional work. But before taking the decision he likes to know precisely by what percentage of time the counter clerk is idle.

Table 13·1

Frequency distribution of inter-arrival times

Time between arrivals (minutes)	*Frequency* %	*Cumulative frequency* (%)
1	25	35
2	25	60
3	20	80
4	12	92
5	8	100

Table 13·2

Frequency distribution of service times

Service time (minutes)	*Frequency* (%)	*Cumulative frequency* (%)
1·0	20	20
1·5	35	55
2·0	25	80
2·5	15	95
3·0	5	100

EXAMPLE 13·1·2

A production shop has a group of 20 automatic machines being maintained by a crew of five mechanics. It is observed that quite often the machines have to wait for repairs for long spells of time, resulting in loss of production. To save the downtime of the machines, the management is interested in employing additional repairmen.

The problem is of determining the proper crew size. Fringe benefits of servicemen are Rs. 5 per hour per serviceman and the cost of machine time lost in waiting is Rs. 15 per hour.

From a thorough scrutiny of the previous records of machine break-downs, it is estimated that for a group of 20 similar machines working under similar conditions, the break-downs and service time follow the frequency distribution shown in tables 13·3 and 13·4. The frequency distributions are also shown by the histograms in figures 13·6 and 13·7.

Table 13·3

Break-down per hour	*Frequency (%)*
7 and less	5
9	12
9	25
10	30
11	20
12 and above	8

Table 13·4

Service time (minutes)	*Frequency (%)*
10	5
20	25
30	40
40	25
50	5

13·2. When to use Simulation ?

In the foregoing chapters we have discussed a number of operations research tools and techniques for solving various types of managerial decision making problems. Techniques like linear programming, dynamic programming, queuing theory, network models, etc., are not sufficient to tackle all the important managerial problems requiring data analysis. Each technique has its own limitations.

Linear programming models assume that the data does not alter over the planning horizon. It is one time decision process and assumes average values for the decision variables. If the planning horizon is long, say 10 years, the multiperiod linear programming model may deal with the yearly averaged data, but will not take into account

the variations over the months and weeks with the result that month to month and week to week operations are left implicit. Other important limitation of linear programming is that it assumes the data to be known with certainty. In many real situations the uncertainties about the data are such that they cannot be ignored. In case the uncertainty relates to only a few variables, the sensitivity analysis can be applied to determine its effect on the decision. But, in situations, where uncertainty prevades the entire model, the sensitivity analysis may become too cumbersome and computationly difficult to determine the impact of uncertainty on the recommended plan.

Dynamic programming models, however, can be used to determine optimal strategies, by taking into account the uncertainties and can analyse multiperiod planning problems. In other words, this technique is free from the two main limitations of linear programming. But it has its own shortcomings. Dynamic programming models can be used to tackle very simple situations involving only a few variables. If the number of state variables is a bit larger, the computation task becomes quite complex and involved.

Similar limitations hold good for other mathematical techniques like dynamic stochastic models such as inventory and waiting line situations. Only small scale systems are amenable to these models; moreover, by making a number of assumptions the systems are simplified to such an extent that in many cases the results obtained are only rough approximations.

From the above discussion we conclude that when the characteristics such as uncertainty, complexity, dynamic interaction between the decision and subsequent event, and the need to develop detailed procedures and finely divided time intervals, combine together in one situation, it becomes too complex to be solved by any of the techniques of mathematical programming and probabilistic models. It must be analysed by some other kind of quantitative technique which may give quite accurate and reliable results. Many new techniques are coming up, but, so far, the best available is simulation.

In general, the simulation technique is a dependable tool in situations where mathematical analysis is either too complex or too costly

13·3. What is Simulation ?

Simulation is an imitation of reality. You might have visited some industrial fairs and exhibitions, and noticed a number of simulated environments therein. A children cycling park in the fair with various crossings and signals is a simulated model of the city traffic

system. In the laboratories you perform a number of experiments on simulated models from which you predict the behaviour of the real system in true environments. A simple illustration is the testing of an aircraft model in a wind tunnel from which we determine the performance of the actual aircraft under real operating conditions. Planetarium shows represent a beautiful simulation of the planet system. Environments in a geological garden and in a museum of natural history are other examples of simulation.

In all these examples, it has been tried to imitate the reality to see what might happen under real operating conditions. This imitation of reality which may be in the physical form or in the form of mathematical equations may be called ***simulation***.

The simple examples cited above are of simulating the reality in physical form, which we may refer as (*environmental*) *analogue simulation*. For the complex and intricate problems of managerial decision making, the analogue simulation may not be practicable, and actual experimentation with the system may be uneconomical. Under such circumstances, the complex system is formulated into a mathematical model for which a computer programme is developed, and the problem is solved by using high speed electronic computer. We may refer it as *computer simulation* or *system simulation*.

With this background it will now be in order to define simulation. According to one definition "simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions". Simulation has also been defined as "the use of a system model that has the designed characteristics of reality in order to produce the essence of actual operation". According to Donald G. Malcolm a simulated model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.

13·4. Advantages of the Simulation Technique

The simulation technique, when compared with the mathematical programming and standard probability analysis, offers a number of advantages over these techniques; a few important among them can be summarized as follows :

1. Many important managerial decision problems are too intricate to be solved by mathematical programming and experimentation with the actual system, even if possible, is to too costly and risky.

Simulation offers the solution by allowing experimentation with a model of the system without interfering with the real system. Simulation is, thus, often a bypass for complex mathematical analysis.

2. Through simulation, management can foresee the difficulties and bottlenecks which may come up due to the introduction of new machines, equipment or process. It, thus, eliminates the need of costly trial and error methods of trying out the new concept on real methods and equipment.

3. Simulation has the advantage of being relatively free from mathematics and, thus, can be easily understood by the operating personnel and non-technical managers. This helps in getting the proposed plans accepted and implemented.

4. Simulation models are comparatively flexible and can be modified to accommodate the changing environments of the real situation.

5. Computer simulation can compress the performance of a system over several years and involving large calculations into a few minutes of computer running time.

6. The simulation technique is easier to use than mathematical models and is considered quite superior to the mathematical analysis.

7. Simulation has advantageously been used for training the operating and managerial staff in the operation of complex plans. It is always advantageous to train people on simulated models before putting into their hands the real system. Simulated exercises have been developed to impart the trainee sufficient exercise and experience. A simulated exercise familiarizes the trainee with the data required and helps in judging what information is really important. Due to his personal involvement into the exercise, the trainee gains sufficient confidence, and moreover becomes familiar with data processing on electronic computer.

13·5 Limitations of the Simulation Technique

In spite of all the advantages claimed by the simulation technique, many operations research analysts consider it a method of last resort and use it only when all other techniques fail. If a particular type of problem can be shown to be well represented by a mathematical model, the analytical approach is considered to be more economical, accurate and reliable. On the other hand, in very large and complex problems simulation may suffer from the same deficiencies as other mathematical models. In brief, the simulation technique suffers from the following limitations :

1. Simulation does not produce optimum results. When the model deals with uncertainties, the results of simulation are only reliable approximations subject to statistical errors.

2. Quantification of the variables is another difficulty. In a number of situations it is not possible to quantify all the variables that affect the behaviour of the system.

3. In very large and complex problems, the large number of variables and the involved inter-relationships between them makes the problem very unwieldy and hard to program. The number of variables may be too large and may exceed the capacity of the available computer.

4. For computer problems, simulation is, by no means a cheap method of anylsis. In a number of situations, simulation is comparatively costlier and time consuming.

5. Other important limitations stem from too much tendency to rely on the simulation models. This results in application of the technique to some simple problems which can more appropriately be handled by other techniques of mathematical programming.

13·6 Monte Carlo Method

The Monte Carlo method of simulation owes its development to the two mathematicians, John Von Neumann and Stanislaw Ulam, during World War II when the physicists were faced with the puzzling problem of behaviour of neutrons. How far would neutrons travel through different materials ? The hit and trial experimental solution would have been very costly and time consuming, and the problem was too complicated for theoretical anaylsis. The two mathematicians suggested a solution by submitting the problem to a roulette wheel or wheel of chance. The basic data regarding the occurrence of various events was known, into which the probabilities of separate events were merged in a step by step analysis to predict the outcome of the whole sequence of events. The technique provided an approximate but quite workable solution to the problem. The mathematical techniques they applied had been known for many years, but it was at this stage that it was given the name Monte Carlo. With the remarkable success of the techniques on neutron problem, it soon became popular and found many applications in business and industry. At present it forms a very important tool of operation researcher's tool kit.

Monte Carlo technique has been used to tackle a variety of problems involving stochastic situations and mathematical problems, which cannot be solved with mathematical techniques and where

physical experimentation with the actual system is impracticable. The stochastic situations are usually a long sequence of probabilistic events or steps. We may be able to write mathematical formulae for probability of a particular event, but to write a mathematical relationship for the probabilities of all events in the sequence is a difficult task.

In contrast to mathematical modelling where the results of the analysis yield a direct and overall solution to the problem, in simulation, the behaviour of the system is observed over a sufficiently long period of time, and in the process, the relevant information is collected. The system is first described by listing the various events in the order of their occurrence. An event representing a point in time signifies the end of one or more activities and the beginning of the next activity. As each event occurs, certain actions are taken, resulting in the generation of new events which are further considered in sequence. For example, the arrival of a customer at service facility is the occurrence of an event, and action taken depends upon the availability of facility which may be free or occupied. If free, service begins; if occupied, wait in line, which further generates new activities. If service begins, compute service time; and if to wait in queue, compute waiting time and length of queue, etc. The process is repeated for the next arrival.

The experimentation is performed on a simulated model of the real system. It is a sort of sampling technique in which, instead of drawing samples from a real population, the samples are drawn from a theoretical equivalent of the real population. By making use of roulette wheel or random numbers, Monte Carlo approach determines the probability distribution of the occurrence of the event under consideration, and then samples the data from this distribution.

Important applications of Monte Carlo method are found in waiting line problems, formulating maintenance policies, determining the inventory level, etc.

The procedure and the concept of Monte Carlo technique can well be illustrated with the help of examples. A few very simplified examples have been solved here for this purpose.

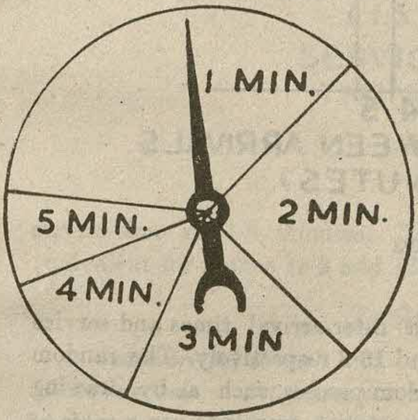
13·7. Solved Examples

EXAMPLE 13·7·1 (Solve example13·1·1).

Solution. The problem is of a waiting line situation. Here the customer arrival time at the check counter and the service time do not follow any specific rules and are random in nature. Had the frequency distribution followed some known laws, the simplest procedure would have been to use the appropriate formulae and calculate

the average time the counter clerk is free. Now let us see how this situation is tackled with the help of simulation technique.

To determine reliable and accurate results, history of the process should be simulated for a very large number of customer arrivals, which can be done with the help of a computer only. But our interest here, is only to demonstrate the technique of simulation, rather of Monte Carlo simulation. To meet the purpose the process has been simulated manually and for only 20 arrivals. This could be done by using the random number or by a roulette wheel. In the case of roulette wheel, which we may call a wheel of chance or a pie diagram, the area of the wheel circle is divided into a number of segments. For simulating the inter-arrival times of the given example the wheel is divided into five segments (Fig. 13·1) representing the various possible inter-arrival times. The area of a segment is proportional to



Fig· 13·1

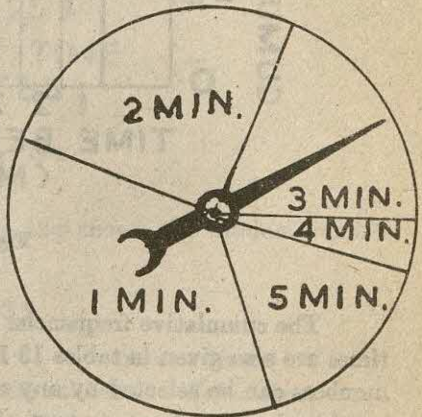


Fig. 13.2

the frequency of occurrence of the represented inter-arrival time. For example, for 35% of the time, inter-arrival time is 1 minute, so the area of the 1 minute segment is 35% of the area of the circle. For each arrival a sharp spin is given to the pointer and the segment in which it comes to rest gives the time between the present and next arrival. A similar roulette wheel, shown in figure 13·2, is constructed to simulate the service times. For each arrival a spin to the pointer gives the service time for that arrival.

In case we have to use the random number table, the cumulative frequency distributions are plotted, as shown in figures 13.3 and 13·4.

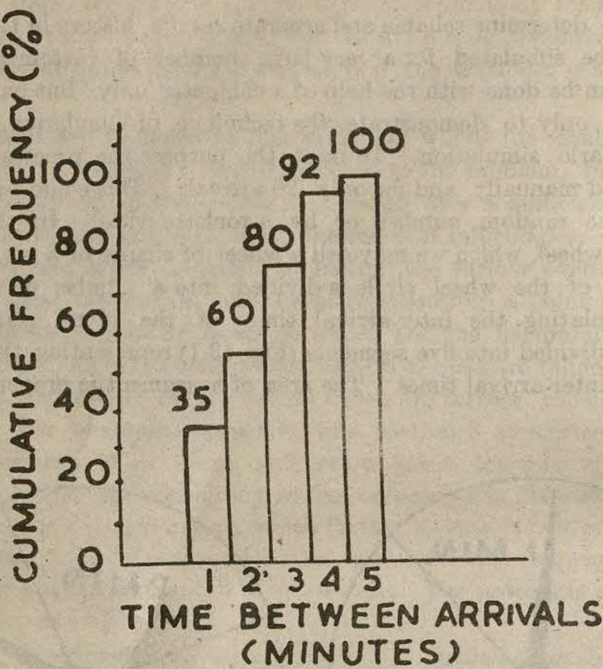


Fig. 13·3

The cumulative frequencies for inter-arrival times and service times are also given in tables 13·1 and 13·2 respectively. The random numbers can be selected by any random process, such as by drawing numbered chips from a box. The easiest way is to use a table of random numbers, like the one given in table C.1 (in appendix). For each arrival, a random number is selected which will represent the percentage of frequency corresponding to which service time can be determined from figure 13·4. For instance, if the random number comes out to be 22 for the first arrival, the service can be simulated by starting from an arbitrary selected random number, and then noting down, in sequence, the last two digit figures in, say, the last column. In a similar way the inter-arrival times can be simulated. For a simple problem like this we can construct a table of random numbers that selects the inter-arrival times, and a table of random numbers for selecting the service times. For example, the random numbers 36 through 60 give an inter-arrival time of 2 minutes. Similarly, for service times, random numbers 21 through 55 give a

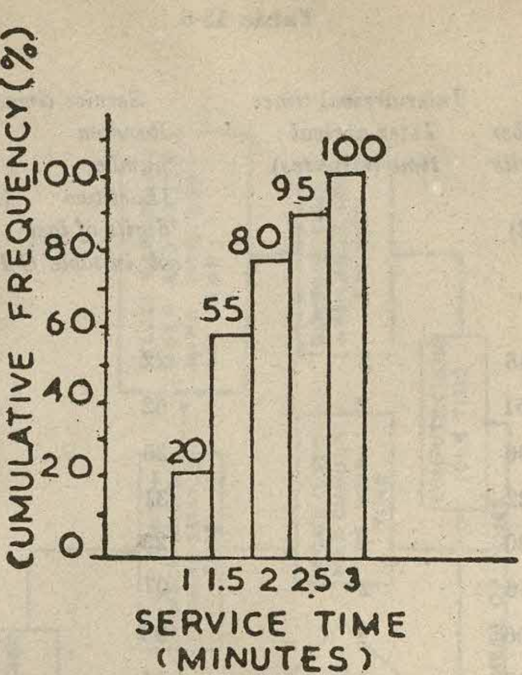


Fig. 13·4

service time of 1·5 minutes. Table 13·5 shows the random number equivalent for figures 13·3 and 13·4.

Table 13 5

Inter-arrival times		*Service times*	
These random numbers	*Select these Inter-arrival times (minutes)*	*These random numbers*	*Select these service time (minutes)*
1—35	1	1—20	1
36—60	2	21—55	1·5
61—80	3	56—80	2
81—92	4	81—95	2·5
93—100	5	96—100	3

Sampling from either the cumulative distributions of figures 13·3 and 13·4, or from table 13·5 will now give customer arrival and service times in proportion to the given distribution, just as if actual customers have been arriving for actual service at the check counter. The simulated history for 20 arrivals is given in table 13·6.

Table 13·6

Arrivals		*Inter-arrival times*	*Service times*	
	Random number (First two digits of last col. in table C.1)	*Inter-arrival time (minutes)*	*Random number (Last two digits of last col. in table C·1)*	*Service time (minutes)*
1	48	2	22	1·5
2	51	2	62	2·0
3	06	1	25	1·5
4	22	1	31	1·5
5	80	3	23	1·5
6	56	2	07	1·0
7	06	1	93	2·5
8	92	4	44	1·5
9	51	2	12	1
10	13	1	26	1·5
11	65	3	93	2·5
12	60	2	01	1
13	51	2	17	1
14	50	2	49	1·5
15	13	1	58	2
16	94	5	98	3
17	57	2	61	2
18	26	1	41	1·5
19	78	3	13	1
20	33	1	59	2

The process of determining the counter clerk's idle time and the customer waiting time can be represented by a block diagram as shown in figure 13·5.

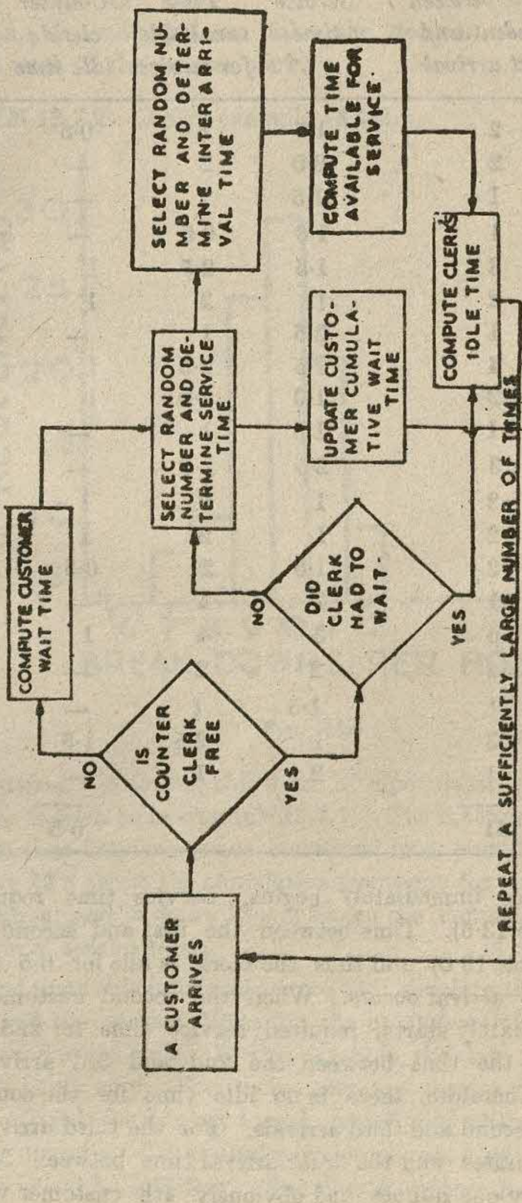

A CUSTOMER ARRIVES
IS COUNTER CLERK FREE
NO
YES
COMPUTE CUSTOMER WAIT TIME
DID CLERK HAD TO WAIT
NO
YES
SELECT RANDOM NUMBER AND DETERMINE SERVICE TIME
SELECT RANDOM NUMBER AND DETERMINE INTER ARRIVAL TIME
UPDATE CUSTOMER CUMULATIVE WAIT TIME
COMPUTE TIME AVAILABLE FOR SERVICE
COMPUTE CLERKS IDLE TIME
REPEAT A SUFFICIENTLY LARGE NUMBER OF TIMES

Fig. 13·5

The computations carried for 20 customer arrivals are shown in table 13·7. When the first customer arrives, counter clerk is

Table 13·7

Arrival number	*Time between present and next arrival*	*Service time*	*Time available for service*	*Counter clerk's idle time*	*Customer waiting time*
1	2	1·5	2	0·5	—
2	2	2·0	2	—	—
3	1	1·5	1	—	—
4	1	1·0	0·5	—	0·5
5	3	1·5	2·5	1	0 5
6	2	1	2	1	—
7	1	2.5	1	—	—
8	4	1·5	2·5	1	1·5
9	2	1·0	2·0	1	—
10	1	1·5	1	—	—
11	3	2·5	2·5	—	0·5
12	2	1	2	1	—
13	2	1	2	1	—
14	2	1·5	2	0·5	—
15	1	2	1	—	—
16	5	3	4	1	1
17	2	2	2	—	—
18	1	1·5	1	—	—
19	3	1	2·5	1·5	0 5
20	1	2	1	—	—
	41			9·5	4·5

free and service immediately begins. Service time required is 1·5 minutes (table 13·6). Time between the first and second arrivals is 2 minutes (table 13·6) and thus the clerk is idle for 0 5 minute before the next arrival occurs. When the second customer arrives, service immediately starts, required service time for 2nd arrival is 2 minutes and the time between the 2nd and 3rd arrivals is also 2 minutes. Therefore, there is no idle time for the counter clerk between the second and third arrivals. For the third arrival, service time is 1·5 minutes and the inter-arrival time between 3rd and 4th arrivals is only one minute and obviously 4th customer will have to wait for 0·5 minute before the service can begin. This way the

computations are continued. For 20 customer arrivals the counter clerk is idle for 9·5 minutes, while the customers have to wait for 4·5 minutes. The total time taken by 20 arrivals is 41 minutes (summing up the inter-arrival times). Therefore, the counter clerk is idle for about $\left(\frac{9{\cdot}5\times 100}{41}\right)=23\%$ of the time. Based on this specific information, the manager can take the decision of assigning some additional work to the counter clerk.

EXAMPLE 13·7-2 : Solve example 13·1-2.

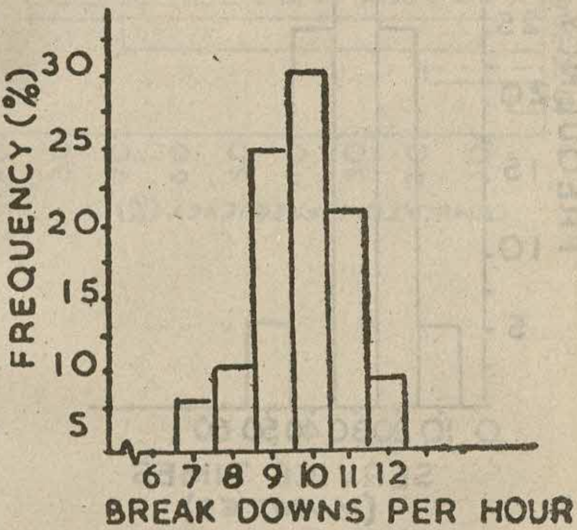


Fig. 13·6

Solution. Knowing this much of input data, we can proceed in the same fashion as in example 13·7-1. The machine break-down and service time frequencies are converted into cumulátive frequencies. Figure 13·8 shows the cumulative frequency for number of break-downs per hour and in figure 13·9 is shown the cumulative frequency distribution of service times.

Based upon this information, we can simulate the breakdowns per hour and the service times for the breakdown machines for any length of production time. In this example, the data has been simulated for 12 hours only. For more accurate results the simulation should be carried for a much longer period but that will need an electronic computer. To illustrate the procedure, this 12 hour manual simulation would be sufficient. For simulating the behaviour of the system, random number table C-1 is used. In the random number

table each number is of ten digits. As our probabilities vary from zero to 100, we will use only the two digits of the selected numbers.

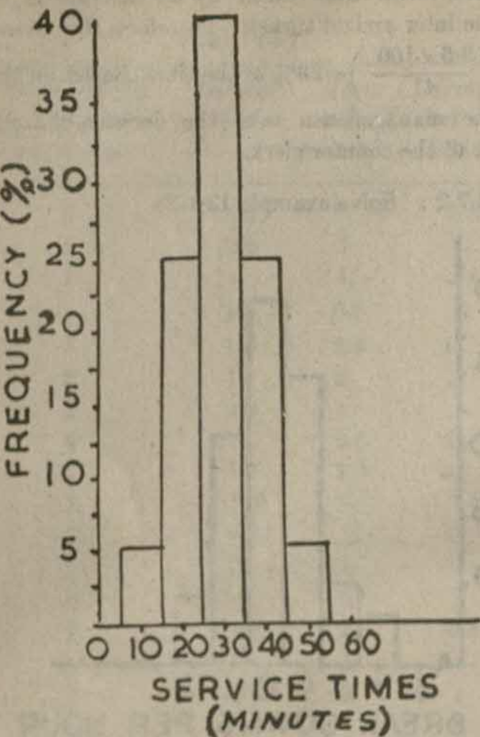


Fig. 13·7

For example, for simulating the study for the first hour, we arbitrarily select the first two digits of the random numbers in the first column of table C.1. These digits for the first number are 21. Corresponding to the cumulative frequency of 21, the number of break-downs per hour is 9. This can be determined from figure 13·8 or a table of random numbers equivalent to figure 13·8 can be constructed. Now for 9 break-downs, we have to simulate the service times. Again, at random, we select a number in the random number table and note down nine, two digit numbers in sequence from that column. These numbers turn out to be 11, 71, 65, 41, 35, 17, 91, 07 and 34. Then, corresponding to each number, the service time is found from the cumulative service time distribution (figure 13·9). For instance, corresponding to random number 11, which represents the cumulative frequency, service time is 20 minutes and is continued from hour to hour. For each hour, the number of break-downs and their service

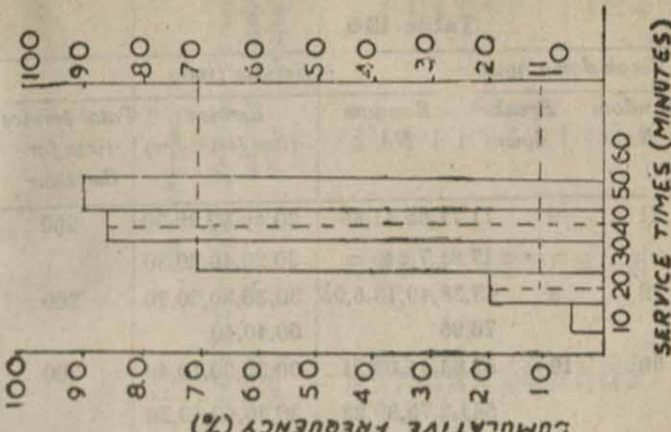


Fig. 13·9

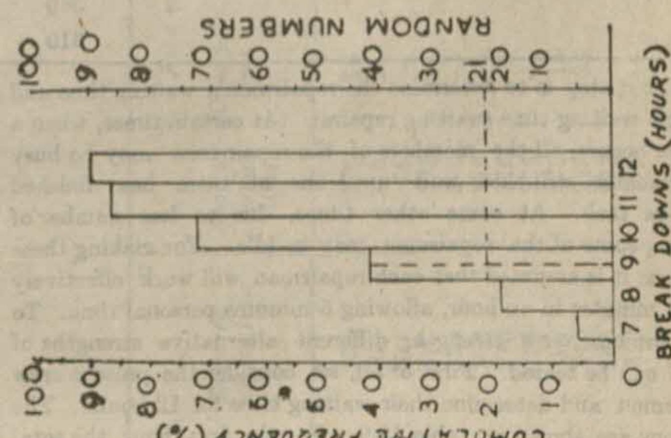


Fig. 13·8

times are determined as explained for the first hour. A simulated sample of first four hours of machine repair times is shown in table 13·8.

Table 13·8

Hours	*Break-down/hour*		*Service times*		
	Random No.	*Break-down*	*Random No.*	*Service time (minutes)*	*Total service time for the hour*
1st	21	9	11,71,65,41,35	20,40,30,30,30	260
			17,91,7,34	30,20,40,20,30	
2nd	12	8	43,38,49,13,5,95	30,30,30,20,20	260
			76,95	50,40,40	
3rd	69	10	57,63,41,03,91	30,30,30,10,40	300
			58,52,75,89,23	30,30,40,40,20	
4th	11	9	36,59,39,19,21	30,30,30,20,20	280
			74,86,90,64	40,40,40,30	
5th					320
6th					290
7th					270
8th					305
9th					260
10th					240
11th					340
12th					310

The next step is to determine the repairman's waiting time and the machine waiting time (waiting repairs). At certain times, when a break-down occurs, all the members of the repair crew may be busy and the machine will then wait until one of them has finished his previous task. At some other times, due to less number of breakdowns, some of the repairmen may be idle. For making these computations it is assumed that each repairman will work effectively for only 55 minutes in an hour, allowing 5 minutes personal time. To determine the best crew strength, different alternative strengths of 4, 5, 6 or 7 can be tested. First of all, we consider the present crew of 5 servicemen and determine their waiting time for 12 hours. The computations are shown in table 13·9. For the first hour the total service time required is 260 minutes (table 13·8). Allowing 25 minutes (5×5) of personal time for the crew, the total time available for service is $300-25=275$ minutes, which means in the first hour,

Table 13·9

Hour No.	Required service time	Five repairmen: Available service time	Crew idle time	Machine waiting time	Cumulative machine waiting time	Six repairmen: Available service time	Crew idle time	Machine waiting time	Cumulative machine waiting time
1	260	275	15	—	—	330	70	—	—
2	260	275	15	—	—	330	70	—	—
3	300	275	—	25	25	330	30	—	—
4	280	275	—	5	30	330	50	—	—
5	320	275	—	45	75	330	10	—	—
6	290	275	—	15	90	330	40	—	—
7	270	275	5	—	85	330	60	—	—
8	305	275	—	30	115	330	25	—	—
9	260	275	15	—	100	330	70	—	—
10	240	275	35	—	65	330	90	—	—
11	340	275	—	65	130	330	—	10	10
12	310	275	—	35	165	330	20	—	—
					880				10

servicemen will be idle for $275-260=15$ minutes. Similarly, in the second hour there is 15 minutes idle time or waiting time for the servicemen. In the third hour required service time is 300 minutes (table 13·8), available time again is 275 minutes, so that machines will wait for $300-275=25$ minutes, which we can call machine waiting time. In fourth hour again, the machine waiting time is 5 minutes which makes the cumulative machine waiting time of $25+5=30$ minutes for this hour. The computations are carried on from hour to hour and idle times of the repairmen and the cumulative waiting times of the machines are calculated. The total times for which the machines have to wait is 880 minutes.

Similarly, computations are done for a crew of six repairmen (table 13·9). We note that with six repairmen, the machine waiting time comes down to only 10 minutes. If the crew size is further increased, it will add only to the idle time of the repairmen.

It is given that the wages including the fringe benefits of servicemen are Rs. 5 per hour per serviceman and the cost of machine time lost in waiting is Rs. 15 per hour.

Therefore, with a crew of five repairmen,

cost of time lost by machines in waiting for 880 minutes	=Rs. 220·00
Wages of five servicemen for 12 hours	=Rs. 300·00
Total cost	=Rs. 520·00
With a crew of six repairmen, cost of time lost by machines in waiting for 10 minutes	=Rs. 2·50
Wages of six servicemen for 12 hours	=Rs. 360·00
Total cost	=Rs. 362·50

The comparison of the total costs shows that the best crew size is of six repairmen. For more accurate and reliable results, the study should be carried over a much longer period on a computer.

13·8 Additional Examples

EXAMPLE 13·8-1

A manufacturing firm has 25 semi-automatic machines in one section. Machines work for eight hours a day and a repair squad of 5 mechanics attends to their maintenance. The machines are such

that only one mechanic can work on a machine at one time. It has been determined from the past break-down history that there are ten per cent chances that a machine will break-down in any given hour. The times required for repairs and their probability distribution is given in the table below.

Table 13·10

Time required to repair each machine (minutes)	*Probability (%)*
15	5
20	10
25	20
30	35
35	22
40	8

The management of the firm is interested in knowing whether this strength of repair squad is optimum and if not, what is the optimum number. The cost of idle machine time to the company is Rs. 10 per hour, while the wages paid to a repairman are Rs. 5 per hour. Company allows 10 minutes per hour as each repairman's personal time.

EXAMPLE 13·8-2

In the child welfare section of a hospital, two specialists attend to the outdoor patients daily from 2 P.M. to 5 P.M. There is a general complaint from the public that they have to wait too long. The duty doctors also complain that due to excess of patients they have to sit beyond 5 P.M. on many of the days. From the data collected over the past months, the following distributions of patients' arrivals and the check up times are determined :

Table 13·11

Number of patients per day	*Frequency (%)*	*Check-up time per patient (minutes)*	*Frequency (%)*
20	40	8	20
25	30	12	40
30	20	15	35
40	10	20	5

What should be the number of doctors on duty so that their average busy time does not exceed 3 hours/day?

EXAMPLE 13·8-3

A printing press receives a different number of orders each day. The time required for composing and printing varies from order to order. There is sufficient number of printing machines and the orders usually do not have to wait for printing. The critical time is that of composing. The manager of the press is interested in knowing the number of composers he should have so that the sum of the cost of composer idle time and the cost of losing orders is minimized. The following data regarding the number of orders per day and the composing times is available.

Table 13·12

No. of orders per day	*Frequency (%)*	*Composing time per order (hours)*	*Frequency (%)*
3	10	2	10
5	20	3	25
8	35	4	30
10	25	5	25
12	10	6	10

The press works for eight hours per day, but a composer can work effectively for only seven hours a day. An order is accepted only if it can be processed within two days. The wages of the composer are Rs. 3 per hour, while the cost of orders back ordered comes to Rs. 5 per hour.

EXAMPLE 13·8-4

At a toll office, a sample of 100 arrivals of vehicles gives the following frequency distribution of the inter-arrival and service times :

Table 13·13

Inter-arrival time (minutes)	*Frequency (%)*	*Service time (minutes)*	*Frequency (%)*
1·0	2	1·5	10
1·5	5	—	—
2·0	9	2·0	22
2·5	25	—	—
3·0	22	2·5	40
3·5	11	—	—
4·0	10	3·0	20
4·5	6	—	—
5·0	3	3·5	8
5·5	2	—	—

There is one clerk at the toll office. Simulate the process for 25 arrivals and estimate the average per cent vehicle waiting time and the average per cent idle time of the clerk.

EXAMPLE 13·8·5

A coffee house in the busy market of city operates counter service. The proprietor of the coffee house has approached you with the problem of determining the number of bearers he should employ at the counter. He wants that the average waiting time of the customer should not exceed 2 minutes. After recording the data for a number of days, the following frequency distribution of inter-arrival time of customers and the service time at the counter are established.

Table 13·14

Inter-arrival time (minutes)	*Frequency* (%)	*Service time (minutes)*	*Frequency* (%)
0	5	1·0	5
0·5	35	2·0	25
1·0	25	3·0	35
1·5	15	4·0	20
2·0	10	5·0	15
2·5	7		
3·0	3		

Simulate the system for about 30 arrivals for various alternate numbers of bearers.

EXAMPLE 13·8·6

The common maintenance problem with a heavily loaded machine is the failure of its bearings. There are three bearings which cause the trouble and keep the machine down for a considerable part of time. It looks that the present practice of replacing a bearing as and when it fails is not a good policy. It is decided to evaluate the following three alternate policies : 1. The present policy of replacing a bearing as and when it fails. 2. Replace all the three bearings when any of them fails. 3. Replace all the bearings which have been in use for 1,000 hours or more when a bearing fails.

It has been observed that it takes 7 hours on the part of one mechanic to replace one bearing. If two bearings are replaced,

maintenance time is 9 hours and it takes 11 hours to replace all the three bearings.

The wages of a maintenance mechanic are Rs. 5 per hour, while machine down time cost is Rs. 4 per hour. The cost of each bearing is Rs. 10.

From the past performance the following frequency distribution of the actual working lives of bearings is available :

Table 13·15

Bearing life (hours)	*Frequency %*
600	2
700	6
800	8
900	14
1,000	17
1,100	20
1,200	15
1,300	10
1,400	7
1,500	1

Determine the best policy of replacing the bearings by simulating approximately 10,000 hours of service time for each of the three policies.

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14

Basic Steps in PERT and CPM

14·1. Introduction

The complexities of the present-day management problems and the business competitions 'has added to the pressure on the rbains of decision makers. In a large and complex project involving a number of interrelated activities, requiring a number of men, machines and materials, it is not possible for the management to make and execute an optimum schedule just by intution based on the organisational capabilities and work experience. Managements are, thus, always on the look out for some methods and techniques which may help in planning, scheduling and controlling the project. A project may be defined as a combination of interrelated activities which must be executed in a certain order before the entire task can be completed. The aim of planning is to develop a sequence of activities of the project, so that the project completion time and cost are properly balanced and that the excessive demand of key resources is avoided. To meet the object of systematic planning, the managements have evolved a number of techniques applying network strategy. PERT (Programme Evaluation and Review Technique) and CPM (Critical Path Method) are two of the many network techniques which have been widely used for planning, scheduling and controlling the large and complex projects.

14·2. Historical Background

Up to the end of 18th century, the decision making was intuitive and depended primarily on the managerial capabilities, experience and academic background of managers. It was only in the early 1900's, that the pioneers of scientific management started developing the scientific management techniques. Henry L. Gantt,

during World War I, developed the Gantt chart for production scheduling. The Gantt chart was later modified to bar chart which was used as an important tool in both the project and production scheduling.

With the development of computers, many new scheduling techniques came into existence. In 1957, the network techniques of PERT and CPM were developed almost concurrently. Most of the formal efforts in the development of CPM were initiated by the E.I. du Pont de Numours Company in 1956 through its IEC (Integrated Engineering Control) group. In 1957, the Remington Rand Application group joined the Du Pont IEC team. In the beginning CPM was used for planning and scheduling the constructional projects. It was also used for scheduling the maintenance shutdowns. The construction industry in general and the petrochemical industry in particular were the major areas of CPM applications.

PERT was developed by U.S. Navy for scheduling the research and development work for the polaris missiles programme whose activities were subject to a considerable degree of uncertainty. That is why the principal feature of PERT is that its activity time estimates are probabilistic. The activity times in CPM applications were relatively less uncertain and were, thus, of deterministic nature. Following the success in polaris programme, the use of PERT became popular in the U.S. Aerospace industry, large weapon systems, Atomic energy programmes and in many other fields.

With the passage of time, PERT and CPM applications started overlapping and now they are used almost as a single technique and the difference between the two is only of the historical or academic interest. With slight modifications both have given rise to several other network techniques, such as PEP (Programme Evaluation Procedure), RAMPS (Resource Allocation and Multi Project Scheduling), LESS (Least Cost Estimating and Scheduling) and SCANS (Scheduling and Control by Automated Network Systems), etc.

14·3. Phases of Project Scheduling

Project scheduling can be divided into three phases : planning, scheduling and control.

Planning

Planning phase consists of

1. Setting the objectives of the project and the assumptions to be made.

2. Development of W.B.S. (Work Breakdown Structure). Depending upon the objective of the management, the extent of control desired and the availability of computational aids, the project is broken down into clearly definable activities.
3. Determination of time estimates for these activities, and
4. Establishment of inter-dependence relationship between the activities.

Scheduling

The scheduling phase covers the determination of

1. Start and finish times for each activity.
2. Critical path on which the activities require the special attention, and
3. Slack and floats for the non-critical paths.

Controlling

Controlling phase is the follow up to the planning and scheduling and involves

1. Making periodical progress reports.
2. Reviewing the progress.
3. Analysing the status of the project, and
4. Management decisions regarding updating, resource allocation, etc.

14·4. Work Breakdown Structure (W. B. S.)

A project is a combination of interrelated activities which must be performed in a certain order for its completion. The process of dividing the project into these activities is called the Work Breakdown Structure (W. B. S.). The *activity* or *a unit of work*, also called *work content* is a clearly identifiable and manageable work unit.

Let us consider a very simple situation to illustrate the W.B.S. A group of students is given the project of designing, fabricating and testing a small centrifugal pump. The project can be broken down into the following sub-parts :

(*i*) design,
(*ii*) fabrication,
(*iii*) testing.

The network at this level of detail will look as shown in figure 14·1.

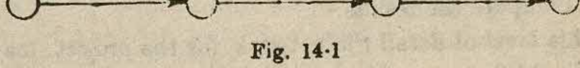


Fig. 14·1

The work units can further be broken down into smaller work contents as shown below.

A. Design.
B. Make drawings.
C. Make patterns.
D. Make moulds.
E. Do casting of parts.
F. Do machining of parts.
G. Assemble.
H. Design test rig.
I. Fabricate test rig.
J. Perform the test.

The network at this level of detail may look like the one shown in figure 14·2.

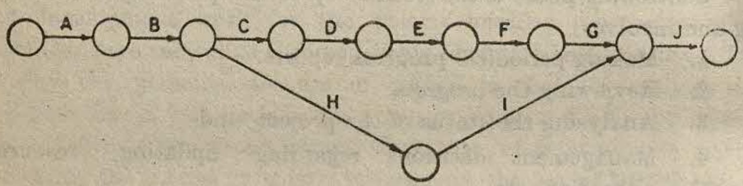


Fig. 14·2

Some of these activities can still be further sub-divided and the work can be carried in parallel on a number of activities. The various activities can be listed as

A. Design.
B. Make drawings.
C. Make pattern of impeller.
D. Make pattern of casing.
E. Make mould for impeller.
F. Make mould for casing.
G. Do casting of all the parts.
H. Machine impeller.
I. Machine casing and other parts.
J. Assemble the pump.
K. Design test rig.
L. Fabricate test rig.
M. Perform the test.
N. Compute the results.

At this level of detail the network for the project may look as shown in fig. 14·3.

The level of detail depends upon the objective of the management, the extent of control desired and the availability of the compu-

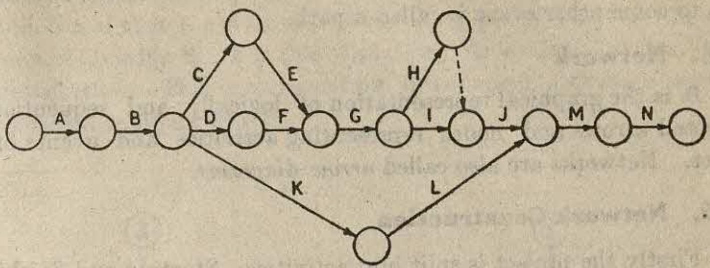


Fig. 14·3

tational aids. The larger the details, better will be the control and more involved will be the computations.

14·5. Network Logic

Some of the terms commonly used in networks are defined below.

14·5·1. Activity

It is a physically identifiable part of a project which consumes time and resources. Activities are obtained by the work breakdown into smaller work contents. In the network, activity is represented by an arrow, the tail of which represents the start and the head, the finish of the activity. The length, shape and direction of the arrow has no relation to the size of the activity.

14·5·2. Event

The beginning and end points of an activity are called events or nodes. Event is a point in the time and does not consume any resources. It is generally represented by a numbered circle. The head event, called the *j*th event, has always a number higher than the tail event, called the *i*th event, *i.e.*, $j > i$. For example,

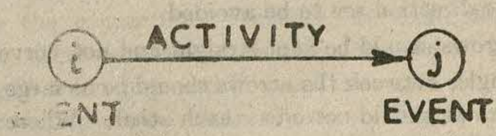


Fig. 14·4

'Making the pattern of impeller' is an activity.

'Start making the pattern of impeller' is an event.

'Pattern making completed' is an event.

14·5·3. Path

An unbroken chain of activity arrows connecting the initial event to some other event is called a path.

14·5·4. Network

It is the graphical representation of logically and sequentially connected arrows and nodes representing activities and events of a project. Networks are also called *arrow diagrams.*

14·5·5. Network Construction

Firstly the project is split into activities. Starting and finishing events of the project are then decided. After deciding the precedence order, the activities are put in a logical sequence by using the graphical notations. While constructing the network, in order to ensure that the activities fall in a logical sequence, the following questions are checked :

(*i*) What activities must be completed before a particular activity starts ?

(*ii*) What activities follow this ?

(*iii*) What activities must be performed concurrently with this ?

Activities which must be completed before a particular activity starts are called the *predecessor activities* and those which must follow a particular activity are called *successor activities.*

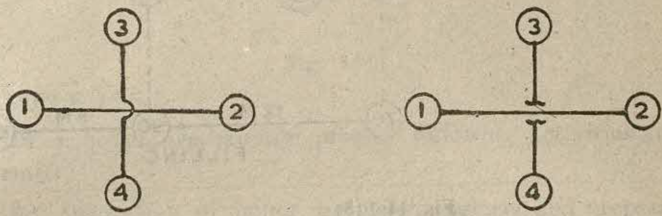
While drawing the network following points should be kept in mind :

1. Each activity is represented by one and only one arrow. But in some situations where an activity is further subdivided into segments, each segment will be represented by a separate arrow.
2. Time follows from left to right. Arrows pointing in opposite direction are to be avoided.
3. Arrows should be kept straight and not curved or bent.
4. Angles between the arrows should be as large as possible.
5. Arrows should not cross each other. Where crossing cannot be avoided, the crossing methods shown in figure 14·5 should be adopted.

14·5·6. Dummy

An activity which only determines the dependency of one activity over the other, but does not consume any time is called a dummy activity. Dummies are usually represented by dotted line arrows

To illustrate the use of dummy, refer to figure 14·6 (*a*) and assume that the start of activity C depends upon the completion of activities A and B and that the start of activity E depends only on the completion of activity B. For this situation, figure 14·6 (*a*) is a faulty representation. This is corrected by introducing a dummy activity D as shown in figure 14·6(*b*).



(*a*) Pipe line cross over (*b*) Broken arrow cross over.

Fig. 14·5

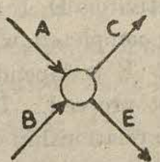


Fig. 14·6 (*a*)

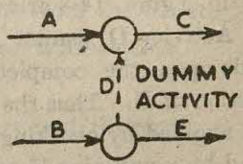


Fig. 14·6 (*b*)

14·5-7. Partial Dependency

In certain situations the starting of an activity depends upon the partial completion of a predecessor activity. In such cases the predecessor activity is further broken into two parts and dummy is used to make the connection. Consider, for example, a pipe line laying project. There are three activities : digging of 20 metre long trench, laying 20 metre pipe and filling of the trench. Here, laying of pipe can be started after 8 metre of trench digging and similarly after 8 metre of pipe laying, filling can be started. The situation which is faultly represented in figure 14·7 (*a*) is corrected by introducing dummies as shown in figure 14·7 (*b*).

Fig. 14·7 (*a*)

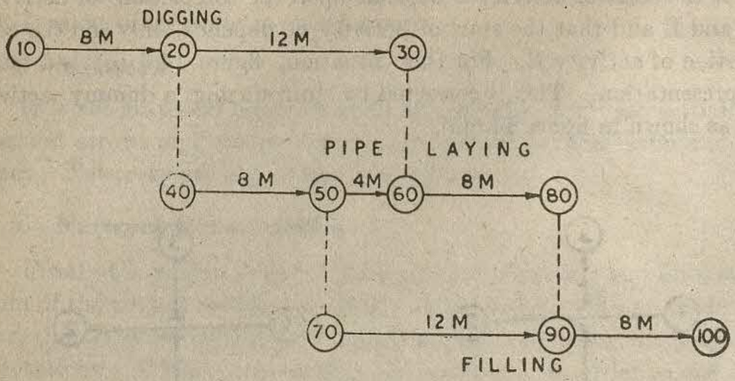


Fig. 14·7 (*b*)

14·5·8. Looping

Sometimes, due to faulty net-work sequence a condition illustrated in figure 14·8 arises. Here the activities D, E and F form a loop. Activity D cannot start until F is completed, which, in turn, depends upon the completion of E. But E is dependent upon the completion of D. Thus the network cannot proceed. This situation can be avoided by checking the precedence relationship of the activities and by numbering them in a logical sequence

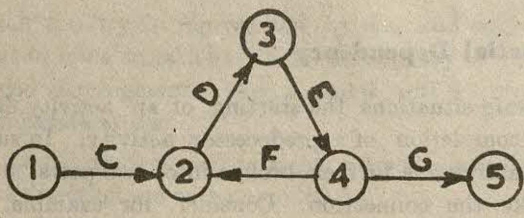


Fig. 14·8

14·5·9. Dangling

It is the situation where activities other than the initial and the final activities do not have any successor activity as shown in figure 14·9. To avoid this situation, it should be kept in mind that all events except the first and the last of the whole project must have at least one entering and one leaving activity.

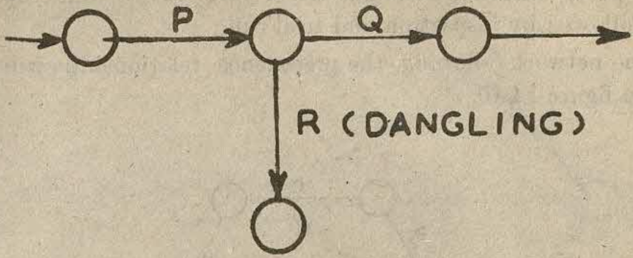


Fig. 14·9

EXAMPLE 14·5·1

In a boiler overhauling project following activities are to be performed :

A. Inspection of boiler by boiler engineer and preparation of list of parts to be replaced/repaired.

B. Collecting quotations for the parts to be purchased.

C. Placing the orders and purchasing.

D. Dismantling of the defective parts from the boiler.

E. Preparation of necessary instructions for repairs.

F. Repair of parts in the workshop.

G. Cleaning of the various mountings and fittings.

H. Installation of the repaired parts.

I. Installation of the purchased parts.

J. Inspection.

K. Trial run.

Assuming that the work is assigned to the boiler engineer who has one boiler mechanic and one boiler attendant at his disposal, draw a network showing the precedence relationships.

Solution. When we look at the list of activities, we note that activity A (inspection of boiler) is to be followed by dismantling (D) and only after that it can be decided which parts can be repaired and which will have to be replaced. Now the repairing and purchasing can go side by side. But the instructions for repairs may be prepared after sending the letters for quotations. Note that it becomes a partial constraint. Also the cleaning of the boiler which is to be done by the attendant can be started after activity D. Now we assume that repairing will take less time than purchasing. But the installation of repaired parts can be started only when the cleaning is complete. This results in the use of a dummy activity. After the installation of

repaired parts, installation of purchased parts can be taken up. This will be followed by inspection and trial run.

The network showing the precedence relationships will look as shown in figure 14·10.

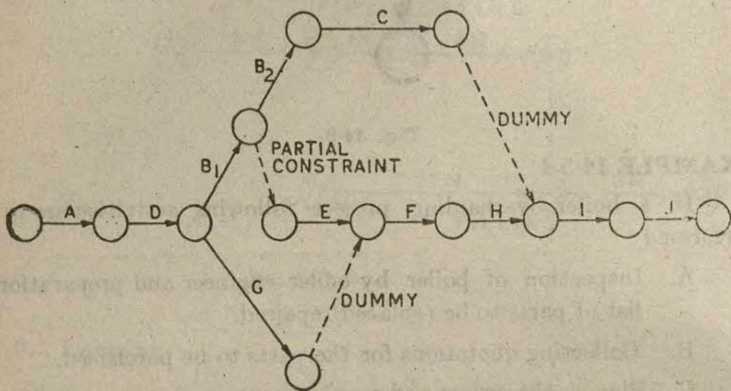


Fig. 14·10.

EXAMPLE 14·5·2

Following are the activities which are to be performed for a building site preparation. Determine the precedence relationship and draw the network.

A. Clear the site.
B. Survey and layout.
C. Rough grade.
D. Excavate for sewer.
E. Excavate for electrical manholes.
F. Install sewer and backfill.
G. Install electrical manholes.
H. Construct the boundary wall.

Solution. Looking at the list of activities we can fix the following precedence order :

B succeeds A and C succeeds B *i.e.*, B>A; C>B.

D and E can start together after the completion of C, *i.e.*, D, E>C.

F will follow D and G will follow E, *i.e.*, F>D; G>E.

H can start after C *i.e.*, H>C.

Thus the precedence relationships are :

B>A; C>B; D, E, H>C; F>D and G>E.

The project can be represented in the form of a network as shown in figure 14·11.

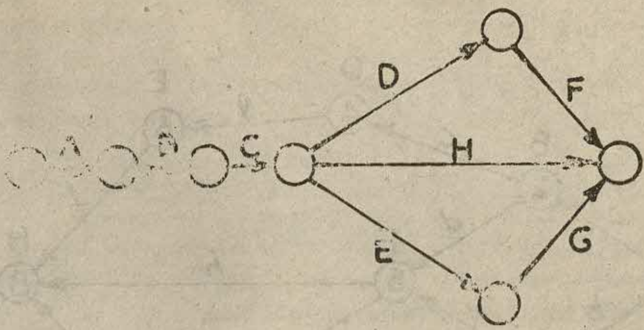


Fig. 14·11.

14·6. Numbering the Events (Fulkerson's Rule)

After the network is drawn in a logical sequence, every event is assigned a number which is placed inside the node circle. The number sequence should be such as to reflect the flow of the network. The rule devised by D.R. Fulkerson is used for the purpose of numbering. It involves the following steps :

1. The initial event which has all outgoing arrows with no incoming arrow is numbered '1'.
2. Delete all the arrows coming out from node '1'. This will convert some more nodes (at least one) into initial events. Number these events 2, 3, ...
3. Delete all the arrows going out from these numbered events to create more initial events. Assign the next numbers to these events.
4. Continue until the final or terminal node, which has all arrows coming in with no arrow going out, is numbered.

To illustrate the numbering technique let us consider the network shown in figure 14·12.

Event A is initial event and is numbered 1. Delete the arrows *a* and *b*. This will create two more (B & C) initial events. Number these 2 & 3. Now delete the arrows coming out from nodes 2 and 3, *i.e.*, delete the arrows *c*, *d*, *e* and *f*. This converts D and F into initial events. Number these nodes as 4 and 5. Then delete the arrows *g*, *h* and *i*, the two nodes E and G become the initial nodes and they are numbered 6 and 7. Then the last event or terminal event is numbered as 8.

This continuous numbering may be all right when the project is very small and the network is not liable to any modifications later

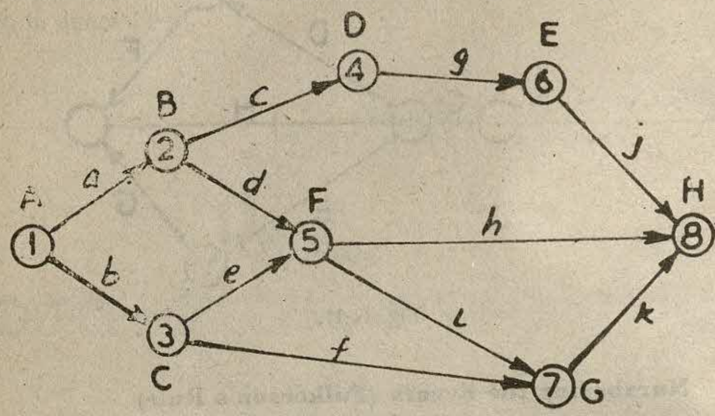


Fig. 14·12

on. But in large networks, where extensive modifications may have to be made, there should be scope of adding more events and numbering them without causing any inconsistency or loops. This is achieved by *skip numbering*. One way is to assign the numbers such as 10, 20, 30, 40...or 4, 8, 12, 16..., etc. The second way is to leave some numbers such as 7, 8, 9; 17, 18, 19; 27, 28, 29,... and allot them to the events added afterwards. There can be still more ways of doing skip numbering.

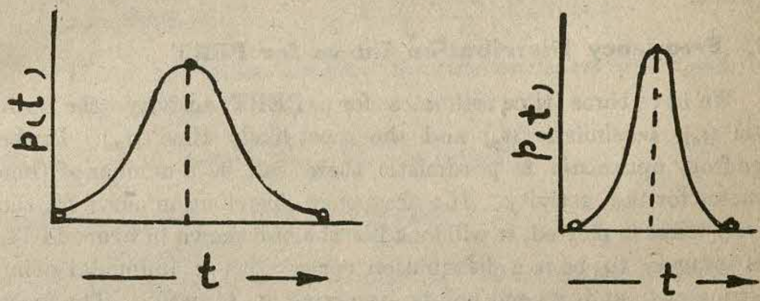
14·7. Measure of Activity

Each task or activity takes some time for its completion. This time duration depends upon the nature of the activity. Some activities are rarely performed and no data exists for their time durations. Their time consumption involves a considerable degree of uncertainty. Such activities are called *'variable activities'* and stochastic modelling techniques are applied in their time estimation. Under this category fall the activities which demand creative ability, such as, research, design and development work and the activities which are performed under uncertain environments, such as, construction work during rainy season.

On the other hand, there are activities for which the associated time duration can be accurately estimated. Such activities are called *deterministic in nature or deterministic type.* These activities are usually repetitive in nature. Also it is presumed that (*i*) skilled persons experienced in method study are available to do the job

(*ii*) sufficient additional resources are available to allow uninterrupted activity. Above all, it is the assumption of confidence that all will go well. Figure 14·13 shows frequency distribution curves for the two types of activites. In Figure 14·13 (*a*) the dispersion of the curve is more and hence more is the uncertainty. In Figure (*b*), for deterministic type activity, the dispersion is less and the system tends to be more deterministic.

The projects which comprise of the variable type activities associated with probabilistic time estimates, employ PERT version of the networks and the projects comprising of deterministic type of activities are handled by CPM version of networks. This is the main difference between the two techniques. The other difference between the two is that PERT is event-oriented while the CPM is activity-oriented.



(*a*) Fig. 14·13 (*b*)

t = activity time

$p_{(t)}$ = relative frequency of occurrence or probability density function.

14·7-1. Time Units

Any convenient time unit can be used, but it must be consistent throughout the network. Depending upon the project length and level of detail, time unit may be working days, shifts or weeks. Full time units are usually used, for instance, an activity estimated at 3 days and 6 hours will be assigned 4 days.

14·7-2. Time Estimates

The CPM system of networks omits the probabilistic considerations and is based on a *Single Time Estimate* of the average time required to execute the activity.

The PERT system is based on *Three Time Estimates* of the performance time of an activity. They are

(*i*) *The Optimistic Time Estimate.* The shortest possible time required for the completion of an activity, if all goes extremely well.

(*ii*) *The Pessimistic Time Estimate.* The maximum possible time the activity will take if everything goes bad. However, major catastrophes such as earthquakes, floods, storms and labour troubles are not taken into account while estimating this time.

(*iii*) *The Most Likely Time Estimate.* The most likely time is the time an activity will take if executed under normal conditions.

For determining the single time estimates used in CPM, some historical data may be available, but the best way of predicting the time estimates is by intelligent guessing. The experienced person who may be an engineer, foreman, or worker having sufficient technical competence is asked to guess the various time estimates. For estimation the activity should be taken randomly, so that the guess of the assessor is not prejudiced by the predecessor and the successor activities.

14·8. Frequency Distribution Curve for PERT

We have three time estimates for a PERT activity, the optimistic (t_o), pessimistic (t_p) and the most likely time (t_m). In the range from optimistic to pessimistic, there can be a number of time estimates for the activity. If a frequency distribution curve for the activity times is plotted, it will look like the one shown in figure 14·14. It is assumed to be a β-distribution curve with a unimodal point occurring at t_m and its end points occurring at t_0 and t_p. The most likely time need not be the midpoint of t_0 and t_p and hence the frequency distribution curve may be skewed to the left, skewed to the rigth or symmetric.

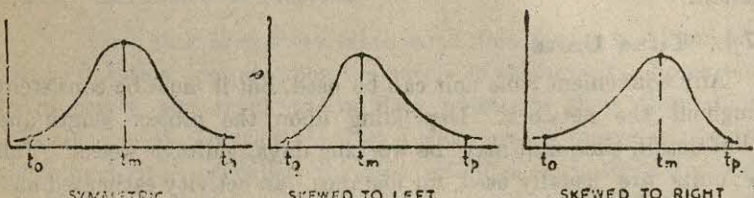


Fig. 14·14. β-Distribution curve.

Though the β-distribution curve is not fully described by the mean (μ) and the standard deviation (σ), yet in PERT the following relations are approximated for μ and σ :

$$\text{Mean, } \mu = \frac{t_0 + 4t_m + t_p}{6}.$$

$$\text{Standard deviation, } \sigma = \frac{t_p - t_0}{6}.$$

By definition, variance $V=\sigma^2=\left(\frac{t_p-t_0}{6}\right)^2$.

Expected time or average time of an activity is taken equal to mean. This is the time which the activity is expected to consume while executed.

$$t_E=\mu=\frac{t_0+4t_m+t_p}{6}.$$

What is the probability that the activity will be completed in this expected time ? Variance is the measure of this uncertainty. Greater the value of variance, the larger will be the uncertainty.

Let the time estimates for a particular activity be : $t_0=5$ days, $t_m=7$ days and $t_p=9$ days.

Then the expected time, $t_E=\frac{5+7+9}{6}=7$ days,

and Variance, $V=\left(\frac{9-5}{6}\right)^2=0{\cdot}444.$

If for the same activity the time estimates were $t_0=4$ days $t_m=7$ days and $t_p=10$ days,

then $t_E=\frac{4+4\times 7+10}{6}=7$ days,

and Variance, $V=\left(\frac{10-4}{6}\right)^2=1{\cdot}00.$

Comparing the two estimates, we note that the expected time is same in both the cases. But the variance is more in the second case. Hence the uncertainty of completion of *activity in 7 days is more than that in 9 days.*

EXAMPLE 14·8-1

To make the procedure of using three time estimates clear, let us consider the network shown in Figure 14·15. For each activity

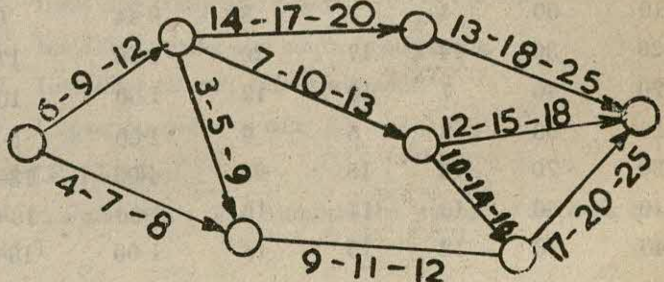


Fig. 14·15

the three time estimates t_o, t_m and t_p are given along the arrows in the t_o—t_m—t_p order.

Let us first number the events. Using Fulkerson's rule, we start from the left and number the initial event as 10. Then, assuming the arrows emerging from node 10 to be deleted, we create one more initial event and number it 20. This procedure is repeated until all the events are numbered.

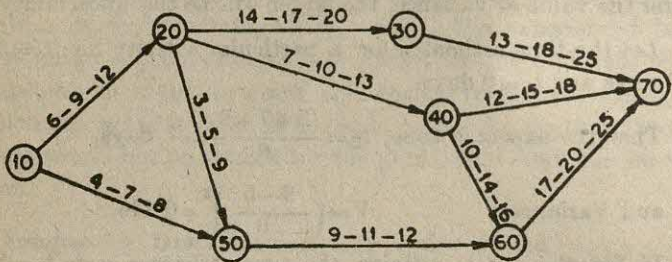


Fig. 14·16.

Now we put the events in a tabular form and calculate the variance and expected times. These calculations can be carried on the network itself also.

Table 14·1

Activity $i\,j$: Predecessor event i	Activity $i\,j$: Successor event j	t_o	t_m	t_p	$\sigma = \frac{(t_p - t_o)}{6}$	$t_E = \frac{(t_o + 4t_m + t_p)}{6}$
10	20	6	9	12	1.00	9.0
10	50	4	7	8	0·44	6·7
20	30	14	17	20	1·00	17·0
20	40	7	10	13	1·00	10·0
20	50	3	5	9	1·00	5·33
30	70	13	18	25	4·00	13·50
40	60	10	14	16	1·00	13·67
40	70	12	15	18	1·00	15·00
50	60	9	11	12	0·25	10·83
60	70	17	20	25	1·78	20·33

The entry in the tabular form starts with the initial event, by entering first number (10 in this case) in the first row under the column predecessor event *i*. Then the activities emerging out from the initial event (here 10-20 and 10-50) are entered in the ascending order. Then, we go to the next higher number (here 20) in the predecessor event column and enter all the activities emerging out from this event i.e., 20-30, 20 40 and 20-50. This procedure is repeated until all the events are entered.

14·9. Additional Examples

EXAMPLE 14·9·1

A prize distribution function is to be organised. The date has been fixed with the chief guest. Prepare a W.B.S. and draw the precedence network. Also number the events.

EXAMPLE 14·9·2

The project of a pumping set installation has been broken down into the following major activities. Determine the precedence relationship and draw a network for the project. Using Fulkerson's rule number the events.

Activity	*Description*
A.	Preliminary investigation.
B.	Selection of site.
C.	Boring and installing suction side equipment.
D.	Building foundation.
E.	Compacting foundation.
F.	Erecting building.
G.	Installing pump and motor.
H.	Installing wiring and control equipment.
I.	Installing delivery side equipment.
J.	Inspection and trial run.

EXAMPLE 14·9·3

Using Fulkerson's rule number the events of network given in figure 14·3.

EXAMPLE 14·9·4

Calculate the variance and expected activity times for the acti-

vities of the network shown in figure 14·17. Enter calculations in the tabular form.

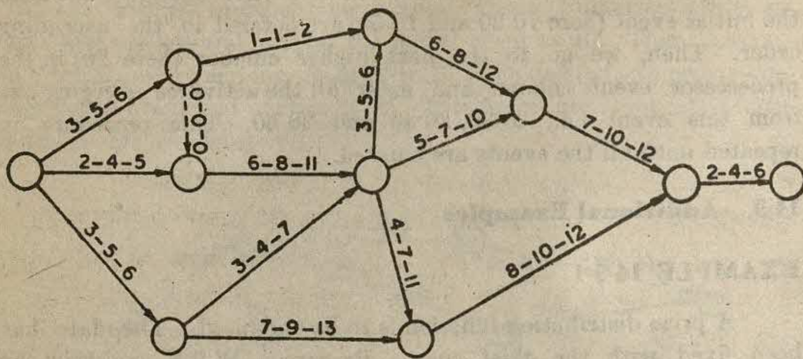


Fig. 14·17.

EXAMPLE 14·9·5

From the data given below, draw a network and calculate the expected task times and their variance.

Task	A	B	C	D	E	F	G	H	I	J	K
Least time	4	5	8	2	4	6	8	5	3	5	6
Largest time	8	7	12	7	10	15	16	9	7	11	13
Most likely time	5	7	11	3	7	9	12	6	5	8	9

Precedence relationships are : A, C, D can start immediately. E>B, C; F, G>D; H, I>E, F ; J>I, G; K>H; B>A. *(P.U. April, 1976)*

15

PERT Computations

15·1. Introduction

Though both PERT and CPM networks consist of events and activities, yet the emphasis is more on events in a PERT network and on activities in a CPM network. In a network there are a number of paths joining the initial and the final events of the project. The path which takes the maximum of time is called the *critical path* and the activities on the critical path are called *critical activities.* The procedure for identifying the critical path in both the PERT and CPM networks is similar. The critical path calculations consist of two phases : the forward pass computations and the backward pass computations.

15·1. Forward Pass Computations

Calculations begin at the initial event and move towards the end event. Initial event is assigned zero time, and then proceeding to the next event in sequence, the time at which that event is expected to occur at the earliest is calculated. This is called the *Earliest expected time* for that event and is denoted by T_E.

To illustrate the procedure, let us consider the network shown in figure 15·1.

Event 10 is at zero time, the time at which event 20 can occur at the earliest is 6 days after the start of the project. Thus for event 20, $T_E=6$. There are two chains leading into event 30. Event 30 can occur only when activities 10-30 and 20-30 are completed. Activity 10-30 takes 8 days, while activity 20-30 which can start after 6 days from zero time takes three days for completion and hence will be completed, at the earliest, 9 days after the start of the project, *i.e.*, for event 30, $T_E=9$.

Generalising,

T_E^j = Maximum of all $[T_E^i + t_E^{ij}]$ for all $i\,j$ leading into the event,

where T_E^j is the earliest expected time of the successor event j,

T_E^i is the earliest expected time of the predecessor event i, and

t_E^{ij} is the expected time of activity ij.

Thus T_E^{30} = Maximum of $[T_E^{10} + t_E^{10-30}),\ (T_E^{20} + t_E^{20-30})]$

$=$ Maximum of $[(0+8),\ (6+3)]$

$=9$

T_E^{40} = Maximum of $[(T_E^{20} + t_E^{20-40}),\ (T_E^{30} + t_E^{30} + t_E^{30-40})]$

$=$ Maximum of $[(6+8),\ (9+4)]$

$=15.$

$T_E^{50} = T_E^{30} + t_E^{30-50}$ (There is only one activity 30-50 leading into event 50)

$=9+10=19.$

T_E^{60} = Maximum of $[T_E^{30} + t_E^{30-60}),\ (T_E^{40} + t_E^{40-60})$ and $(T_E^{50} + t_E^{50-60})]$

$=$ Maximum of $[(9+12),\ (15+8)$ add $(19+6)]$

$=25.$

Similarly, $T_E^{70} = 33$,

and $T_E^{80} = 40.$

All these computations can be carried on the network itself.

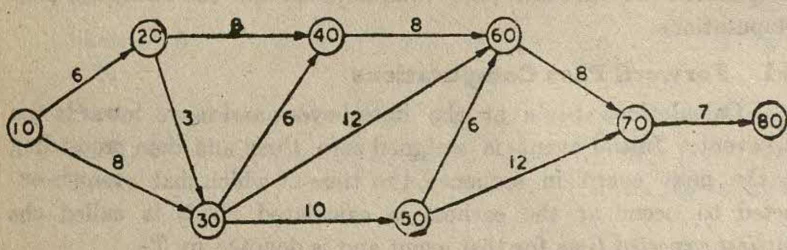


Fig. 15·1

15·3. Backward Pass Computations

Calculations begin from the end node of the project and proceed towards the start node. To start the calculations, the time of occurrence for the last node is decided. This is the time at which the project must be completed. This is called the '*Contractual Obligation Time*' and is denoted by T_S. If not known, the contractual obligation time is taken to be equal to the earliest expected time for the end event. The objective of the backward pass is to calculate the '*Latest Allowable Occurrence Time*', the time at which a particular event must occur at the latest. This is denoted by T_L.

Referring to the network shown in figure 15·1 and taking $T_S = T_E^{80} = T_L^{80} = 40$ days, event 70 must occur at the latest after **33** (40 − 7) days from the start, so that the project is completed on schedule. Therefore $T_L^{70} = 33$. As there is only one arrow leading back into the event 60, $T_L^{60} = 33 - 8 = 25$. There are two paths leading back into event 50. Through the path 80-70-50 event 50 should occur at the latest after $33 - 12 = 21$ days, through the path 80-70-60-50 event 50 should occur at the latest after $25 - 6 = 19$ days. Looking into the two we decide that event 50 must occur 19 days after the start so that the project is completed on schedule. Therefore

$$T_L^{50} = 19.$$

Generalising,

T_L^i = Minimum of all $[(T_L^j - t_E^{ij})]$ for all ij emerging from i,
where T_L^i is the latest allowable occurrence time for event i,
T_L^j is the latest allowable occurrence time for event j, and
t_E^{ij} is the expected time for activity ij.

Thus T_L^{40} = Minimum of $[(T_L^{70} - t_E^{40\text{-}70}), (T_L^{60} - t_E^{40\text{-}60})]$
= Minimum of $[(33 - 10), (25 - 8)]$
= 17.

T_L^{30} = Minimum of $[(T_L^{40} - t_E^{30\text{-}40}), (T_L^{60} - t_E^{30\text{-}60})$ and $(T_L^{50} - t_E^{30\text{-}50})]$
= Minimum of $[(17 - 6), (25 - 12)$ and $(19 - 10)]$
= 9

Similarly, $T_L^{20} = 6$,

and $T_L^{10} = 0$.

All these computations can be carried on the network itself. The network with the T_E and T_L values for all events is shown in figure 15·2.

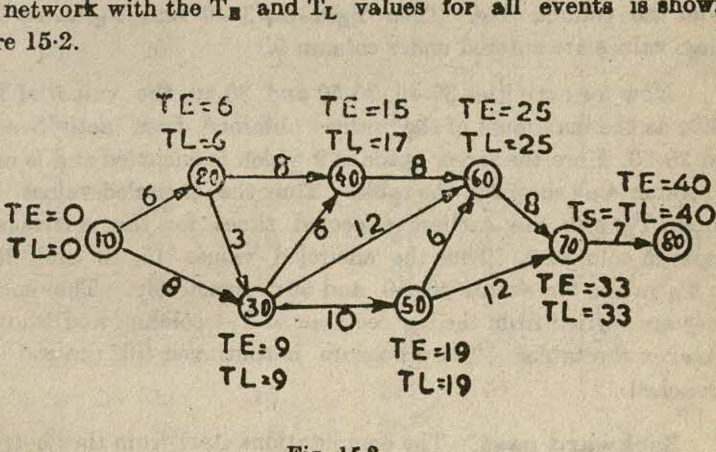


Fig. 15·2

15·4 Computations in Tabular Form

The T_E and T_L values for the events can be calculated in a tabular form also. As shown in table 15·1 the forward pass calculations start from the top of the table and proceed downwards, while the backward pass calculations start from the bottom of the table and proceed upwards. Columns 1 and 2 are for the predecessor and successor events of an activity. Column 3 is for the expected activity times. In columns 4 and 5 are entered the T_E values for event i and j of the activity ij and columns 6 and 7 are for the corresponding T_L values.

Forward Pass. The same relation as discussed in the Forward pass computations is used to compute the T_E values.

i.e., T_E^j = Maximum of all $[T_E^i + t_E^{ij}]$ for all ij leading into the event j.

Starting with the initial event 10, T_E^{10} = zero; it is entered in column 4 in the first row. Then for the first activity 10–20, the successor event is 20; $T_E^{20} = T_E^{10} + t_E^{10-20} = 0 + 6 = 6$ is entered in the first row and column 5 for T_E^j values. As there is only one activity leading into the event 20, the values of $T_E^{20} = 6$ is the maximum value and is encircled.

For activity 10-30, the value under T_E^i column (No. 4) is the value for $T_E^{10} = 0$, and the value under T_E^j column (No. 5) is the value of $T_E^{30} = T_E^{10} + t_E^{10-30} = 0 = 8 = 8$.

For the activity 20–30, the value under T_E^i column (No. 4) is the value of T_E^{20} which has been determined in first row as $T_E^{20} = 6$. For both of activities 20-30 and 20-40, value of $T_E^{20} = 6$ and is entered under the column T_E^i. Then $T_E^{30} = 6 + 3 = 9$ and $T_E^{40} = 6 + 8 = 14$. These values are entered under column 5.

Now for activities 30-40, 30-50 and 30-60, the value of T_E^{30} is taken as the maximum of the values obtained from activities 10-30 and 20-30. Here the larger value is 9 which is encircled and is carried to column 4 as shown in the table. Thus the encircled values in the column T_E^j give the earliest expected times for the corresponding events in column 2. Thus the encircled values 15, 19 and 25 are the T_E values for events 40, 50 and 60 respectively. The encircled times are carried from the T_E^j column to T_E^i column and shown by arrows in the tables. The procedure is continued till the end event is reached.

Backward pass. The computations start from the bottom of the table and are based on the relation,

T_E^i = Minimum of all $[T_E^j - t_E^{ij}]$ for all ij emerging from event i.

At the bottom of the table, the last activity ij is activity 70-80. $T_L^{80} = T_E^{80} = 40$ and is entered in the last row under the T_E^j column. Then, $T_E^{70} = T_E^{80} - t_E^{70\text{-}80} = 40 - 7 = 33$ is entered under T_L^i column and this value is encircled, as there is no other event successor to 70. This value 33 is then carried to column T_L^j and is entered against the activities 60-70 and 50-70.

Then $T_L^{60} = 33 - 8 = 25$. Since there is only one event successor to 60, this value of 25 is encircled and is carried to column T_L^j as shown in table 15·1.

For the predecessor event 50, there are two time values :

$$T_E^{50} = T_E^{70} - t_E^{50\text{-}70} = 33 - 12 = 21,$$

$$\text{and } T_E^{50} = T_E^{60} - t_E^{50\text{-}60} = 25 - 6 = 19.$$

The minimum of the two *i.e.*, $T_E^{50} = 19$ is encircled and is carried to T_L^j column and entered against the activities with 50 as the successor event.

The procedure is continued till the start event is reached.

Table 15·1

1	2	3	4	5	6	7
Activity i j						
Predecessor event (i)	*Successor event j*	t_E^{ij}	T_E^i	T_E^j	T_L^i	T_L^j
10	20	6	0	6	0	6
10	30	8	0	8	1	9
20	30	3	6	9	6	9
20	40	8	6	14	9	17
30	40	6	9	15	11	17
30	50	10	9	19	9	19
30	60	12	9	21	13	25
40	60	8	15	23	17	25
50	60	6	19	25	19	25
50	70	12	19	31	21	33
60	70	8	25	33	25	33
70	80	7	33	40	33	40

15·5. Slack

Slack and float both refer to the amount of time by which a particular event or activity can be delayed without affecting the tin e schedule of the network. The term slack refers to events and is used with PERT networks, while the term float refers to activities and is used with CPM networks.

In PERT language, slack is defined as the difference between the latest allowable time and the earliest expected time of an event.

For an event j, slack $S_j = T_L{}^j - T_E{}^j$.

Slack calculations can be performed either on the network itself or in the tabular form. In the network shown in figure 15·3 there is no slack for events 4, 8, 16 and 24 and hence no delay on the path 4, 8, 16, 24 can be allowed. For event 12 slack time of 4 days exists. So the activity leading to event 12 can be started late or work on it can be slackened up to 4 days.

Calculations of slack for the network shown in figure 15· are given in table 15·2. Note that where any event is repeated, only the

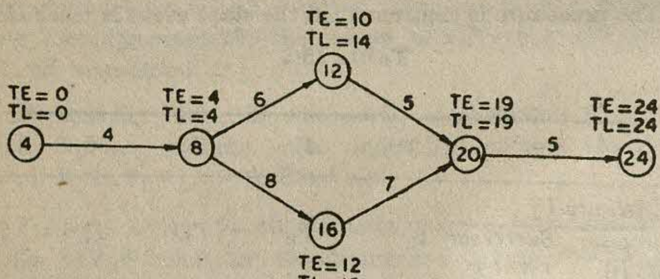


Fig. 15·3

Table 15·2

Predecessor event i	Successor event j	$t_E{}^{ij}$	$T_E{}^i$	$T_E{}^j$	$T_L{}^i$	$T_L{}^j$	$S_j = T_L{}^j - T_E{}^j$
4	8	4	0	4	0	4	0
8	12	6	4	10	8	14	4
8	16	8	4	12	4	12	0
12	20	5	10	15	14	19	4
16	20	7	12	19	12	19	0
20	24	5	19	24	19	24	0

row in which the value of T_E^j is encircled is to be considered. For the starting event the slack is usually zero.

15·6. Critical Path

After knowing the T_E and T_L values for various events in the network, the critical path activities can be identified by applying the following conditions :

(*i*) T_E and T_L values for the tail event of the critical activity are the same *i.e.*, $T_E^i = T_L^i$.

(*ii*) T_E and T_L values for the head events of the critical activities are the same *i.e.*, $T_E^j = T_L^j$.

Or we can say that both the predecessor and successor events of the critical activity have zero slacks.

(*iii*) For the critical activity, $T_E^j - T_E^i = T_L^j - T_L^i = t_E^{ij}$.

The path forming an unbroken chain of critical activities from the start event to the end event is called the *critical path*. On the critical path all events have zero slacks. In the network the critical path is shown by thick lines.

When we apply the above conditions to the network shown in figure 15·3, we find that activities 4-8, 8-16, 16-20 and 20-24 are critical activities. The path 4-8-16-20-24 is, thus, the critical path .

In the execution of the project, it is the critical path which needs optimum care. Any delay in any activity on this path will upset the whole project and will delay its completion. That is why the name critical path is assigned to this path.

EXAMPLE 15·1

Consider the network shown in figure 15·4. The activity times in days are given along the arrows. Calculate the slacks for the events and determine the critical path. Put the calculations in tabular form.

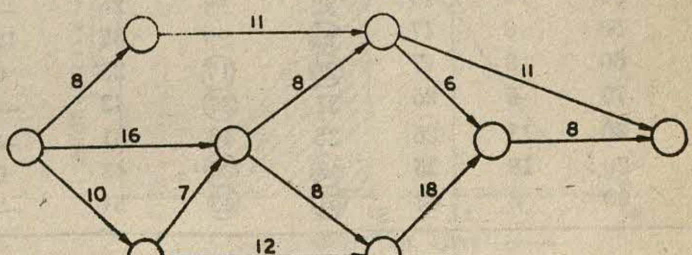


Fig. 15·4.

Solution. The events of the network are first numbered and then T_E and T_L values are calculated. Figure 15·5 shows the network with the numbers in the node circles and T_E, T_L and S values along the nodes.

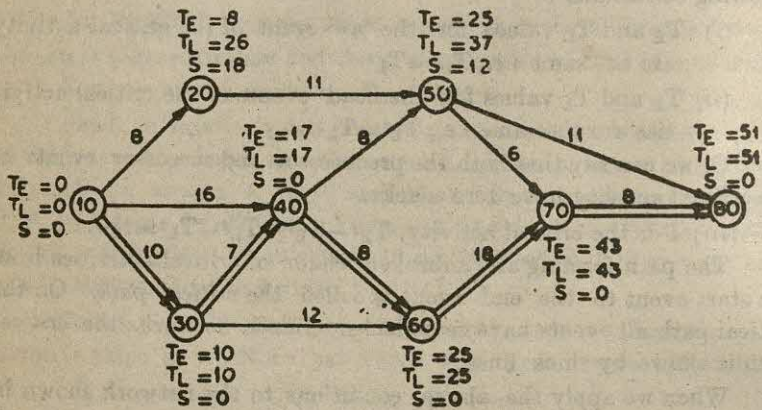


Fig. 15·5

Table 15·3

Prodecessor event i	*Successor event j*	t_{Ei}^{ij} *(days)*	T_E^i	T_E^j	T_L^i	T_L^j	*Slack* S_j $T_L^j - T_E^i$ *(days)*
10	20	8	0	8	[illegible]	26	18
10	30	10	0	10	[illegible]	10	0
10	40	16	0	16	[illegible]	17	—
20	50	11	8	19	[illegible]	37	—
30	40	7	10	17	10	17	0
30	60	12	10	22	13	25	—
40	50	8	17	25	29	37	12
40	60	8	17	25	17	25	0
50	70	6	25	31	37	43	—
50	80	11	25	36	40	51	—
60	70	18	25	43	25	43	0
70	80	8	43	51	43	51	—

From the computations carried on the network as well as in table 15·3, we find that slack of 18 days and 12 days exists for the events 20 and 50 respectively, while there is zero slack for all other

events. Now, to determine the critical path we apply the conditions of section 15·5. Note that activity 30-60 is not a critical activity, though the slack at both the end events is zero. This does not satisfy the condition $T_E{}^j - T_E{}^i = T_L{}^j - T_L{}^i = t_E{}^{ij}$. This activity can be expanded by three days. Thus the critical path is 10-30-40-60-70-80, and the same is known with heavy lines on the network.

15·7. Probability of Meeting the Scheduled Dates

After identifying the critical path and the occurrence times of all activities, the question arises "what is the probability that a particular event will occur on or before the scheduled date ?" This particular event may be any event in the network which marks a significant state in the project and affects the subsequent project activities.

Let us recall that the expected time of an activity is the weighted average of the three time estimates, the optimistic (t_0), the most likely (t_m) and the pessimistic (t_p).

i.e., $$t_E{}^{ij} = \frac{t_0 + 4t_m + t_p}{6}.$$

The probability that the activity *i-j* will be completed in time $t_E{}^{ij}$ is fifty per cent. In the frequency distribution curve for the activity *i-j*, the vertical line through t_E will divide the area under the curve in two equal parts, as shown in fig. 15·6. For completing the activity in any other time t_K, the probability will be

$$p = \frac{\text{Area under AEK}}{\text{Area under AEB}}$$

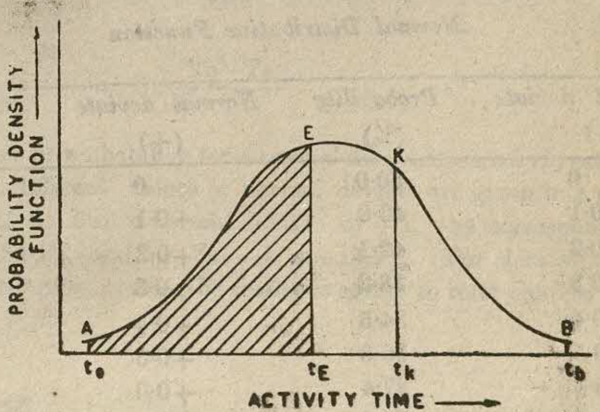


Fig. 15·6

A project consists of a number of activities. All activities, as we know, are independent random variables and hence length of the project up to a certain event through a certain path is "also a random' variable. But the point of difference is that expected project length T_E does not have the same frequency distribution as the expected activity time t_E. While a β-distribution curve approximately represents the activity time frequency distribution, the project expected time T_E follows approximately what is called a normal distribution curve. The standard normal distribution curve has an area equal to unity and a standard deviation of one, and is symmetrical about the mean value as shown in figure 15·7.

The probability of completing a project in time T_S is given by

$$p(T_S)=\frac{\text{Area under ACS}}{\text{Area under ACB}}.$$

Thus $p(T_S)$ depends upon the location of T_S. Taking T_E as the reference point, the distance T_ET_S can be expressed in terms of standard deviation. The value of the standard deviation for a network is calculated as

Standard deviation for network,

$$\sigma=\sqrt{\text{Sum of the variances along critical path}}$$

Variance for an activity, $V=\left(\frac{t_p-t_o}{6}\right)^2$.

i.e., σ for network $=\sqrt{\Sigma\sigma^2_{ij}}$.

Table 15.4

Normal Distribution Function

Normal deviate (+)	*Probability* (%)	*Normal deviate* (+)	*Probability* (%)
0	50·0	0	50·0
—0·1	46·0	+0·1	54·0
—0·2	42·1	+0·2	57·9
—0·3	38·2	+0·3	61·8
—0·4	34·5	+0·4	65·5
—0·5	30·8	+0·5	69·2
—0·6	27·4	+0·6	72·6
—0·7	24·2	+0·7	75·8
—0·8	21·2	+0·8	78·8

Normal deviate (+)	*Probability* (%)	*Normal deviate* (+)	*Probability* (%)
−0·9	18·4	+0·9	81·6
−1·0	15·9	+1·0	84·1
−1·1	13·6	+1·1	86·4
−1·2	11·5	+1·2	88·5
−1·3	9·7	+1·3	90·3
−1·4	8.1	+1·4	91·9
−1·5	6·7	+1·5	93·3
−1·6	5·5	+1·6	94·5
−1·7	4·5	+1·7	95·5
−1·8	3·6	+1·8	96·4
−1·9	2·9	+1·9	97·1
−2·0	2.3	+2·0	97·7
−2·1	1·8	+2·1	98·2
−2·2	1·4	+2·2	98·6
−2·3	1·1	+2·3	98·9
−2·4	0·8	+2·4	99·2
−2.5	0·6	+2·5	99·4
−2·6	0·5	+2·6	99·5
−2·7	0.3	+2·7	99·7
−2·8	0·3	+2.8	99·7
−2·9	0·2	+2·9	99·8
−3·0	0·1	+3·0	99·9

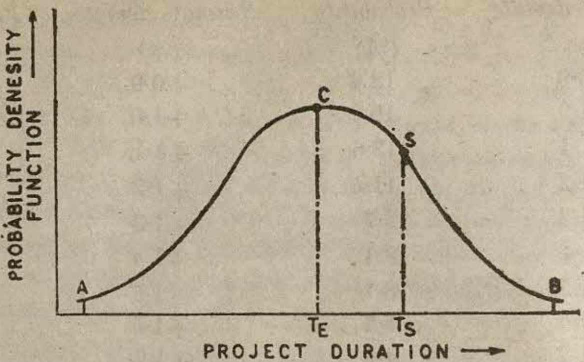
Since the standard deviation for a normal curve is 1, the σ calculated above is used as a scale factor for calculating the normal deviate.

$$\text{Normal deviation, } Z = \frac{T_S - T_E}{\sigma}.$$

The values of probability for a normal distribution curve, corresponding to the different values of normal deviate are given in a simplified table-15·4. For a normal deviate of +1, the corresponding probability is 84·1% and for $Z = -1$, $p = 15.9\%$. For details of the frequency distribution curves the reader is asked to refer chapter 9 of the book.

EXAMPLE 15·2

To illustrate the procedure let us consider the network shown



Fig· 15·7

in fig. 15·8. The three time esitmates, the expected activity durations and the variance are shown along the arrows. The earliest

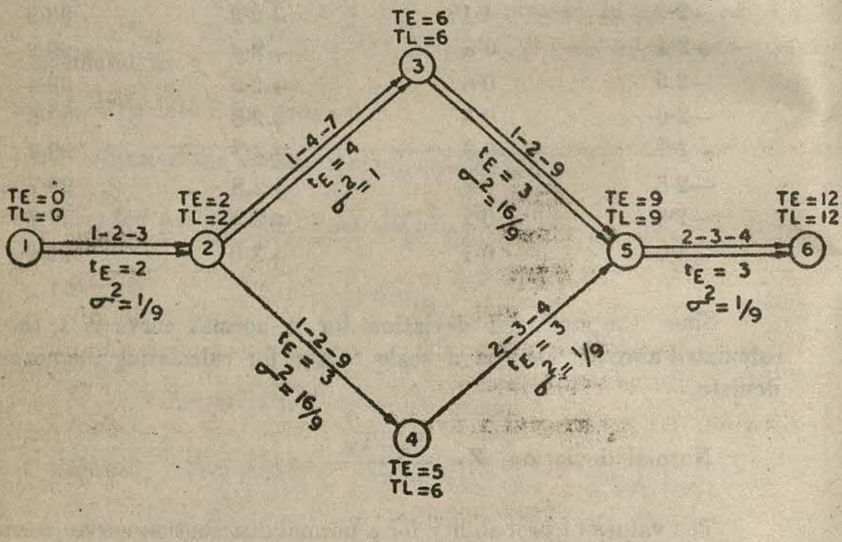


Fig. 15·8

expected times and the latest allowable occurrence times are computed and put along the nodes. We identify that the path 1-2-3-5-6 is the critical path and expected length of project is 12 days. Now let us consider three cases :

1. What is the probability of completing the project in 12 days ?

Here, $T_E = 12$ days, $T_S = 12$ days.

$\therefore$ σ for network $= \sqrt{\Sigma V_{ij}}$

$$= \sqrt{1/9+1+16/9+1/9} = 1\cdot73.$$

This $\sigma = 1\cdot73$ is used as scale factor to calculate the normal deviate, Z.

$$Z = \frac{T_S - T_E}{\sigma} = \frac{12-12}{\sigma} = \text{Zero}.$$

From the probability function table 15·4, the probability corresponding to $Z=0$, is 50%. Thus the probability of completing the project in 12 days is fifty per cent.

2. What is the probability of completing the project in 14 days ?

Here, $T_E = 12$ days, $T_S = 14$ days.

$\therefore$ σ as calculated above $= 1\cdot73$.

$\therefore$ Normal deviate, $Z = \dfrac{14-12}{17\cdot3} = 1\cdot16.$

The corresponding probability is 87·7%.

3. What is the probability of completing the project in 10 days ?

Here, $Z = \dfrac{T_S - T_E}{\sigma}$

$$= \frac{10-12}{1\cdot73} = -1\cdot16.$$

The corresponding probability is 12·3%.

The knowledge of the probability of project completion at the scheduled dates helps the management in determining whether the allocated resources are just sufficient, or are in excess, or the work will have to be expedited by adding more resources. The probability may also indicate a situation totally unsatisfactory or a situation too much comfortable and excess of resources. For probability up to 0·3 : Replacing of the project is necessary,

0·3 to 0·4 : Close scrutiny is required,

0·4 to 0·65 : Satisfactory,

More than 0·65 : Excess resources and replanning.

EXAMPLE 15·3

What is the probability that event 3 of network shown in fig. 15·8 will occur 7 days after the start ?

Solution. (*a*) Here $T_S = 7$, $T_E = 6$.

$\therefore$ Standard deviation, $\sigma=\sqrt{1/9+1}=1{\cdot}05.$

$\therefore$ Normal deviate, $\quad Z=\dfrac{T_S-T_E}{\sigma}=\dfrac{7-6}{1{\cdot}06}=\cdot 94.$

From Table 15·4, probability corresponding to Z=0·94 is 0·824 or 82·4%.

EXAMPLE 15·4

Consider the network shown in Fig. 15·9. The three time estimates for activities are given along the arrows. Determine the critical path. What is the probability that the project will be completed in 20 days ?

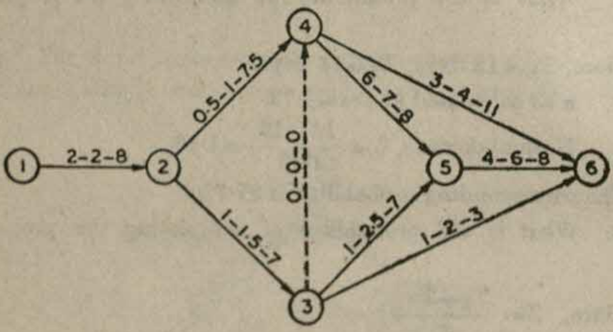


Fig. 15·9

Solution. First step is to number the events. In this network the events are already numbered. The calculations can be performed on the network itself or in the tabular form. After calculating the expected times and the variances of the activities they are put along the arrows as shown in Fig. 15·10. By carrying the forward pass

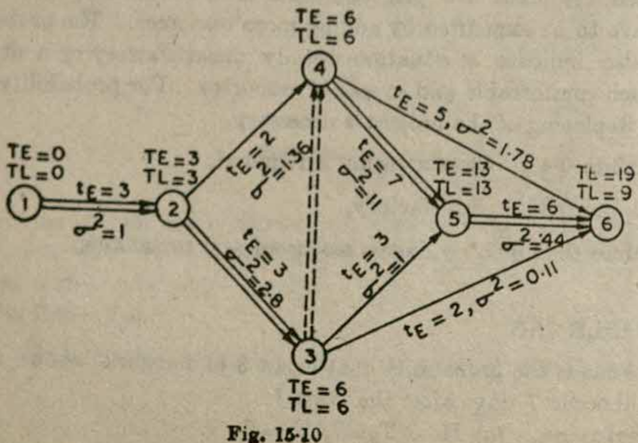


Fig. 15·10

and backward pass computations T_E and T_L values are determined for all the events. By applying the conditions of critical activities, it is determined that 1-2-3-4-5-6 is the critical path.

Expected time duration of the project, $T_E = 19$ days.

Contractual obligation time, $T_S = 20$ days.

Standard deviation for the project $\sigma = \sqrt{\Sigma \sigma^2_{ij}}$ for all ij on the critical path.

$\therefore \quad \sigma = \sqrt{1+2{\cdot}8+0+0{\cdot}11+0{\cdot}44} = 2{\cdot}08$

$\therefore \quad$ Normal deviate, $Z = \dfrac{T_S - T_E}{\sigma} = \dfrac{20-19}{2{\cdot}08} = \dfrac{1}{2{\cdot}08} = {\cdot}48.$

From the probability table, probability $= 68{\cdot}46\%$

15·8. Additional Examples

EXAMPLE 15·8·1

For the network shown in Fig. 15·11, the scheduled completion time is 32 days. Determine the slack times for the events and identify the critical path.

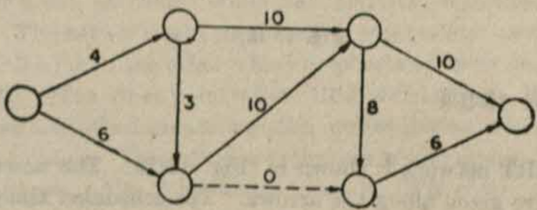


Fig. 15·11

EXAMPLE 15·8·2

For the data given below, find the following :

(*i*) The expected task times and their variances.

(*ii*) The earliest expected and latest allowable occurrence times of each event.

(*iii*) The critical path.

(*iv*) The probability that each task will be completed on schedule.

Task :	A	B	C	D	E	F	G	H	I	J	K	L
Least time (days) :	3	1	2	6	8	0	5	6	1	3	8	2
Most likely time (days) :	5	2	4	8	12	0	7	9	2	6	15	4
Greatest time (days) :	6	3	6	12	17	0	9	12	3	8	20	6

Precedence relationship :

A and B can start immediately;

C, D>A; E>B, C; F, H>E; G>D, F; J>G; I, K>H; L>J, I.

EXAMPLE 15·8·3

In the PERT network shown in Fig. 15·12, the activity time estimates (in weeks) are given along the arrows. If the scheduled completion time of the project is 23 weeks, calculate the latest possible occurrence times on the basis of the scheduled date of final event. Calculate the slack for each event and identify the critical path. What is the probability that the project will be completed on the scheduled date ?

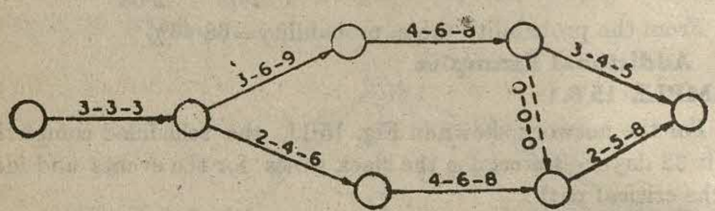


Fig. 15·12

EXAMPLE 15·8·4

A PERT network is shown in Fig. 15·13. The activity times (in hours) are given along the arrows. The scheduled times for some

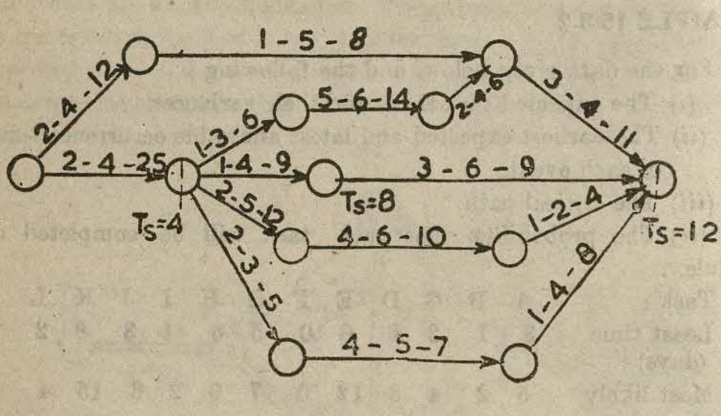


Fig. 15·13

important events are given along the nodes. Determine the critical path and the probabilities of meeting the scheduled dates for the specified events. Tabulate the results and determine the slacks.

16

CPM Computations

16·1. How CPM Differs from PERT ?

It has already been discussed that there are two basic elements of a network, the 'Event' and the 'Activity'. Event represents a significant point on time, while the activity is a time consuming element. The network technique which emphasises more on events is called PERT and the other which emphasises more on activities is called CPM. The other point where CPM differs from PERT is that CPM does not take into account the uncertainties while estimating its activity times. This is indeed the main difference between the two techniques. Also it is assumed that in CPM costs are related to time.

These differences between the two techniques are only of historical importance, and today PERT and CPM actually are used as one technique of project scheduling.

16·2. Some CPM Terms

There are some terms which are more clearly associated to activities than to events. While the earliest expected time, latest allowable occurrence time and slack corresponded to events, the terms like earliest start time, earliest completion time, the latest start time, the latest completion time and floats correspond to activities. We shall now define these terms.

16·2·1. Earliest Start Time. Earliest start time of an activity is the earliest occurrence time of the event from which the activity emanates. Earliest start time for activity ij will be denoted by $E_S{}^i$ and is the same as the earliest occurrence time of event i.

$$i.e., \quad E_S{}^i = T_E{}^i.$$

16·2·2. Latest Completion Time. This is the latest occurrence time of the event at which the activity terminates. Latest completion time for activity ij will be denoted by $L_C{}^j$, and

$$L_C{}^j = T_L{}^j.$$

16·2·3. Latest Start time. This is the latest completion time of the activity minus the activity duration. For activity ij, the latest start time,

$$\begin{aligned} L_S{}^{ij} &= L_C{}^j - t_E{}^{ij} \\ &= T_L{}^j - t_E{}^{ij}. \end{aligned}$$

16·2·4. Earliest Completion Time. This is the earliest start time of the activity plus the duration of activity. For activity ij, the earliest completion time,

$$\begin{aligned} E_C{}^{ij} &= E_S{}^i + t_E{}^{ij} \\ &= T_E{}^i + t_E{}^{ij}. \end{aligned}$$

To illustrate how these time elements are calculated, let us consider the network shown in Fig. 16·1. The activity durations are given along the arrows. The T_E and T_L values which are synonymous to $E_S{}^i$ and $L_C{}^j$ are given along the node circles. The calculations for these values, as explained under PERT computations, are carried in two passes, the forward pass and the backward pass.

Taking the start time of the first activity as zero,

$$E_S{}^1 = T_E{}^1 = 0.$$

Earliest start time of activities emanating from event 2 is the earliest occurrence time of event 2.

i.e., $$E_S{}^2 = T_E{}^2 \mp E_S{}^1 + t_E{}^{1-2} = 0 + 3 = 3.$$

Similarly,

$$\begin{aligned} E_S{}^3 &= E_S{}^2 + t_E{}^{2-3} \\ &= 3 + 3 = 6. \end{aligned}$$

Now, event 4 has two incoming arrows 2—4 and 3—4.

$$\begin{aligned} \therefore \quad E_S{}^4 = T_E{}^4 &= \text{Maximum } [(T_E{}^2 + t_E{}^{2-4}), (T_E{}^3 + t_E{}^{3-4})] \\ &= \text{Maximum } [(3+2), (6+0)] \\ &= 6. \end{aligned}$$

The process continues until the end event is reached.

Backward pass computations begin with the end event and proceed towards the initial event. In this pass the latest completion times are calculated.

From the network shown in figure 16·1,

$$T_E{}^6 = T_L{}^6 = 19,$$

or $$L_C{}^6 = E_S{}^6 = 19.$$

Since there is only one activity leading back into the event 5,

$$L_C^5 = T_L^5 = L_C^6 - t_E^{5-6}$$
$$= 19 - 6 = 13.$$

Then $\quad L_C^4 = T_L^4 = \text{Minimum}\,[(T_L^6 - t_E^{4-6}),\ (T_L^5 - t_E^{4-5})]$

$$= \text{Minimum}\,[(19-5),\ (13-7)]$$
$$= 6.$$

This process is continued till the start event is reached.

After computing the E_S and L_C times, the values of latest start (L_S^{ij}) time is obtained by subtracting the activity duration from the L_C^j values and the earliest completion time (E_C^{ij}) is obtained by adding the activity duration to E_S^i values.

i.e., $L_S^{ij} = L_C^j - t_E^{ij}$,

and $E_C^{ij} = E_S^i + t_E^{ij}$.

As the time elements involved are more, it is convenient to do these computations in a tabular form, as shown in table 16·1. The earliest start and completion times are calculated in the forward pass, which begins from the top of the table and proceeds downwards.

For activity 1-2, the earliest start time, $E_S^1 = 0$ and the earliest completion time, $E_C^{1-2} = E_S^1 + t_E^{1-2} = 0 + 3 = 3$.

For activities 2-3 and 2-4, the earliest start time is the earliest finish time of activity 1-2 which is 3. Thus

For activity 2-3, the earliest start time $E_S^2 = 3$;

$\therefore$ Earliest completion time $E_C^{2-3} = 3 + 3 = 6$.

For activity 3-4, $\quad E_S^3 = 6$;

$$\therefore \quad E_C^{3-5} = 6 + 0 = 6.$$

For activity 3-5, $\quad E_S^3 = 6$;

$$\therefore \quad E_C^{3-5} = 6 + 3 = 9.$$

For activity 3-6, $\quad E_S^3 = 6$;

$$\therefore \quad E_C^{3-6} = 6 + 2 = 8.$$

For activity 4—5, there are two predecessor activities, namely, 2—4 and 3—4, giving their completion times as $E_C^{2-4} = 5$ and $E_C^{3-4} = 6$ respectively. The maximum of the two is the earliest start time of the activities emerging from node 4.

Thus for activity 4-5, $\quad E_S^4 = 6$;

$$\therefore \quad E_C^{4-5} = 6 + 7 = 13,$$

and for activity 4-6, $E_s^4=6$;

$$\therefore \quad E_C^{4-6}=6+5=11.$$

The process is continued until the end event is reached.

During the backward pass, the total start and the latest completion times are calculated. The backward pass computations begin from the bottom of the table and proceed upwards.

For activity 5-6, $L_C^6=19$;

$$\therefore \quad L_S^{5-6}=L_C^6-t_E^{5-6}=19-6=13.$$

For activity 4-6, $L_C^6=19$;

$$\therefore \quad L_S^{4-6}=19-5=14.$$

For activity 4-5, the latest completion time is the latest start time for activity 5-6, and $L_C^5=13$;

$$L_S^{4-5}=13-7=6.$$

For computing the time elements for activities terminating into node 4, the latest completion time L_C^4 is the minimum of latest start times of activities 4-5 and 4-6 emanating from node 4.

$$\therefore \quad L_C^4=\text{Minimum } [L_S^{4-5}, L_S^{4-6}]$$

$$=\text{Minimum } [6, 13]=6.$$

Now for activity 2-4, $L_C^4=6$;

$$\therefore \quad E_S^{2-4}=6-2=4,$$

and for activity 3-4, $L_C^4=6$;

$$\therefore \quad E_S^{3-4}=6-3=3.$$

The process continues until the first activity is reached.

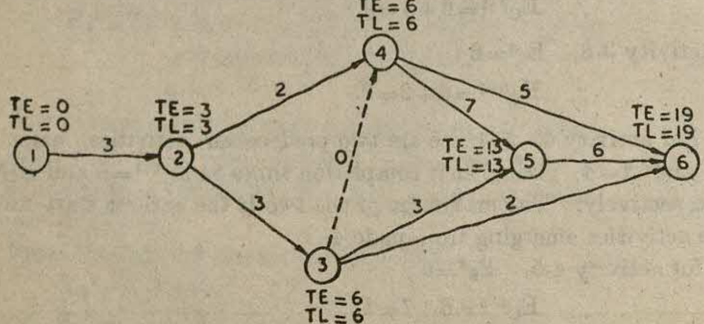


Fig. 16·1.

Table 16·1

Activity ij	Activity duration t_E^{ij}	Earliest: Start time E_S^i	Earliest: Completion time E_C^{ij}	Latest: Start time L_S^{ij}	Latest: Completion time L_C^j	Float: Total $L_S^{ij}-E_S^i$	Float: Free $E_S^j-E_C^{ij}$	Float: Independent $E_S^j-L_C^i-t_E^{ij}$
1	2	3	4	5	6	7	8	9
1—2	3	0	3	0	3	0	0	0
2—3	3	3	6	3	6	0	0	0
2—4	2	3	5	4	6	1	1	1
3—4	0	6	6	6	6	0	0	0
3—5	3	6	9	10	13	4	4	4
3—6	2	6	8	17	19	11	11	11
4—5	7	6	13	6	13	0	0	0
4—6	5	6	11	14	19	8	8	8
5—6	6	13	19	13	19	0	0	0

Total Float, $T_F = \text{Column } 6 - \text{Column } 5 - \text{Column } 2$
$= \text{Column } 5 - \text{Column } 3 = L_S^{ij} - E_S^i$.

Free float, $F_F = E_S^j - E_C^{ij}$.

16·3. Critical Path

With the knowledge of E_S and L_C times, which are the same as the T_E and T_L times, the critical path can be identified as discussed in PERT computations in section 15.5. For the network shown in figure 16·1, the critical path is 1-2-3-4 5-6.

16·4. Float

It has already been discussed that the float of an activity has the same significance as the slack of the events. Slack corresponds to events and hence to PERT, while the float corresponds to activities and hence to CPM. There are three types of floats : the total float, free float and the independent float.

Total Float. Total float is defined as the difference between the maximum time available to perform the activity and the activity duration time. The maximum time available for any activity is from the earliest start time to the latest completion time. Thus for an activity *ij*,

maximum time available $= L_C{}^j - E_S{}^i$.

If $t_E{}^{ij}$ is the duration of the activity,

then, total float. $T_F{}^{ij} = (L_C{}^j - E_S{}^i) - t_E{}^{ij}$

$$= (L_C{}^j - t_E{}^{ij}) - E_S{}^i$$

$$= L_S{}^{ij} - E_S{}^i$$

Thus the total float of an activity is the difference of its latest start time and its earliest start time. The activities which have zero float (total float) are called *critical activities* and the path of critical activities, the *critical path.*

Free Float. This is based on the assumption that all activities start as early as possible. In that case, the time available for an activity *ij* is equal to $E_S{}^j - E_S{}^i$ and free float,

$$F_F{}^{ij} = (E_S{}^j - E_S{}^i) - t_E{}^{ij}$$

$$= E_S{}^j - (E_S{}^i + t_E{}^{ij})$$

$$= E_S{}^j - E_S{}^{ij}.$$

Thus, the free float for an activity is the difference between the earliest start time for the successor activity and the earliest completion time for activity under consideration. For example, for activity 2-4, of network shown in Fig. 16·1, $E_S{}^j = E_S{}^4 = 6$,

and $$E_S{}^{ij} = E_S{}^{2\text{-}4} = 5.$$

$$\therefore \quad F_F{}^{2\text{-}4} = 6 - 5 = 1.$$

Also, as the earliest start time $E_S{}^j$ of an activity *jk* is the latest completion time $L_C{}^j$ of its predecessor activity *ij*, free float can be determined as

$$F_F{}^{ij} = L_C{}^j - E_C{}^{ij}$$

Independent Float. This is based on the assumption that the predecessor event occurs at its latest possible time and the successor event at its earliest possible time. In other words, the activity predecessor to the activity under consideration finishes at its latest possible time, while the successor activity starts at the earliest possible time. Then, the time available for the activity is $E_S{}^j - L_C{}^i$.

$\therefore$ Independent float, $I_F{}^{ij} = E_S{}^j - L_C{}^i - t_E{}^{ij}$.

For example, for activity 1-2 in fig. 16·1,

$$E_S{}^j = E_S{}^2 = 3,$$
$$L_C{}^i = L_C{}^1 = 0,\ t_E{}^{ij} = 3,$$
$$\therefore \quad I_F{}^{1\text{-}2} = 3 - 0 - 3 = 0.$$

For activity $3-5$, $E_S{}^5 = 13$,

$$L_C{}^3 = 6,$$
$$t_E{}^{3\text{-}5} = 3,$$
$$\therefore \quad I_F{}^{3\text{-}5} = 13 - 6 - 3 = 4.$$

From the above discussion we notice that if the activity is expanded to absorb the total float, the activity becomes critical. If the activity is expanded to absorb the free float, it does not affect the succeeding activity but affects the predecessor activity, as that must complete at the earliest possible time. The use of the independent float does not affect any of the predecessor or successor activities.

Also it is noticed that for an activity having zero total float, free and independent floats are also zero or may be negative. And for an activity having zero free float, the total float may or may not be zero, but the independent float is zero or negative. For the activity with zero independent float, free and total floats may or, may not be zeros. In the example considered above the total float free float and independent float are the same. This is mainly because all the nodes of the project network happen to be on the critica path.

The knowledge of floats helps the management in determining the flexibility of the schedule and the extent to which the resources will be utilised on different activities. This helps in diverting the resources from the non-critical activities to the critical activities which can result in shortening the project period and in the saving of costs. The utility of the floats will be clear when we discuss the cost analysis of the project.

EXAMPLE 16·1

Consider the network shown in figure 16·2. Determine the total, free and independent floats and identify the critical path.

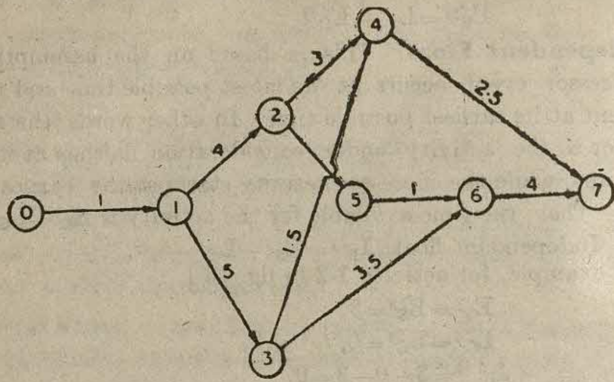


Fig. 16·2

Solution. Calculations are carried in the following steps :

Step 1.

The earliest start time $E_S{}^i$ or the earliest expected time T_E are calculated by the forward pass computations. The values are put along the nodes. At the same time the earliest completion times of activities ($E_C{}^{ij}$) are calculated. The values are entered in table 16·2.

Step 2.

The latest completion times of activities ($L_C{}^j$) or latest allowable occurrence times of events (T_L) are calculated by the backward pass. The values are put along the nodes on the network. At the same time latest start times of activities are calculated and the values are entered in table 16·2.

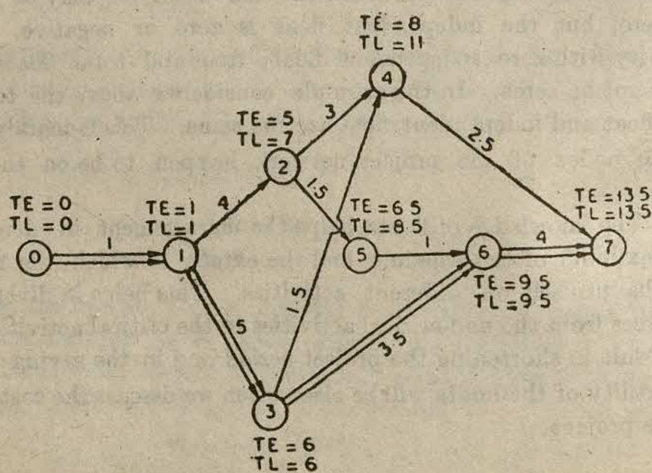


Fig. 16·3

Table 16·2

Activity ij	*Activity duration* t_E^{ij}	*Earliest* *Start time* E_S^{i}	*Earliest* *Completion time* E_C^{ij}	*Latest* *Start time* L_S^{ij}	*Latest* *Completion time* L_C^{j}	*Floats* *Total* $T_F = L_S^{ij} - E_S^{i}$	*Floats* *Free* $F_F = E_S^{j} - E_C^{ij}$	*Floats* *Independent* $I_F = E_S^{j} - L_C^{i} - t_E^{ij}$
1	2	3	4	5	6	7	8	9
0—1	1	0	1	0	1	0	0	0
1—2	4	1	5	3	7	2	0	0
1—3	5	1	6	1	6	0	0	0
2—4	3	5	8	8	11	3	0	−2
2—5	1·5	5	6·5	7	8·5	2	0	0
3—4	1·5	6	7·5	9·5	11	3·5	0·5	0·5
3—6	3·5	6	9 5	6	9·5	0	0	0
4—7	2·5	8	10·5	11	13·5	3	3	0
5—6	1	6·5	7·5	8·5	9·5	2	2	0
6—7	4	9·5	13·5	9·5	13·5	0	0	0

Step 3.

By applying the equation $T_F^{ij}=L_S^{ij}-E_S^i$, the values of total float are calculated. This is directly the difference of values in columns 5 and 3 of table 16-2.

Step 4.

Free float is calculated by applying the relation,

$$F_E^{ij}=E_S^j-E_C^{ij}.$$

There is no column for E_S^j. The values are seen in the column E_S^i against the appropriate row.

For example, for activity 1-2,

$E_S^j=5$. This value is obtained from column E_S^i against the activity 2-4 or 2-5.

Further, E_C^{ij} against the activity 1-2 is $=5$.

$\therefore \qquad F_F^{1\text{-}2}=0$

Step 5.

Independent float is calculated by applying the relation,

$$I_F^{ij}=E_S^j-L_C^i-t_E^{ij}.$$

For example, for activity 2-4, $E_S^j=E_S^4=8$;

$$L_C^i=L_C^2=7 \text{ and } t_E^{ij}=3$$

$$I_F^{2\text{-}4}=8-7-3=-2.$$

Step 6.

The activities with zero total float are identified. The path on which these activities fall is the critical path. In this example the critical path is $0-1-3-6-7$.

16·5. Negative Float and Negative Slack

The latest allowable occurrence time (T_L) for the end event in a CPM network is usually assumed to be equal to the earliest expected time (T_E) for that event. But in a PERT network, there is specified a date by which the project is expected to be complete. This is called the *scheduled completion time* T_S and for the backward pass computation, T_L for the end event is taken equal to T_S. Now there may be three cases : $T_S>T_E$, $T_S=T_E$ and $T_S<T_E$.

When $T_S>T_E$, a positive float results and the events have positive slacks.

When $T_S=T_E$, a zero float results and critical events have zero slacks.

When $T_S < T_E$, a negative float results and critical events have negative slacks.

So, when $T_S < T_E$, the critical activity will not have zero float. In such cases the critical path is the path of least float.

EXAMPLE 16·2

Consider PERT network given in fig. 16.4. Determine the float of each activity and identify the critical path if the scheduled completion time for the project is 20 weeks. Also identify the sub-critical path.

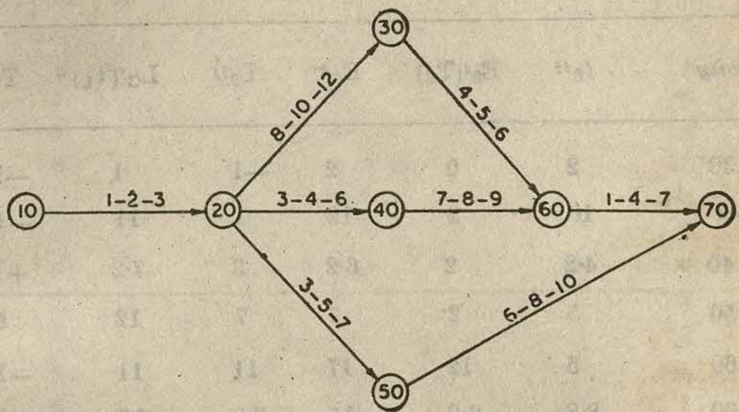


Fig. 16·4

Solution. After calculating the expected activity times, the earliest expected times of the events are calculated. For the end event $T_E=21$. The scheduled completion time is 20 weeks. Taking $T_L=T_S=20$ for the end event, the latest occurrence times of events are calculated by the backward pass.

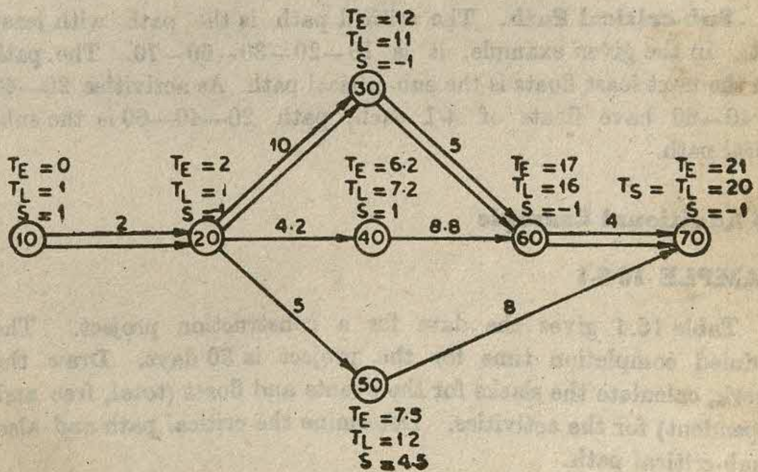


Fig. 16·5

The slacks for the events are also shown along the nodes. The most negative slack is -1 and it is for the events 10, 20, 30, 60 and 70. The path joining these events 10—20—20—60—70 is the critical path.

For determining the floats, the calculations have been done in the tabular form in table 16·3.

Table 16·3

Activity *ij*	t_E^{ij}	$E_S^i(T_E)$	E_C^{ij}	L_S^{ij}	$L_C^j(T_L)$	T_F
10—30	2	0	2	—1	1	—1
20—30	10	2	12	1	11	—1
20—40	4·2	2	6·2	3	7·2	+1
20—50	5	2	7	7	12	- 5
30—60	5	12	17	11	11	—1
40—60	8·8	6·2	15	7·2	16	+1
50—70	8	7	15	12	20	+5
60—70	4	17	21	16	20	—1

Sub-critical Path. The critical path is the path with least float. In the given example, it is 10—20—30—60—70. The path with the next least floats is the sub-critical path. As activities 20—40 and 40—60 have floats of +1 each, path 20—40—60 is the sub-critical path.

16·6 Additional Example

EXAMPLE 16·6·1

Table 16.4 gives the data for a construction project. The scheduled completion time for the project is 36 days. Draw the network, calculate the slacks for the events and floats (total, free and independent) for the activities. Determine the critical path and also the sub-critical path.

Table 16·4

Job No.	*Immediate predecessor*	*Time (days)*
A	—	0
B	A	4
C	B	2
D	C	4
E	D	6
F	C	1
G	F	2
H	F	3
I	D	2
J	D, G	4
K	I, J, H	10
L	K	3
M	I	1
N	I	2
O	I	3
P	E	2
Q	P	1
R	C	1
S	O, T	2
T	M, N	3
U	T	1
V	Q, R	2
W	V	5
X	S, U, W	0

17

Cost Analysis, Contracting and Updating

17·1. Project Cost

Each activity of the project consumes some resources and hence has cost associated with it. In most of the cases cost of an activity will vary to some extent with the amount of time consumed by the activity. The cost of total project, which is the aggregate of the activity costs will also depend upon the project duration time. Thus by increasing the costs, the project duration can be cut down to some extent. The aim is always to strike a balance between the costs and time, and to obtain an optimum project schedule. An optimum project schedule implies lowest possible cost and the minimum possible time for the project. The total cost of any project consists of the direct and indirect costs involved in its execution.

Direct Cost

The cost is directly dependent upon the amount of resources involved in the execution of the individual activities. This is the cost of the materials, equipment and labour required to perform the activity. If the activity is performed by a sub-contractor, the price of the sub-contract will be the activity direct cost. It can be seen from the direct cost-time relationship shown in fig. 17·1 that the activity can be varied by varying the cost. Point N, referred to as the '*Normal Point*' corresponds to the normal activity time. If the duration is further increased, the decrease in cost is insignificant. As the activity is compressed, the direct cost goes on increasing. If it is compressed beyond C, the cost increases very rapidly for insignificant change in the activity duration time. This point is called '*crash point*

Crash time is thus the minimum activity duration to which an activity can be compressed by increasing the resources and hence by increasing the direct cost.

To make the calculations simple, the direct cost-time curve is approximated to either a single straight line or multi-straight lines (segments), as shown in fig. 17·2(*a*) and 17·2(*b*) respectively. The slope of the line (or a segment of line) gives the increase in direct cost per unit time for expediting the activity. This is called cost slope.

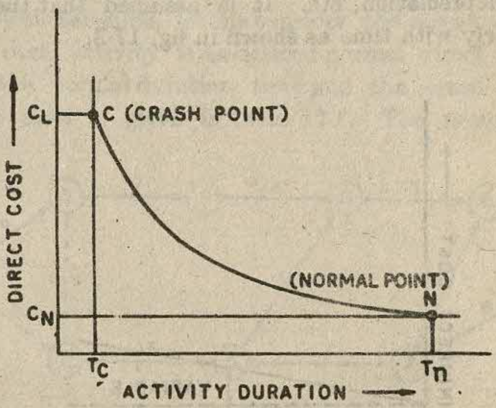


Fig. 17·1

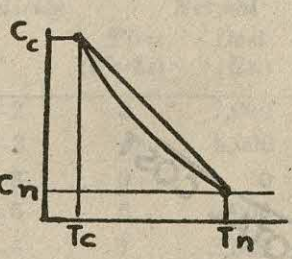


Fig. 17·2 (*a*)

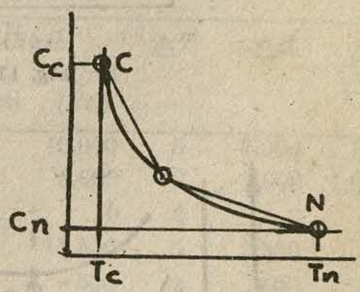


Fig. 17·2 (*b*)

For the straight line approximation in figure 17·2 (*a*),

$$\text{cost slope} = \frac{C_C - C_n}{T_n - T_C} = \frac{\Delta C}{\Delta T}.$$

The choice between the single straight line and segmented approximation depends upon the non-linearity of the cost curve. Also the segmented approximation is adopted only when the activity can be broken down into sub-activities. Above all, it is the judgement of the executive whether to approximate the curve by segmer and

go through involved calculations or to use a little rough single slope approximation and save calculation work.

Indirect Cost

Project indirect cost can further be sub-divided into two parts: fixed indirect cost and variable indirect cost. The fixed indirect cost is due to the general and administrative expenses, licence fee, insurance cost and taxes and does not depend upon the progress of the project. The variable indirect cost depends upon the time consumed by the project and consists of overhead expenditure, supervision, interest on capital and depreciation, etc. It is assumed that the indirect cost increases linearly with time as shown in fig. 17·3.

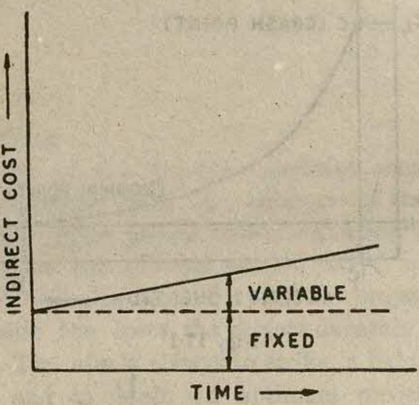


Fig. 17·3

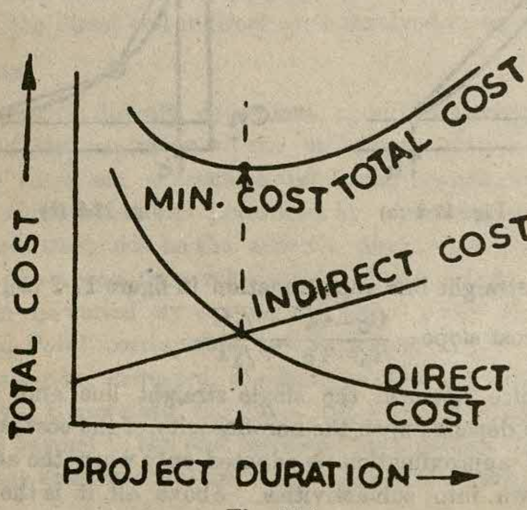


Fig. 17·4

The sum of the direct and indirect costs gives the *total project cost.* As the direct cost decreases with time and indirect cost increases with time, the total project cost curve will have a point where the total cost will be minimum (Figure 17·4). The time corresponding to this point is called the *optimum duration time* and the cost, the *optimum cost.*

17·2. Crashing the Network

EXAMPLE 17·2·1

To explain the process of crashing a network to reach the optimum project schedule, let us consider the network shown in fig. 17·5. With each activity is associated normal direct cost and crash direct cost, the normal duration time and the crash duration time. The complete data is given in table 17·1. The network has been

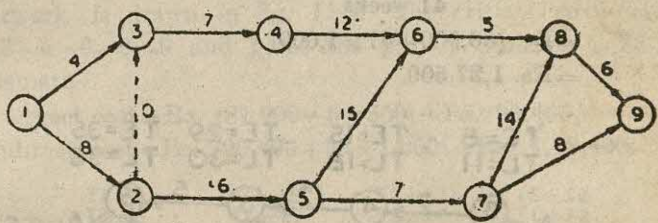


Fig. 17·5

Tabe 17·1

Activity	*Normal* Time (*Weeks*)	*Normal* Cost (*Rs.*)	*Crash* Time (*Weeks*)	*Crash* Cost (*Rs.*)	ΔT	ΔC	$\frac{\Delta C}{\Delta T}$
1—2	8	7,000	3	10,000	5	3,000	600
1—3	4	6,000	2	8,000	2	2,000	1,000
2—3	0	0	0	0	0	0	0
2—5	6	9,000	1	11,500	5	2,500	500
3—4	7	2,500	5	500	2	500	250
4—6	12	10,000	8	16,000	4	6,000	1,500
5—6	15	12,000	10	16,000	5	4,000	800
5—7	7	12,000	6	14,000	1	2,000	2,000
6—8	5	10,000	5	10,000	0	0	—
7—8	14	6,000	7	7,400	7	1,400	200
7—9	8	6,000	5	12,000	3	6,000	2,000
8—9	6	6,000	4	7,800	2	1,800	900
Total		86,500		1,15,700			

Indirect cost = Rs. 1,000 per week.

drawn for normal conditions and the times shown along the arrows are normal duration times. Assuming the straight line approximation for the activity cost curves, the cost slopes for the different activities are calculated.

$$\text{Cost slope} = \frac{\triangle C}{\triangle T} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}.$$

To determine the optimum project schedule, the first step is to identify the critical path. By carrying the forward pass and backward pass calculations, the earliest and latest times for each event are determined. These values are shown along the nodes in Fig. 17·6. Path 1—2—5—7—8—9 is the critical path and the project duration associated with it is 41 weeks. Total cost of the normal project is the sum of the direct and indirect costs.

i.e., total project cost = Direct cost of all activities + Indirect cost for 41 weeks

= Rs. (86,500 + 41 × 1,000)

= Rs. 1,27,500.

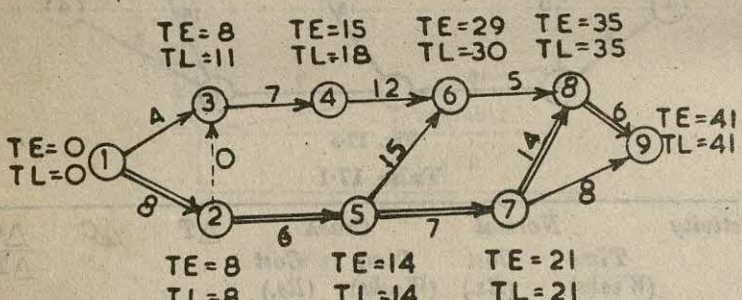


Fig. 17·6.

Next step is to identify those activities along the critical path which can be crashed. Crash the least expensive activity, *i.e.*, the activity with the least slope. Activity 7—8 with cost slope of 200 is the least expensive and can be crashed by seven days at an additional cost of Rs. 1,400 only. The crashed network is shown in Fig. 17·7. The new critical path is 1—2—5—6—8—9 which gives the project duration of 40 days. Thus by crashing activity 7—8 by seven days, the project duration has been reduced by one day. For this critical path,

direct project cost = Rs. (86,500 + 7 × 200)

= Rs. 87,900,

indirect cost = Rs. 40 × 1,000 = Rs. 40,000.

∴ Total project cost = Rs. 1,27,900.

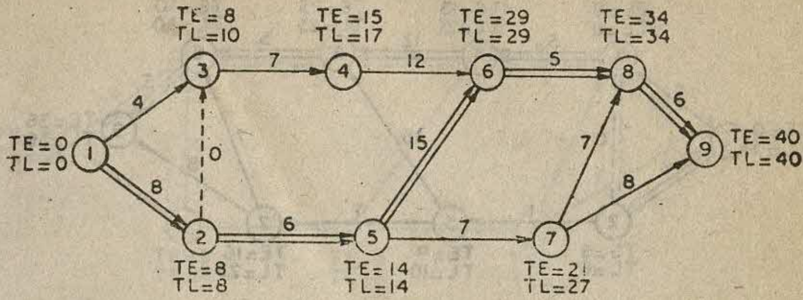


Fig. 17·7

In the new critical path, the activity with the lowest cost slope is activity 2—5 with cost slope of 500. It is crashed by 5 weeks and the network is drawn in fig. 17·8. The critical path shifts to 1—2—3—4—6—8—9 and gives the project duration as 38 weeks. For this path

direct cost = Rs. $(87{,}900+5\times500)$ = Rs. 90,400,

indirect cost = Rs. $(90{,}400+38\times1{,}000)$ = Rs. 1,28,400.

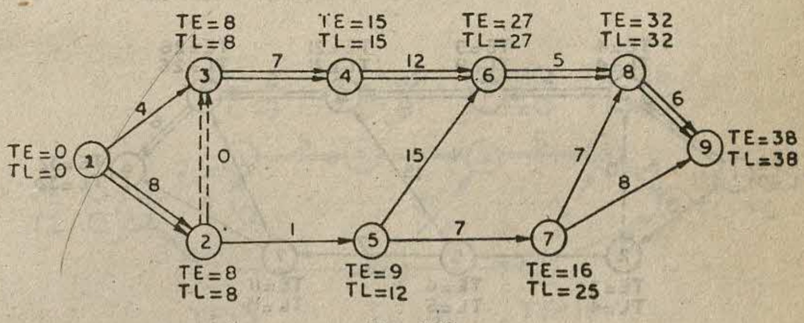


Fig 17·8

The least expensive activity on the critical path of figure 17·8 is 3—4 with cost slope of 250. This activity can be crashed by two weeks. After crashing, the new network will appear as shown in figure 17·9 with critical path as 1—2—3—4—6—8—9 and project duration of 36 weeks. The associated costs are

direct cost = Rs. $(90{,}400+2\times250)$ = Rs. 90,900,

indirect cost = Rs. (36×1.000) = Rs. 36,000,

and total project cost = Rs. 1,26,900.

In the new critical path the least expensive activity is 1—2 with a cost slope ot 600 and can be crashed by 5 weeks. This

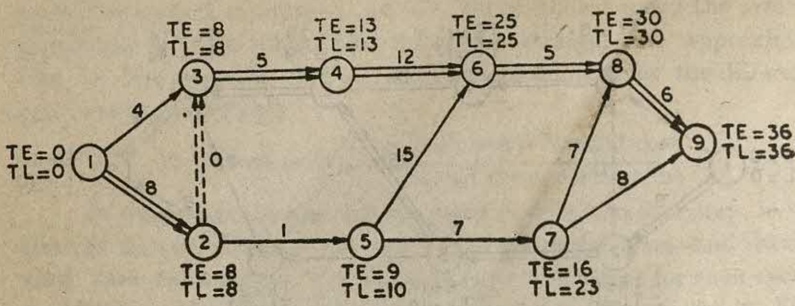


Fig. 17·9

changes the critical path to 1—3—4—6—8—9 as shown in figure 17·10. For this path,

project completion time $=32$ weeks,

direct cost $=$ Rs. $(90{,}900+5\times 600)=$ Rs. 93,900,

indirect cost $=$ Rs. $(32\times 1{,}000)=$ Rs. 32,000,

and total cost $=$ Rs. $(93{,}900+32{,}000)$

$=$ Rs. 1,25,900.

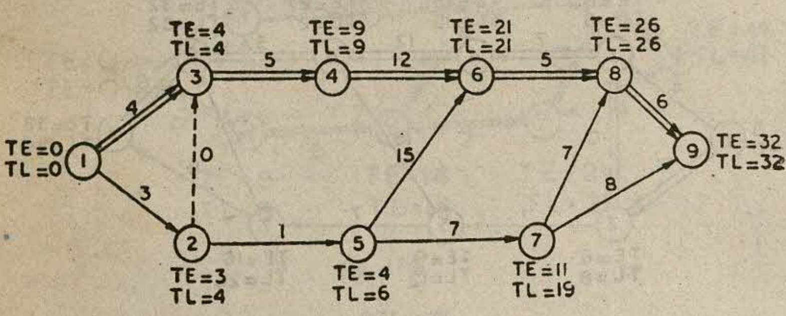


Fig. 17·10

The next activity to be crashed is 8—9. It can be crashed by two weeks at a cost of Rs. (2×900). The critical path remains unchanged but the project duration reduces to 30 weeks. This is shown in fig. 17·11, for which,

direct cost $=$ Rs. $(93{,}900+2\times 900)$

$=$ Rs. 95,700

indirect cost $=$ Rs. $(30\times 1{,}000)=$ Rs. 30,000,

and total cost $=$ Rs. 1,25,700.

Out of the remaining uncrashed activities on the critical path 1—3—4—6—8—9, activity 1—3 with cost slope of 1,000 is the least expensive and this can be crashed by 2 weeks. With this the critical

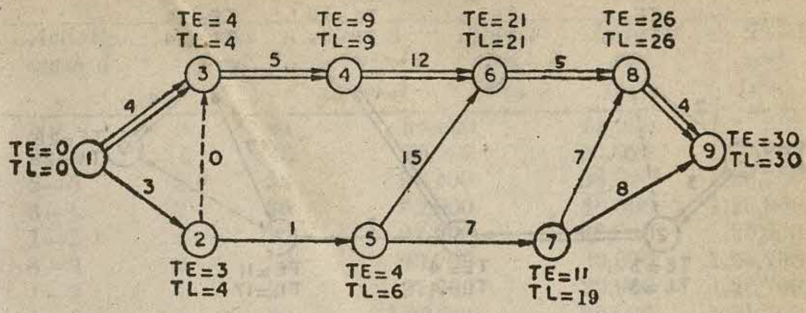


Fig. 17·11

path shifts to 1—2—3—4—6—8—9 and the project duration time comes down to 29 weeks (fig. 17·12). The costs associated with this path are

direct cost = Rs. (95,700 + 2 × 1,000)

= Rs. 97,700,

indirect cost = Rs. 29,000,

and total cost = Rs. 1,26,700.

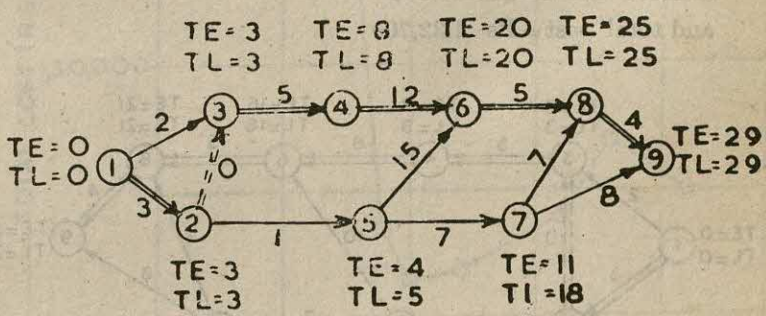


Fig. 17·12

In this critical path of fig. 17·12, there is only one activity 4—6 which is uncrashed. This can be crashed by 4 weeks at a cost of Rs. (4 × 1,500) = Rs. 6,000 only. The network appears as shown in fig. 17·13, with the critical path as 1—2—5—6—8—9 and with the project completion time of 28 weeks. The associated costs are

direct cost = Rs. (97,700 + 6,000)

= Rs. 1,03,700,

indirect cost = Rs. 28,000,

and total cost = Rs. 1,31,700.

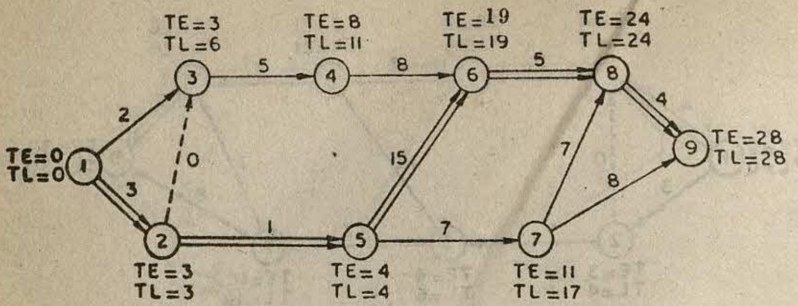


Fig 17.13.

In the above critical path of figure 17·13, activity 5—6 is uncrashed and can be crashed by 5 weeks. Cost slope of this activity is 800. The critical path with this crashing shifts back to 1—2—3—4—6—8—9 with the project duration of 25 days (figure 17·14). For this path,

direct cost =Rs. (1,03,700+5×800)

=Rs. 1,07,700,

indirect cost =Rs. 25,000

and total cost=Rs· 1,32,700

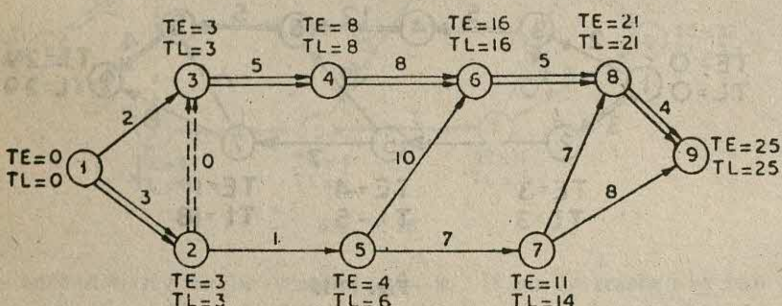


Fig. 17·14

As all the activities on the critical path, which could be crashed have been crashed, no further crashing of the project network is possible. The crashing of non-critical activities does not alter the project duration time and hence is of no use. Now to determine the optimum project duration, the total project cost is plotted against the duration time. The summary of calculations carried out is given in table 17·2 and the total project cost curve is shown in fig. 17·15.

Table 17·2

Activity crashed	*Weeks saved*	*Project duration (Weeks)*	*Direct cost (Rs.)*	*Indirect cost (Rs.)*	*Total cost (Rs.)*
Nil	0	41	86,500	41,000	1,27,500
7—8	1	40	87,900	40,000	1,27,900
2—5	2	38	94,400	38,000	1,28,400
3—4	2	36	90,900	36,000	1,26,900
1—2	4	32	93,900	32,000	1,25,900
8—9	2	30	95,700	30,000	1,25,700
1—3	1	29	97,700	29,000	1,26,700
4—6	1	28	1,03,700	28,000	1,31,700
5—6	3	25	1,07,700	25,000	1,32,700

From this curve in fig. 17·15 (or directly from table 17·2) we find that the minimum cost corresponding to the project duration of 30 weeks is Rs. 1,25,700.

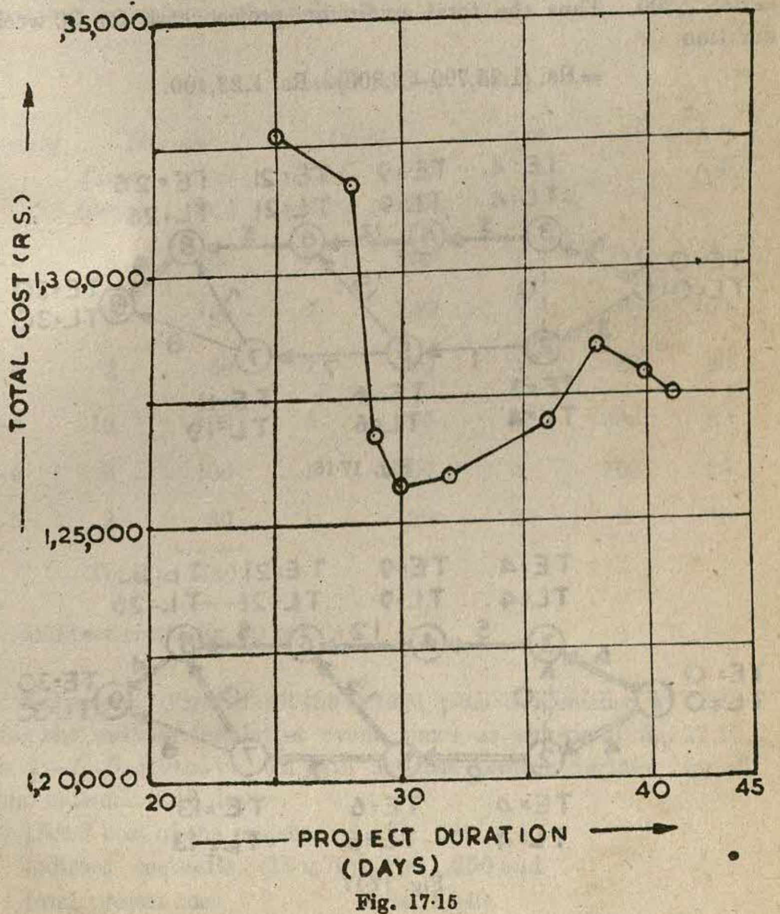
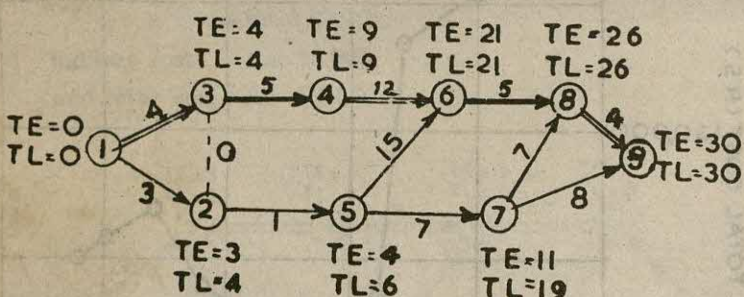


Fig. 17·15

Next, we have to see if the project cost can be reduced further without affecting the project duration time. This can be done by *expanding* or *un-crashing* the activities which do not lie on the critical path. Looking at the network of fig. 17·16, we find that activities 1—2, 2—5, and 7—8 can be uncrashed. Uncrashing should start with the activity having the maximum cost slope. An activity is to be expanded only to the extent that it itself may become critical, but should not affect the original critical path. Thus expanding of activity 1—2 by one day makes 1—2 critical without affecting the critical path. Activity 2—5 can also be expanded by one day only. This makes the path 1—2—5—6 also critical. Next activity 7—8 is expanded by six days. This makes two more activities 5—7 and 7—8 critical. The final network is shown in figure 17·17. But for activity 7—9, the whole network has become critical. Saving in cost is due to direct cost only and is Rs. $(1\times600+1\times500+6\times200)$ =Rs. 2,300. Thus the total minimum project cost for 30 weeks duration

$$=\text{Rs. }(1{,}25{,}700-2{,}300)=\text{Rs. }1{,}23{,}400.$$



Fig, 17·16

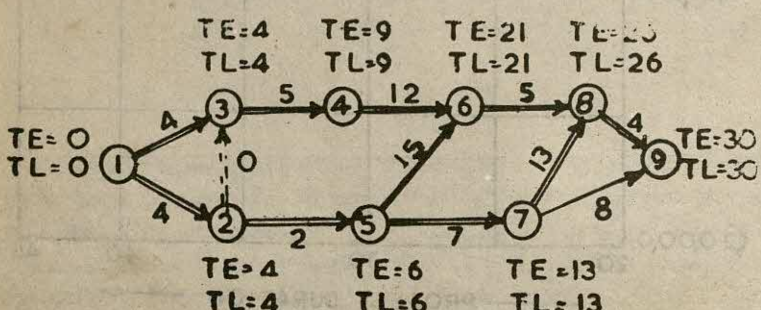


Fig. 17·17

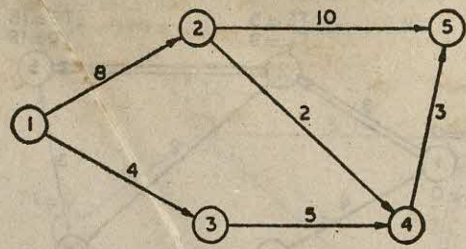


Fig. 17·18

EXAMPLE 17.2-2

Consider the network shown in fig. 17·18 for which the utility data is given in table 17·3. Crash systematically the activities and determine the optimum project duration and cost.

Table 17·3

Activity	*Normal*		*Crash*		$\triangle T$	$\triangle C$	$\frac{\triangle C}{\triangle T}$
	Time (days)	*Cost (Rs.)*	*Time (days)*	*Cost (Rs.)*			
1—2	8	100	6	200	2	100	50
1—3	4	150	2	350	2	100	100
2—4	2	50	1	90	1	40	40
2—5	10	100	5	400	5	300	60
3—4	5	100	1	200	4	100	25
4—5	3	80	1	100	2	20	10
	Total :	580					

Indirect cost = Rs. 70 per day.

Solution. First of all the critical path is identified by determining the earliest and latest event times as shown in fig. 17·19. Path 1—2—5 is the critical path and the project duration for all normal network is 18 days.

Direct cost of the project = Rs. 580,
indirect cost = Rs. (18×70) = Rs. 1,260 and
total project cost = Rs. 1,840.

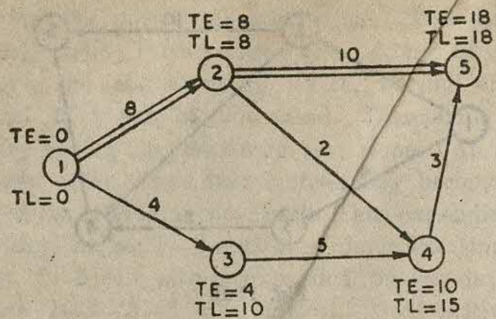


Fig. 17·19

In this example instead of crashing all the activities, we will crash only those activities whose cost slope is less than the indirect cost. In case an activity with cost slope greater than indirect cost is crashed, the saving in indirect cost will be less than the cost of crashing.

Along the critical path 1—2—5, both activities 1—2 and 2—5 have cost slopes less than the indirect cost. Crashing is started with the activity having the least slope. Therefore activity 1—2 is crashed by two days. The new network is shown in fig. 17·20. The project duration is reduced to 16 days while the critical path remains unchanged.

Direct cost = Rs. $(580+2\times 50)$ = Rs. 680, and
total project cost = Rs. $(680+16\times 70)$ = Rs. 1,800.

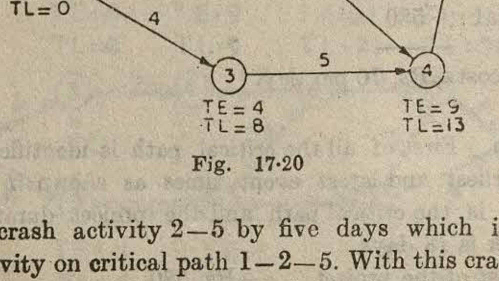


Fig. 17·20

Next crash activity 2—5 by five days which is the only uncrashed activity on critical path 1—2—5. With this crashing the critical path shifts to 1—3—4—5 as shown in fig. 17·21, for which project duration = 12 days.

Direct cost = Rs. $(680+5\times 60)$ = Rs. 980,

indirect cost = Rs. (12×70) = Rs. 840, and
total cost = Rs. $(980 + 840)$ = Rs. 1,820.

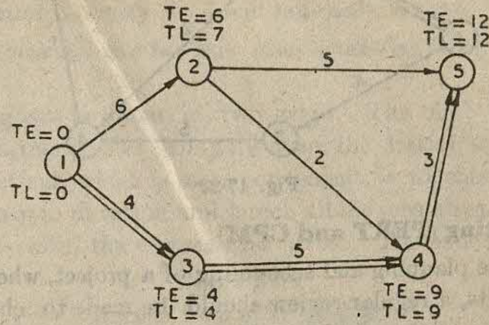


Fig. 17·21

Activities on the new critical path with slope less than Rs. 70 are 3—4 and 0—5 with slopes of 25 and 10 and can be crashed by 4 and 2 days respectively. Crash activity 4—5 by two days. Critical path changes to 1—2—5 as shown in figure 17·22. For this path,

direct cost = Rs. $(980 + 2 \times 10)$
= Rs. 1,000,
total cost = Rs. $(1{,}000 + 11 \times 70)$
= Rs. 1,770.

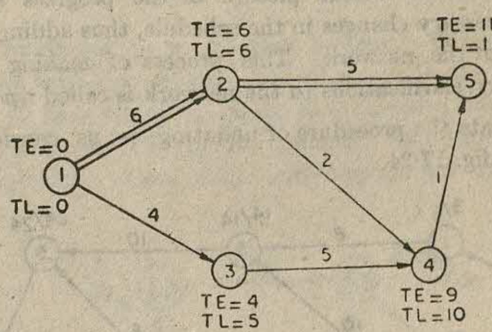


Fig. 17·22

The project cannot be further contracted as the activities on the critical path are already crashed.

The activity 4—5 can be expanded by one day without affecting the project duration. This will save the crashing cost of 4—5 for one day. Thus the final network obtained is as shown in fig. 17·23, for which,

total project cost = Rs. $(1{,}770 - 10)$ = Rs. 1,760,

and the optimum project duration = 11 days.

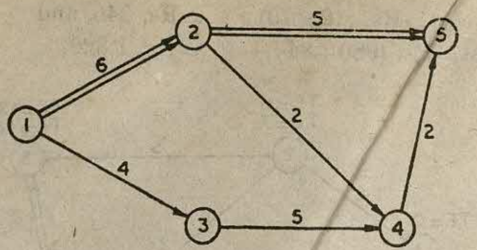


Fig. 17·23

17·3. Updating (PERT and CPM)

After the planning and scheduling of a project, when the actual execution starts, a regular review should be made to check the continuous validity of the schedule. In the actual execution it generally happens that the time schedule developed for the project is not exactly followed. Some of the jobs take more time than estimated and some others are completed in period lesser than estimated. There may be a number of reasons for this, such as the non-availability of the resources, breakdown of machinery, labour strikes, wrong estimations of the planner and natural calamities, etc. All these will delay the jobs. On the other hand, jobs may be expedited due to the commissioning of a new machine, development of a better process and wrong estimations of the planner, etc. The review of the situation presents a clear picture of the progress and helps in making the necessary changes in the schedule, thus adding dynamism to the nature of the network. This process of making review and adding necessary clarifications to the network is called *updating*.

To illustrate the procedure of updating, let us consider the network shown in fig. 17·24.

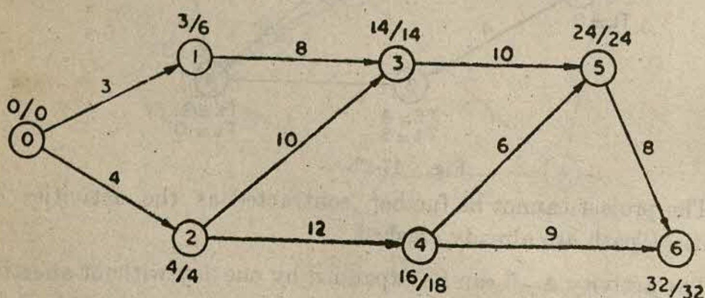


Fig. 17·24

Suppose the progress is checked at the end of 10th day and it is observed that

(*i*) activities 0—1, 0—2 and 1—3 are completed,

(*ii*) activity 2—3 is in progress and will take 6 days more,

(*iii*) activity 2—4 is in progress and will take 7 days more,

(*iv*) also it is estimated that due to the arrival of a new machine, activity 3—5 will take only 6 days.

This information can be put into a tabular form, as shown in table 17·4.

Updating can be done in two ways. The first is to use the revised time estimates and compute from the initial starting event. The second method, which is more convenient, is to change the completed work to zero duration and bunch all the jobs already performed into one arrow called the *elapsed time arrow*. The nodes in the new network are numbered in a different fashion. The time duration assigned to the activities are the revised times. In the new network shown in fig. 17·25, activity 0—20 shows the elapsed time of 10 days. Activities 20—30 and 20—40 are assigned the times they need for their completion. Along other activities are put their new time estimates.

Table 17·4

Activity	*Review time after 10 days More than required (days)*	*Situation*
0—1	9	Completed
0—2	0	Completed
1—3	0	Completed
2—3	6	In progress
2—4	7	In progress
3—5	6	Not started
4—5	6	Not started
4—6	9	Not started
5—6	8	Not started

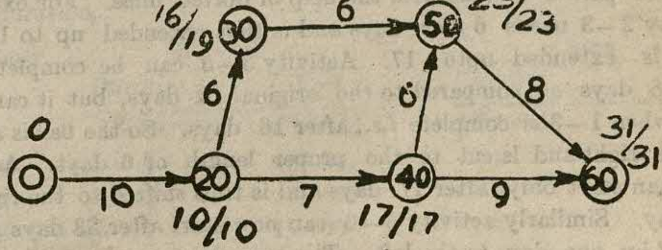


Fig. 17·25

After computing the earliest expected times and the latest allowable times we find that the critical path has changed to

20—40—50—60. The total duration has also come down by one day.

The bar charts also used in the process of updating as shown in fig. 17·26 represent the information regarding the original schedule.

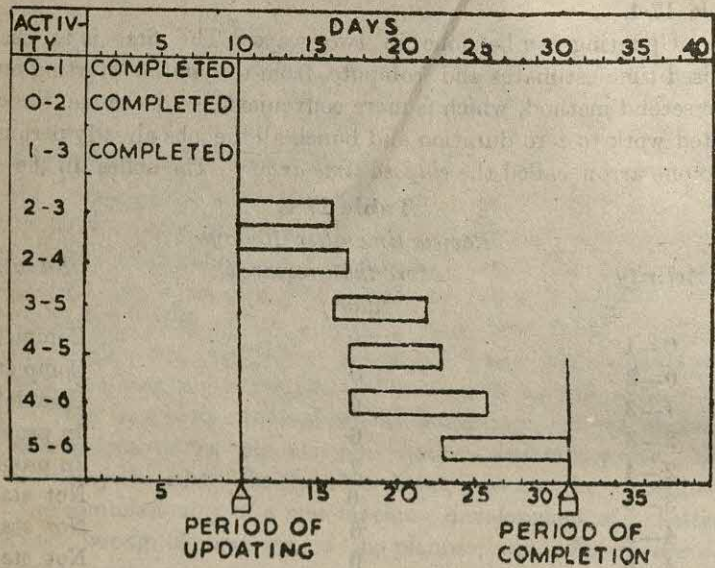


Fig. 17·26

The status of the project at the end of 10 days is shown by shading the bars as shown in the figure. The updating line shows that activities 1—3, 2—3, and 2—4 are in progress, but from the review of the project it is observed that 1—3 has already been completed. So, this is shaded through the total length. The changes in the lengths of bar to indicate the increase or decrease in activity duration period is done with the help of dotted lines. For example, activity 2—3 needs 6 more days and is thus extended up to 16, and 2—4 is extended upto 17. Activity 3—5 can be completed in only 6 days as compared to the original 10 days, but it can start only when 1—3 is complete *i.e.*, after 16 days. So the bar is shifted to the right and is cut to the proper length of 6 days. Activity 4—5 can start only after 17 days and is thus shifted to the right by one day. Similarly activity 5—6 can now start after 23 days and is shifted by one day to the left. The bar chart corresponding to the new network will look as shown in Fig. 17·27.

How often the updating should be done ? There is no special rule to decide about the frequency of updating. This depends upon

the nature and the size of the project and upon the attitude of the management. However, a general observation can be made that frequency of updating may be less at the initial stages but should be more frequent near the completion of the project. Some slippages at the beginning can be absorbed, but a slip near completion will delay the project. In small projects, as the time for absorbing the slippages is less, more frequent updating is called for.

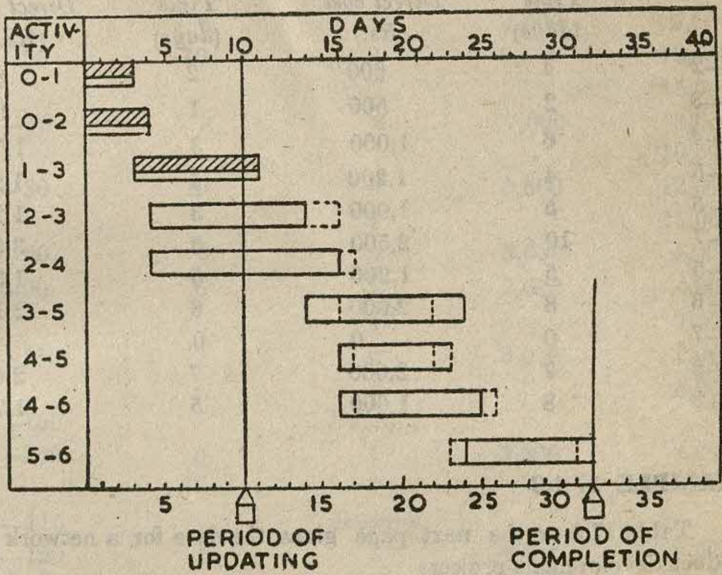


Fig. 17·27

17·4. Additional Examples

EXAMPLE 17·4·1

The utility data for a network is given below. Crash the network to minimum project duration and determine the project cost for that duration.

Activity	*Normal*		*Crash*	
	Duration	*Cost*	*Duration*	*Cost*
	(Weeks)	*(Rs.)*	*(Weeks)*	*(Rs.)*
0—1	1	5,000	1	5,000
1—2	3	5,000	2	12,000
1—3	7	11.000	4	17,000
2—3	5	10,000	3	12,000
2—4	8	8.500	6	12,500
3—4	4	8,500	2	16,500
4—5	1	5,000	1	5,000

EXAMPLE 17·4·2

The utility data for a network is given below. The activity durations are in days and the cost in rupees. The indirect cost per day is Rs. 250. Determine the optimum project schedule.

Activity	*Normal*		*Crash*	
	Time (days)	*Direct cost (Rs.)*	*Time (days)*	*Direct cost (Rs.)*
1—2	4	600	2	800
1—3	2	500	1	900
2—4	6	1,000	3	1,750
2—5	4	1,200	4	1,200
3—5	5	1,000	3	1,200
3—7	10	2,500	5	3,500
4—5	5	1,300	5	1,300
5—6	8	2,000	6	2,100
5—7	0	0	0	0
6—8	7	2,000	7	2,000
7—8	8	1,600	5	1,780

EXAMPLE 17·4·3

Table 17·5 on the next page gives the data for a network of a product development project.

The marketing department realises that if the product is introduced in the market ahead of its completion date, additional profits are available. What would be the additional cost to crash the project by 6 weeks ?

EXAMPLE 17·4·4

For the network shown in fig. 17·28 a review of the project after 15 days reveals the following situation :

(*a*) Activities 1—2, 1—3, 2—3, 2—4 and 3—4 are completed.

(*b*) Activities 3—5 and 4—6 are in progress and need 2 and 4 days more respectively.

(*c*) The revised estimate shows that activity 8—9 will take only 8 days, but activity 7—9 will need 10 days.

Formulate a new network after updating and determine the new critical path. Show the progress of the project on a bar chart and indicate the modification based on updating.

Table 17·5

Activity	*Time*		*Cost*	
	Normal (Weeks)	*Crash (Weeks)*	*Normal (Rs.)*	*Crash (Rs.)*
10—20	3	2	2,000	4,500
10—30	6	5	2,000	5,000
		4		9,000
20—40	6	5	4,000	9,000
20—50	10	8	9,500	12,000
		7		15,000
30—70	9	8	7,000	8,000
		7		10,500
30—80	10	8	9,500	12,000
		7		15,000
40—60	6	5	3,500	4,000
40—90	6	4	2,000	4,000
		3		7,000
60—110	7	6	3,000	4,000
		5		8,000
70—100		Dummy		
80—100	5	4	3,500	4,000
		2		8,000
90—110		Dummy		
100—120	6	3	6,000	9,900
110—120	5	4	2,000	4,000
120—130	2	1	2,500	4,000

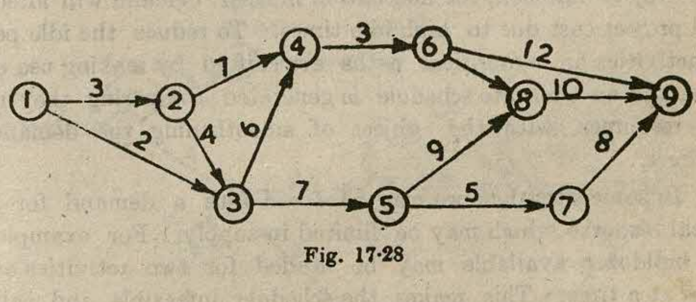


Fig. 17·28

18

Resource Scheduling and General Remarks

18·1. Resource Scheduling

During the development of PERT and CPM networks we have generally assumed that sufficient resources are available to perform the various activities. At a certain time the demand on a particular resource is the cumulative demand of that resource on all the activities being performed at that time. Going according to the developed plan, the demand on a certain type of resource may fluctuate from very high at one time to a very low at another. If it is a material or unskilled labour which has to be procured from time to time, the fluctuation in demand will not much affect the cost of the project. But if it is some personnel who cannot be hired and fired during the project or machines which are to be hired for the total project duration, the fluctuation in their demand will affect the total project cost due to high idle times. To reduce the idle period, the activities on non-critical paths are shifted by making use of the floats, and an alternate schedule is generated comparing the important resources, with the object of smoothening the demand on resources.

In some situations we may be faced with a demand for some critical resource which may be limited in supply. For example, the only bulldozer available may be needed for two activities at two places at a time. This makes the schedule infeasible and calls for a re-examination with the object of generating an alternate plan with feasible scheduling of the limited resource.

Thus the object of resource scheduling is two-fold : it aims at

bringing down the costs and at the same time reduces pressure on the limited resources in conflicting demands.

Depending upon the type of constraint, the resource scheduling situation may be of two types :

(*a*) The constraint may be the total project duration. In this case the resource scheduling only smoothens the demand on resources in order that the demand of any resource is as uniform as possible. In this case the resource scheduling is called *'Resource Smoothening'* or *'Load Smoothening'*.

(*b*) The second type of constraint may be on the availability of certain resources. Here the project duration is not treated as an invariant, but the demand on certain specified resources should not go beyond the specified level. This operation of resource allocation is called *'Resource Levelling'* or *'Load Levelling'*.

18·2. Load Smoothening

The first step in resource smoothening is to determine the maximum requirement. One way is to draw the time scaled version of the network and assign resource requirements to activities. Then, below the time scaled network, the cumulative resource requirements for each time unit are plotted. The result is a *'Load Histogram'*. The load histogram which is also known as *'force curve'* may be plotted on the basis of early start times or the late start times of the activities. These load histograms establish the framework within which smoothening or levelling must occur.

Let us illustrate the load smoothening operation by considering the network shown in figure 18·1. To keep the example simple, only one category of resources, that is, the crew size has been taken. The manpower required for each activity is as given in table 18·1.

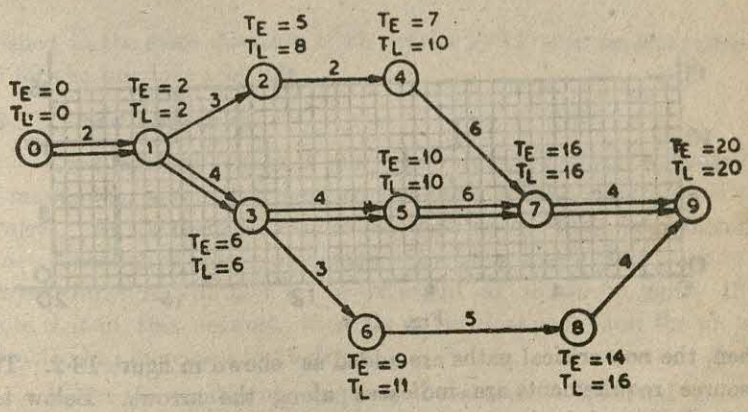


Fig. 18·1

The earliest and latest occurrence times of events have been calculated and are indicated along the nodes. The critical path is identified as 0-1-3-5-7-9, with the total project duration of 20 weeks. In the time scaled version of the network which is also called *squared network*, first of all the critical path is drawn along a straight line.

Table 18·1

Activity	*Crew size (Men)*
0—1	4
1—2	3
1—3	3
2—4	5
3—5	3
3—6	4
4—7	3
5—7	6
6—8	2
7—9	2
8—9	9

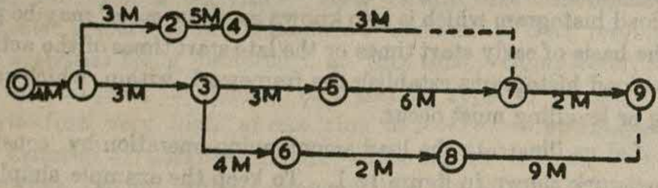


Fig. 18·2

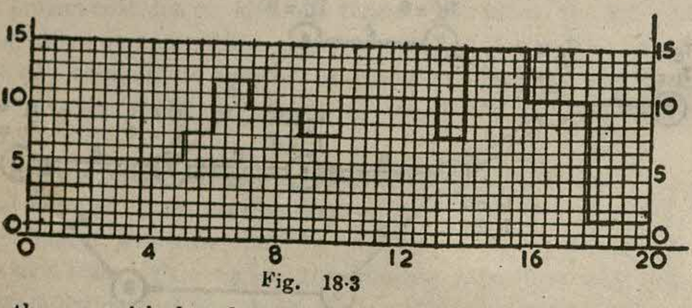


Fig. 18·3

Then, the non-critical paths are added as shown in figure 18·2. The resource requirements are indicated along the arrows. Below the squared network is shown the load histogram (figure 18·3). This is based on the earliest start times, and is obtained by vertically

summing up the manpower requirements for each week. We observe that the maximum demand of 15 men occurs in 15th and 16th weeks. To smoothen the load the activities will have to be shifted depending upon the floats. Path 3—6—8—9 has a float of two weeks, and the activities 6—8 and 8—9 are shifted to the right, so that the start of each is delayed by two weeks. Similarly, activity 4—7 can be shifted to the right so that it starts on 10th day instead of starting on 7th day. After making the necessary shifting, the network is drawn as shown in figure 18·4. The load histogram for this network drawn in figure 18·5 indicates that the maximum manpower required is 11 men. Thus with the new schedule, the same project can be accom-

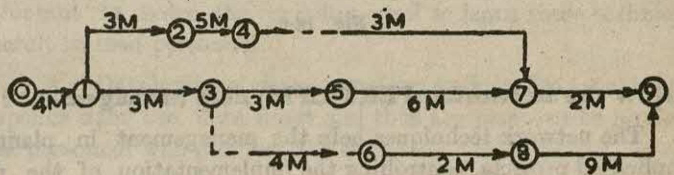


Fig. 18·4

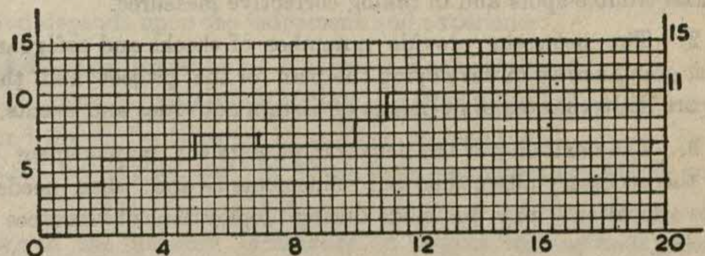


Fig. 18·5

plished in the same duration of 20 weeks by 11 men as compared to 15 for the previous schedule.

18·3. Load Levelling

Load levelling is done if the constraint is on the availability of manpower, say, only 9 men are available for the execution of the project. As the demand cannot be reduced to 9 by smoothening, a new scheduling situation occurs. To bring down the peak to 9, project duration will have to be extended as shown in figure 18·6. Note that in this network there is no critical path and the project duration has increased to 25 days.

When the number of categories of resources considered is more than one, a compromise will have to be made to the minimum level of each resource in the resource smoothening.

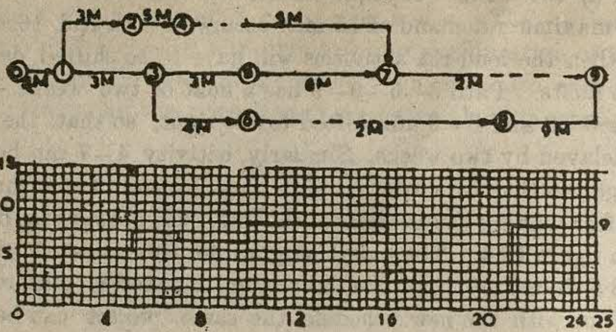


Fig. 18·6

18·4. How the networks (PERT/CPM) help Management ?

1. The network techniques help the management in planning the complicated projects, controlling the implementation of the plan and keeping the plan uptodate. They also help in locating the potential trouble spots and in taking corrective measures.

2. The networks provide a number of checks and safeguards against going astray in developing the plan for the project and thus there are little chances of oversight of certain activities and events.

3. The flexibility of the networks permits the management to make the necessary alterations and refinements as and when needed. These allocations may be made during deployment of resources or reviewing.

4. The networks clearly designate the responsibilities of different supervisors. Supervisor of an activity knows his time schedule precisely and also the supervisors of other activities with whom he has to co-ordinate.

5. These techniques help the management in achieving the objective with minimum of time and least cost and also in predicting the probable project duration and the associated cost.

6. Applications of PERT and CPM have resulted in saving of time, which directly results in saving of cost. Saving in time or early completion of the project results in earlier return of revenue and introduction of the product or process ahead of the competitors, resulting in increased profits.

7. Application of network techniques has resulted in better managerial control, improved utilisation of resources, improved communication and progress reporting and better decision making.

18·5. Difficulties in using network methods

Following are some of the problems faced in the managerial use of network methods :

1. Difficulty in securing the realistic time estimates. In the case of new and non-repetitive type projects, the time estimates produced are often mere guesses.

2. The natural tendency to oppose changes results in the difficulty of persuading the management to accept these techniques.

3. The planning and implementation of networks require personnel trained in the network methodology. Managements are reluctant to spare the existing staff to learn these techniques or to recruit trained personnel.

4. Developing a clear logical network is also troublesome. This depends upon the data input and thus the plan can be no better than the personnel who provides the data.

5. Determination of the level of network detail is another troublesome area. The level of detail varies from planner to planner and depends upon the judgement and experience.

18·6. Applications of Network Techniques

The list containing PERT and CPM applications is very large and the applications are expanding to many new areas. Following are a few typical areas in which these techniques are widely accepted:

1. *Construction Industry* : It is one of the largest areas in which the network techniques of project management have found application.

2. *Manufacturing* : The design, development, and testing of new machines, installing machines and plant layouts are a few examples of how it can be applied to the manufacturing function of a firm.

3. *Maintenance planning* : Shutdown and Maintenance of power plants, chemical plants, steel furnaces and overhauling of large machines have been carried out by using PERT.

4. *Research and Development* : R and D has been the most extensive area where PERT has been used for development of new products, processes and systems.

5. *Administration* : Networks have been used by the administration for streamlining paperwork system, for making major administrative system revisions, for long range planning and developing staffing plans, etc.

6. *Marketing* : Networks have been used for advertising programmes, for development and launching of new products and for planning their distribution.

7. *Inventory planning* : Installation of production and inventory control, acquisition of spare parts, etc., have been greatly helped by network techniques.

18·7. PERT and CPM – Cost and Scope

One of the obstacles in the acceptance of network techniques by the managements is the difficulty of precisely determining the cost of applying these techniques. It is always essential to know as to what will these techniques cost and whether it will be worthwhile to use them. Of course, every project is not PERT-able and all the PERT-able projects many not warrant its use. The cost of applying PERT and CPM techniques includes the salaries and other benefits paid to the PERT/CPM team, computer charges and some overhead expenses. The cost differs from project to project and from firm to firm. One firm quotes that a project of 400 events needs 96 man-hours for the network preparation, while another quotes that it needs 5 men for 3 weeks. A few firms regard the costs too high, while others consider it moderate or even quite low. The general observation is that the cost is not a major deterrent to the use of network techniques. This is due to the reason that there are direct and indirect cost savings which are much higher than the extra cost of using networks as compared to the other planning methods.

Managements have started recognising the deficiencies of other planning methods and the planning and operational values of network techniques have become fairly well known. The techniques are finding application in many new areas. These are being extensively used in public and private agencies like the Fertilizer Corporation, Steel Corporation, State Industrial Development Corporations and Electric Generation and Supply Corporations, etc. Many corporations require their contractors to use the techniques in submitting their bids and also in the executing of the works. The knowledge of PERT and CPM is considered essential for a large categories of jobs connected with the engineering and administrative works.

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APPENDICES

APPENDIX A

Review of Vectors and Matrices

A-1. VECTORS

A-1-1. Definition

A vector is an ordered set of real numbers. Let $a_1, a_2, \ldots, a_n$ be any n real numbers called *elements* or *components*, then

$$\mathbf{a}=(a_1, a_2, \ldots, a_n),$$

where $\mathbf{a}$ is called an n-vector (or simply a vector). Generally, a *row* vector is written as $(a_1, a_2, \ldots, a_n)$, while a *column* vector is written as

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

The vector $\mathbf{0}=(0, 0, \ldots 0)$ is called the *null* vector.

A-1-2. Addition (Subtraction) of Vectors

Let $\quad \mathbf{a}=(a_1, a_2, \ldots, a_n)$ and $\mathbf{b}=(b_1, b_2, \ldots, b_n)$ be two vectors in the n-dimensional space. Then

$$\left.\begin{aligned} \mathbf{a}+\mathbf{b}=\mathbf{c}=(a_1+b_1, a_2+b_2, \ldots, a_n+b_n), \\ \mathbf{a}-\mathbf{b}=\mathbf{d}=(a_1-b_1, a_2-b_2, \ldots, a_n-b_n), \end{aligned}\right\} \text{(cummutative law)}$$

$$(\mathbf{a}+\mathbf{b})+\mathbf{e}=\mathbf{a}+(\mathbf{b}+\mathbf{e}), \qquad \text{(associative law)}$$

$$\mathbf{a}+(-\mathbf{a})=\mathbf{0}. \qquad \text{(zero or null vector)}$$

A-1-3. Multiplication of Vectors by Scalers

Given an n-vector $\mathbf{a}$ and a scalar (constant) quantity α, the new vector

$$\begin{aligned} f=\alpha\mathbf{a} &= \alpha(a_1, a_2, \ldots, a_n) \\ &= (\alpha a_1, \alpha a_2, \ldots, \alpha a_n) \end{aligned}$$

is called the *scalar product* of $\mathbf{a}$ and α.

In general, given the n-vectors $\mathbf{a}$ and $\mathbf{b}$ and the scalers α and β,

$$\alpha(\mathbf{a}+\mathbf{b})=\alpha\,\mathbf{a}+\alpha\mathbf{b}, \quad \text{(distributive law)}$$

$$\alpha(\beta\mathbf{a})=(\alpha\beta)\,\mathbf{a}. \quad \text{(associative law)}$$

A.1-4. Inner Product of Vectors

The inner product of two n-vectors $\mathbf{a}$ and $\mathbf{b}$ is the scalar (number) written as $\mathbf{a}\cdot\mathbf{b}$ or simply $\mathbf{ab}$ and is given by

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

For example, if $\mathbf{a}=(2, 3, 4)$ and

$$\mathbf{b}=\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \text{ then}$$

$$\mathbf{a}\cdot\mathbf{b}=2+15+24=41.$$

A.1-5. Linearly Dependent and Independent Vectors

A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, is *linearly dependent* if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i \mathbf{a}_i = 0$$

or

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_n \mathbf{a}_n = 0.$$

In this case, at least one vector can be written as a *linear combination* of the others. For example,

$$\mathbf{a}_1 = k_2\, \mathbf{a}_2 + k_3\, \mathbf{a}_3 + \ldots + k_n\, \mathbf{a}_n.$$

If

$$\sum_{i=1}^{n} \alpha_i \mathbf{a}_i = 0$$

only if all $\alpha_i=0$, then the given set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is said to be *linearly independent*.

For example, the vectors

$$\mathbf{a}_1=(2, 3, 4) \text{ and } \mathbf{a}_2=(6, 9, 12)$$

are linearly dependent since there exist $\alpha_1=3$ and $\alpha_2=-1$ for which

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 = 0.$$

If a set of vectors is linearly independent, any subset of it is also linearly independent. For a linearly dependent set of vectors, its super set is also linearly dependent.

A·1·6 Equality (Inequality) of Vectors

Let $\mathbf{a}=(a_1, a_2, \ldots, a_n)$ and $\mathbf{b}=(b_1, b_2, \ldots, b_n)$ be the two n-vectors. Then

$$\mathbf{a}=\mathbf{b} \quad \text{if } a_i=b_i, \text{ for all } i=1, 2, \ldots, n,$$

$\mathbf{a} > \mathbf{b}$ if $a_i > b_i$, for all $i = 1, 2, \ldots, n$,

$\mathbf{a} < \mathbf{b}$ if $a_i < b_i$, for all $i = 1, 2, \ldots, n$,

$\mathbf{a} \geqslant 0$ implies that $a_i \geqslant 0$, for all $i = 1, 2, \ldots, n$ *i.e.*, each component of vector **a** is non-negative.

A·1-7. Identity (Unit) Vector

An identity vector $\mathbf{e}_i$ is a vector with its ith component unity and all other components zero. For example,

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),$$
$$\mathbf{e}_2 = (0, 1, 0, \ldots, 0),$$
$$\mathbf{e}_i = (0, 0, \ldots, 1, \ldots, 0),$$
$$\mathbf{e}_n = (0, 0, \ldots, 1).$$

A·1-8 Euclidean Space

Euclidean n-space, also called *vector space* is the set (collection) of all n-vectors and is denoted by **v**.

A·1-9. Basis Set

A set of vectors is said to *span* a vector space **v** if every vector of **v** can be expressed as a linear combination of the vectors in the set. A *basis set* for the vector space **v** is a set of linearly independent vectors that spans **v**.

A-2. Matrices

A·2-1. Definition

A *matrix* A of size $(m \times n)$ is a rectangular away (table) of numbers arranged in m rows and n columns. Thus

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{m1} & a_{m2} \quad a_{mn} \end{bmatrix}$$

The (i, j) th element of **A**, denoted by a_{ij}, is the element in the ith row and jth column of **A**. A matrix of size or order $(m \times n)$ is written as

$$\mathbf{A}_{(m \times n)} = [\, a_{ij} \,]_{m \times n} \text{ or simply } [a_{ij}].$$

The elements a_{ij} for $i = j$ are called *diagonal* elements, while the a_{ij} for $i \neq j$ are called *off-diagonal* elements.

A 2-2. Types of Matrices

Square matrix : is a matrix in which $m = n$.

Diagonal matrix : is a square matrix in which $a_{ij} = 0$ for $i \neq j$ *i.e.*, in which off-diagonal elements are all zero.

Identity (Unit) matrix : is a square matrix in which diagonal

elements are all unity and off-diagonal elements are all zero; that is,

$$a_{ij}=1, \text{ for } i=j,$$
$$a_{ij}=0, \text{ for } i\neq j.$$

An identity matrix of order n is denoted by $\mathbf{I}_n$ or simply $\mathbf{I}$. For example, for $n=3$, we have

$$\mathbf{I}_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Row vector : Elements of each row of a matrix define a vector called a *row vector*. Thus a row vector is a matrix with one row and n columns ($1\times n$ matrix).

Column vector : Elements of each column of a matrix define a vector called a *column vector*. Thus a column vector is a matrix with m rows and one column ($m\times 1$ matrix).

Null matrix : is a matrix whose elements are all zero.

Transpose of matrix : The transpose of matrix $\mathbf{A}=[a_{ij}]$, denoted by $\mathbf{A}'$ or $\mathbf{A}^T$ is a matrix obtained by interchanging the rows and columns of $\mathbf{A}$. In other words, the element a_{ij} in $\mathbf{A}$ is equal to the element a_{ji} in $\mathbf{A}^T$ for all i and j.

For example, if

$$\mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

then

$$\mathbf{A}^T=\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Thus if $\mathbf{A}$ is of order $(m\times n)$, $\mathbf{A}^T$ will be of order $(n\times m)$.

Symmetric matrix : The matrix $\mathbf{A}$ is said to be symmetric if $\mathbf{A}^T=\mathbf{A}$ *i.e.*, if $a_{ij}=a_{ji}$ for all i and j.

Equal matrices : Two matrices $\mathbf{A}=[a_{ij}]$ and $\mathbf{B}=[b_{ij}]$ are said to be *equal* matrices if they have the same order and each $a_{ij}=b_{ij}$ for all i and j.

Orthogonal matrix : A square matrix $\mathbf{A}$ is said to be *orthogonal*

if

$$AA^T = AA^T = I.$$

A·2·3 Addition (Subtraction) of Matrices

The *sum* or *difference* of two matrices **A** and **B** of the same order is a matrix **C** (written as $C = \mathbf{A} \pm \mathbf{B}$) where the elements of C are given by

$$c_{ij} = a_{ij} + b_{ij}.$$

The following are the examples of matrix addition and subtraction :

Addition

$$\begin{array}{ccccc} \text{Matrix } \mathbf{A} & & +\text{Matrix } \mathbf{B} & & =\text{Matrix } \mathbf{C} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & + & \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} & = & \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & -6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -7 \\ 2 & 9 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 8 & -10 \\ 3 & 3 & 1 \end{bmatrix}$$

Subtraction

$$\begin{array}{ccccc} \text{Matrix } \mathbf{A} & & -\text{Matrix } \mathbf{B} & & =\text{Matrix } \mathbf{C} \\ \begin{bmatrix} 6 & 5 & 7 \\ 2 & 1 & 8 \end{bmatrix} & - & \begin{bmatrix} 2 & 5 & 9 \\ 1 & 4 & 6 \end{bmatrix} & = & \begin{bmatrix} 4 & 0 & -2 \\ 1 & -3 & 2 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & -4 & 10 \\ 2 & -3 & -7 \end{bmatrix} - \begin{bmatrix} -1 & 5 & 9 \\ -1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -9 & 1 \\ 3 & -1 & -11 \end{bmatrix}$$

If matrices **A**, **B** and **C** are of the same order, then

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}, \qquad \text{(cummutative law)}$$

$$\mathbf{A}\pm(\mathbf{B}\pm\mathbf{C})=(\mathbf{A}\pm\mathbf{B})\pm\mathbf{C}, \qquad \text{(associative law)}$$

$$\mathbf{A}+(-\mathbf{A})=0=(-\mathbf{A})+\mathbf{A},$$

$$\alpha\,(\mathbf{A}\pm\mathbf{B})=\alpha\mathbf{A}\pm\alpha\mathbf{B},$$

$$\alpha\,(\beta\,\mathbf{A})=\beta\,(\alpha\,\mathbf{A})=(\alpha\,\beta)\,\mathbf{A}.$$

A.2·4 Product of Matrices

For two matrices **A** and **B**, the product **AB** is defined if and only if the number of columns of **A** is equal to the number of rows of **B**. If **A** is $(m \times r)$ matrix and **B** is $(r \times n)$ matrix, then $\mathbf{C} = \mathbf{AB}$ is defined and is of size $(m \times n)$. The (i, j)th element of **C** is given by

$$c_{ij} = \sum_{k=1}^{r} a_{ik}\, b_{kj}, \text{ for all } i \text{ and } j.$$

EXAMPLE A·2·4·1.

$$\text{Let } \mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix},$$

$$\text{then} \quad \mathbf{C}=\mathbf{AB}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

$$=\begin{bmatrix} (1\times5+2\times8) & (1\times6+3\times9) & (1\times7+2\times20) \\ (3\times5+4\times8) & (3\times6+4\times9) & (3\times7+4\times10) \end{bmatrix}$$

$$=\begin{bmatrix} 21 & 33 & 47 \\ 47 & 54 & 61 \end{bmatrix}.$$

EXAMPLE A·2·4·2

$$\text{Let } \mathbf{A}=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \text{ and } \mathbf{X}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$\text{then} \quad \mathbf{AX}=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$=\begin{bmatrix} 1\times1+3\times2+5\times3 \\ 2\times1+4\times2+6\times3 \end{bmatrix}$$

$$=\begin{bmatrix} 22 \\ 28 \end{bmatrix}.$$

EXAMPLE A·2.4.3

$$\text{Let } \mathbf{A}=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } \mathbf{Y}=\begin{bmatrix} 2, 3 \end{bmatrix},$$

$$\text{then} \quad \mathbf{YA}=\begin{bmatrix} 2, 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$=\begin{bmatrix} (2\times1+3\times4) & (2\times2+3\times5) & (2\times3+3\times6) \end{bmatrix}$$

$$=\begin{bmatrix} 14 & 19 & 24 \end{bmatrix}.$$

Matrix multiplication satisfies the following properties :

$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC}),$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC},$$
$$\mathbf{C}(\mathbf{A}+\mathbf{B})=\mathbf{CA}+\mathbf{CB},$$
$$\mathbf{IA}=\mathbf{AI}=\mathbf{A},$$
$$\alpha(\mathbf{AB})=(\alpha\mathbf{A})\mathbf{B}=\mathbf{A}(\alpha\mathbf{B}),\ \alpha \text{ is a scalar,}$$

$\mathbf{AB}\neq\mathbf{BA}$ **in general.**

If AB is defined, BA is not necessarily defined.

If **AB**$=0$, it is not necessary that either **A** or **B** must be zero.

A·2·5. Determinant of a Square Matrix

A *determinant* of a square matrix **A**, denoted by $|\mathbf{A}|$, is a number obtained by performing certain operations of the elements of **A**.

If **A** is (2×2) matrix, then

$$|\mathbf{A}|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}.$$

If **A** is (3×3) matrix, then

$$|\mathbf{A}|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}.$$

If **A** is an $(n\times n)$ matrix, then

$$|\mathbf{A}|=\sum_{i=1}^{n} a_{i1}(-1)^{i+1}\,|\mathbf{M}_{i1}|,$$

where $|\mathbf{M}|$ is a submatrix obtained by deleting row i and column 1 of $|\mathbf{A}|$. For example,

$$\text{if } \mathbf{A}=\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix},$$

then $$|\mathbf{A}|=1\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix}-4\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix}+7\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$$

$$=(45-48)-4(18-24)+7(12-15)$$

$$=-3+24-21$$

$$=0.$$

A matrix **A** is said to be *singular* if $|\mathbf{A}|=0$. If $|\mathbf{A}|\neq 0$, the matrix **A** is called *nonsingular*.

Some of the major properties of determinants, are given below.

1. If every element of a column or a row is zero, then the value of the determinant is zero.

i.e., $$|\mathbf{0}|=0.$$

2. The value of the determinant does not change if its rows and columns are interchanged.

i.e., $$|A^T| = |A|.$$

3. If $|B|$ is obtained by interchanging any two rows (columns) of $|A|$, then

$$|B| = -|A|.$$

4. If two rows (columns) of matrix **A** are identical, then

$$|A| = 0.$$

5. If **B** is obtained from **A** by adding to its ith row (column), scalar α times its jth row (column), then

$$|B| = |A|.$$

6. If every element of a row (column) of a determinant is multiplied by a scalar α, the value of determinant is multiplied by α.
7. If **A** and **B** are two square matrices of the same order, then

$$|AB| = |A|\ |B|.$$

A·2·6. Inverse of a Matrix

If **A** is non-singular square-matrix and if there exists a non-singular square matrix **B** such that

$$AB = I = BA,$$

then **B** is called an *inverse* of **A** and is denoted by A^{-1}.

The inverse matrix A^{-1} can be obtained by performing the following two row operations on the original matrix **A**:

1. Multiply or divide any row by a number.
2. Multiply any row by a number and add it to another row.

EXAMPLE A·2·6·1

If $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, find A^{-1}.

Solution. Since $|A| = -1-2 = -3$, **A** is non-singular and hence A^{-1} exists. To find A^{-1} we start with the following matrix:

$$AI = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right].$$

Subtract row 1 from row 2 :

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 1 \end{array}\right]$$

Divide row 2 by -3 :

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

Subtract twice of row 2 from row 1:

$$\left[\begin{array}{cc:cc} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right].$$

Thus $\mathbf{A}^{-1}=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$

The concept of matrix inversion can be used in solving n linearly independent equations. Consider a system of n linear independent equations in n unknowns :

$$\begin{array}{l} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1, \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2, \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=b_n. \end{array}$$

This system of equations can be written as

$$\mathbf{AX}=\mathbf{b},$$

where $\quad \mathbf{A}=\begin{bmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots a_{2n} \\ \vdots & \vdots \quad\quad\quad \vdots \\ a_{n1} & a_{n2}\ldots\ldots a_{nn} \end{bmatrix},$

$$\mathbf{X}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and $\quad \mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$

Since the equations are independent, **A** is non-singular. Thus

$$\mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{b}$$

or $$\mathbf{X}=\mathbf{A}^{-1}\mathbf{b},$$

gives the solution of n unknowns.

APPENDIX B

Review of Finite Differences

B.1 Definition

The calculus of *finite differences* deals with the variations in the value of a function $y=f(x)$ as a result of *finite integral* variations in the value of x. Here x is called independent variable while y is known as dependent variable. If the consecutive values of the independent variable x at an interval of h are

$$a,\ a+h,\ a+2h,\ \ldots$$

then the corresponding values of the dependent variable y are given by

$$f(a),\ f(a+h),\ f(a+2h),\ \ldots\ .$$

The various values of variable x are called *arguments* while the corresponding values of variable y are known as *entries*. The difference between consecutive values of x is known as *interval of difference*; while the difference between the consecutive values of $y=f(x)$ is called *difference*. Unless otherwise stated, the interval of difference (h) is taken as unity.

B.2 Finite Difference Operator

The various differences in the values of dependent variable $y=f(x)$ can be obtained with the help of *difference operators*. The *first difference* of $f(x)$, written as $\triangle f(x)$ is given by the equation

$$\triangle f(x)=f(x+1)-f(x). \qquad [\because\ h=1]$$

Second order difference of $f(x)$, denoted by $\triangle^2 f(x)$ is given by

$$\begin{aligned}\triangle^2 f(x)&=\triangle[\triangle f(x)]\\ &=\triangle[f(x+1)-f(x)]\\ &=\triangle f(x+1)-\triangle f(x)\\ &=f(x+2)-f(x+1)-[f(x+1)-f(x)]\\ &=f(x+2)-2f(x+1)+f(x).\end{aligned}$$

In general, $(n+1)$*th order difference* of $f(x$ is defined as

$$\triangle^{n+1}f(x)=\triangle[\triangle^n f(x)],\ n=0,\ 1,\ 2,\ \ldots\ .$$

It may be easily verified that

$$\begin{aligned}\triangle^{n+1}f(x)&=\triangle^n[\triangle f(x)]\\ &=\triangle^n[f(x+1)-f(x)]\\ &=\triangle^n f(x+1)-\triangle^n f(x),\ n\ (\text{integral})\geqslant 0,\end{aligned}$$

if $\triangle^0=$ identity operator $[\triangle^0 f(x)=f(x)]$

and $\triangle'=\triangle$.

The difference operator $\triangle$ satisfies the following properties :

1. $\triangle[f(x)+g(x)]=\triangle f(x)+\triangle g(x)$.

2. $\quad \triangle[\alpha f(x)]=\alpha \triangle f(x)$.

3. $\quad \triangle[f(x)g(x)]=f(x)\triangle g(x)+g(x+1)\triangle f(x)$

$$=f(x+1)\triangle g(x)+g(x)\triangle f(x)$$

$$=f(x)\triangle g(x)+g(x)\triangle f(x)+\triangle f(x)\triangle g(x).$$

4. $\quad \triangle\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\triangle f(x)-f(x)\triangle g(x)}{g(x+1)g(x)}$.

5. $\quad \triangle\left[\dfrac{1}{f(x)}\right]=-\dfrac{\triangle f(x)}{f(x+1)f(x)}$.

6. $\quad \triangle x^{(n)}=nx^{(n-1)}$,

where $x^{(n)}$ is called the factorial function of degree n and is defined as

$$x^{(n)}=x(x-1)(x-2)\ldots(x-\overline{n-1})$$

$$=x(x-1)(x-2)\ldots(x-n+1).$$

7. $\quad \triangle(x)=(x+1)-x=1$.

8. $\quad \triangle[{}^xC_r]={}^xC_{r-1}$, r constant and ${}^xC_r=\dfrac{x\,!}{r\,!\,(x-r)\,!}$.

9. $\quad \triangle[a^x]=a^x(a-1)$, a constant.

B.3 Applications of Finite Differences

It will require a full volume to cover the various applications of finite differences. Here we give only those applications which are relevant to the subject of operations research.

B.3.1 Conditions for Minima of a Function

The function $f(x)$ is said to have a *local minimum* at $x=x_0$ if the following two conditions are satisfied :

$$f(x_0+1)-f(x_0)>0, \quad \text{i.e.,}\ \triangle f(x_0)>0,$$

$$f(x_0)-f(x_0-1)<0, \quad \text{i.e.,}\ \triangle f(x_0-1)<0.$$

Thus the condition for $f(x)$ to have local minimum at $x=x_0$ is

$$\triangle f(x_0-1)<0<\triangle f(x_0).$$

The function $f(x)$ is said to have an *absolute minimum* at $x=x_0$ if

$$f(x_0)\leqslant f(x) \text{ for all } x.$$

Thus *sufficient* conditions for $f(x)$ to have an absolute minimum at $x=x_0$ are

$$\triangle f(x_0-1)<0<\triangle f(x_0),$$

and

$$\triangle^2 f(x)\geqslant 0 \text{ for all } x.$$

Likewise, the sufficient conditions for $f(x)$ to have an absolute maximum at $x=x_0$ are

$$\triangle f(x_0-1)>0>\triangle f(x_0)$$

and $\qquad \triangle^2 f(x) \leqslant 0$ for all x.

B.3.2. Summation of Series

We know that if $f(x)$ is a continuous function defined in the interval $a \leqslant x \leqslant b$, then from the knowledge of integral calculus

$$\int_a^b f(x)dx = \Big[F(x) \Big]_a^b$$
$$= F(b) - F(a),$$

where $F(x)$ is called antiderivative of $f(x)$. If, however, function $f(x)$ is defined only for integral values of x within the interval $a \leqslant x \leqslant b$, then it can be shown that

$$\sum_{x=a}^{b} f(x) = \Big[F(x) \Big]_a^{b+1} = F(b+1) - F(a),$$

where $F(x)$ is *antidifference* (instead of antiderivative) of $f(x)$

i.e., $\qquad \triangle F(x) = f(x)$.

Proof

$$\because \quad \triangle F(x) = f(x),$$

$$\sum_{x=a}^{b} f(x) = \sum_{x=a}^{b} \triangle F(x)$$
$$= \sum_{x=a}^{b} \Big[F(x+1) - F(x) \Big]$$
$$= F(b+1) - F(b)$$
$$+ F(b) - F(b-1)$$
$$\vdots$$
$$+ F(a+2) - F(a+1)$$
$$+ F(a+1) - F(a)$$
$$= F(b+1) - F(a)$$
$$= \Big[F(x) \Big]_a^{b+1}.$$

It may be noted that the upper limit here is $b+1$ and not b (as in case of integration).

B.3.3 Summation by Parts

We know that in integral calculus, a definite integral of the

form $\int_a^b f(x)dg(x)$ can be integrated by parts as

$$\int_a^b f(x)dg(x) = \Big[f(x)g(x) \Big]_a^b - \int_a^b g(x)df(x)$$

$$= \Big[f(b)g(b) - f(a)g(a) \Big] - \int_a^b g(x)df(x),$$

where $f(x)$ and $g(x)$ are continuous functions of x defined in the interval $a \leqslant x \leqslant b$.

If, however, functions $f(x)$ and $g(x)$ are defined only for integral values of x within the interval $a \leqslant x \leqslant b$, we have an analogous formula in finite difference :

$$\sum_{x=a}^{b} f(x) \,\Delta g(x) = \Big[f(x)g(x) \Big]_a^{b+1} - \sum_{x=a}^{b} g(x+1) \,\Delta f(x).$$

For example, to evaluate $\sum_{x=1}^{k} x\, a^x$ we proceed as follows :

The above formula cannot be applied directly since xa^x is not in the form $f(x)\Delta g(x)$.

However,

$$a^x = \Delta\left(\frac{a^x}{a-1}\right). \qquad [\because \;\; \Delta a^x = a^x(a-1)]$$

$$\therefore \;\; \sum_{x=1}^{k} xa^x = \sum_{x=1}^{k} x\, \Delta\left(\frac{a^x}{a-1}\right)$$

$$= \Big[x \cdot \frac{a^x}{a-1} \Big]_1^{k+1} - \sum_{x=1}^{k} \frac{a^{x+1}}{a-1} \Delta x$$

$$= \Big[x \cdot \frac{a^x}{a-1} \Big]_1^{k+1} - \sum_{x=1}^{k} \frac{a^{x+1}}{a-1} \cdot 1$$

$$= \Big[(k+1)\, \frac{a^{k+1}}{a-1} - \frac{a}{a-1} \Big] - \sum_{x=1}^{k} \frac{a^{x+1}}{a-1}.$$

Now $\quad a^{x+1}=\triangle\left(\dfrac{a^{x+1}}{a-1}\right).$

$$\therefore \sum_{x=1}^{k} xa^x=\left[(k+1)\frac{a^{k+1}}{a-1}-\frac{a}{a-1}\right]-\sum_{x=1}^{k}\triangle\left[\frac{a^{x+1}}{(a-1)^2}\right]$$

$$=\left[(k+1)\frac{a^{k+1}}{a-1}-\frac{a}{a-1}\right]-\frac{1}{(a-1)^2}\Big[a^{x+1}\Big]_1^{k+1}$$

$$=\left[(k+1)\frac{a^{k+1}}{a-1}-\frac{a}{a-1}\right]-\frac{1}{(a-1)^2}\left[a^{k+2}-a^2\right].$$

B·3·4. Differencing Under Summation Sign

In inventory models, the stock level z is a discrete variable and it is often required to find the first difference of some function C (z) of the form .

$$C(z)=\sum_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n ; z).$$

If the function $f(x_1, x_2, \ldots, x_n; z)$ has the same functional form in the whole region of summation and if the boundary of the region is independent of z, then we can apply the formula

$$\triangle[f(x)+g(x)]=\triangle f(x)+\triangle g(x)$$

to each term separately and can obtain

$$\triangle C(z)=\sum_{x_1, x_2, \ldots, x_n} \triangle f(x_1\, x_2, \ldots, x_n; z),$$

where the differences (of R.H.S.) are w.r.t. z. However, if $f(x_1, x_2, \ldots, x_n; z)$ has different forms in different parts of the region of summation, or if the boundary of the region depends upon z, calculation of $\triangle$ C (z) becomes more complicated.

B·3·4·1. Single Summation

Case 1. When x depends upon z and $f(x, z)$ has the same functional form in the region of summation. We have the function

$$C(z)=\sum_{x=a(z)}^{b(z)} f(x, z).$$

$$\therefore\ C(z+1)=\sum_{x=a(z+1)}^{b(z+1)} f(x, z+1)$$

$$=\sum_{x=a(z)}^{b(z)} f(x, z+1)+\sum_{x=b(z)+1}^{b(z+1)} f(x, z+1)-\sum_{x=a(z)}^{a(z+1)-1} f(x, z+1).$$

$$\therefore \quad \Delta C(z)=C(z+1)-C(z)$$

$$=\sum_{x=a(z)}^{b(z)} [f(x, z+1)-f(x, z)]+\sum_{b(z)+1}^{b(z+1)} f(x, z+1)-\sum_{a(z)}^{a(z+1)-1} f(x, z+1)$$

$$=\sum_{a(z)}^{b(z)} \Delta f(x, z)+\sum_{b(z)+1}^{b(z+1)} f(x, z+1)-\sum_{a(z)}^{a(z+1)-1} f(x, z+1).$$

The above analysis assumes that functions $a(z)$ and $b(z)$ increase with z.

Case 2. When x depends upon z and $f(x, z)$ is defined in different forms in different parts of the region of summation. Let the function $f(x, z)$ be defined as

$$f(x, z)=\begin{cases} f_1(x, z) & \text{for } x \text{ in the interval } 0 \leqslant x \leqslant b(z), \\ f_2(x, z) & \text{for } x>b(z), \end{cases}$$

and let it be required to find the difference of the function

$$C(z)=\sum_{x=0}^{\infty} f(x, z).$$

Now $$C(z)=\sum_{x=0}^{b(z)} f_1(x, z)+\sum_{x=b(z)+1}^{\infty} f_2(x, z).$$

Applying the formula of case 1 to both of these sums, we get

$$\Delta C(z)=\sum_{x=0}^{b(z)} \Delta f_1(x, z)+\sum_{x=b(z)+1}^{\infty} \Delta f_2(x, z)+\sum_{b(z)+1}^{b(z+1)} [f_1(x, z+1)-f_2(x, z+1)].$$

If $f_1(x, z+1)=f_2(x, z+1)$ for all values of x in the interval $b(z)+1 \leqslant b(z+1)$, then

$$\triangle C(z)=\sum_{x=0}^{\infty} \triangle f(x, z).$$

B·3·4·2. Double Summation

Case 1. When x_1 and x_2 both depend upon z.

Suppose C (z) is defined by

$$C(z)=\sum_{x_2=a(z)}^{b(z)} \sum_{x_1=C(x_2, z)}^{d(x_2, z)} f(x_1, x_2; z)$$

and that we wish to find $\triangle c(z)$.

To find $\triangle c(z)$, we first define $g(x_2, z)$ by

$$g(x_2, z)=\sum_{x_1=C(x_2, z)}^{d(x_2, z)} f(x_1, x_2; z).$$

Then $$C(z)=\sum_{x_2=a(z)}^{b(z)} g(x_2, z).$$

Now applying the result of case 1 of single summation, we get

$$\triangle C(z)=\sum_{x_2=a(z)}^{b(z)} \triangle g(x_2, z)+\sum_{b(z)+1}^{b(z+1)} g(x_2, z+1)$$
$$-\sum_{a(z)}^{a(z+1)-1} g(x_2, z+1).$$

Applying the same result to $g(x_2, z)$, we get

$$\triangle g(x_2, z)=\sum_{x_1=c(x_2, z)}^{d(x_2, z)} \triangle f(x_1, x_2; z)+\sum_{d(x_2, z)+1}^{d(x_2, z+1)} f(x_1, x_2; z+1)$$
$$-\sum_{c(x_2, z)}^{c(x_2, z+1)-1} f(x_1, x_2; z+1),$$

where all the differences are w.r.t. z.

$$\therefore \quad \triangle c(z)=\sum_{x_2=a(z)}^{b(z)} \sum_{x_1=c(x_2, z)}^{d(x_2, z)} \triangle f(x_1, x_2; z)$$
$$+\sum_{a(z)}^{b(z)} \sum_{d(x_2, z)+1}^{d(x_2, z+1)} f(x_1, x_2; z+1)$$

$$-\sum_{a(z)}^{b(z)} \sum_{c(x_2, z)}^{c(x_2, z+1)-1} f(x_1, x_2; z+1)$$

$$+\sum_{b(z)+1}^{b(z+1)} \sum_{c(x_2, z+1)}^{d(x_2, z+1)} f(x_1, x_2; z+1)$$

$$-\sum_{a(z)}^{a(z+1)-1} \sum_{c(x_2, z+1)}^{d(x_2, z+1)} f(x_1, x_2; z+1).$$

Here again it is assumed that the boundary functions a, b, c, and d are increasing with z.

Case 2. When $f(x_1, x_2; z)$ is defined in different parts of the region of summation. Let the function be defined (as shown in figure B·1) as

$$f(x_1, x_2; z)=\begin{cases} f_1(x_1, x_2; z), \text{ for } [0 \leqslant x_1 \leqslant c(x_2, z);\ 0 \leqslant x_2 \leqslant b(z)], \\ f_2(x_1, x_2; z), \text{ for } [c(x_2, z) < x_1;\ 0 \leqslant x_2 \leqslant b(z)], \\ f_3(x_1, x_2, z), \text{ for } [0 \leqslant x_1 \leqslant d(x_2, z);\ b(z) < x_2], \\ f_4(x_1, x_2; z), \text{ for } [d(x_2, z) < x_1;\ b(z) < x_2]. \end{cases}$$

Then

$$C(z)=\sum_{0}^{\infty} \sum_{0}^{\infty} f(x_1, x_2; z)$$

$$=\sum_{0}^{b} \sum_{0}^{c} f_1+\sum_{0}^{b} \sum_{c+1}^{\infty} f_2+\sum_{b+1}^{\infty} \sum_{0}^{d} f_3+\sum_{b+1}^{\infty} \sum_{d+1}^{\infty} f_4.$$

Now applying the result of case 1, we get

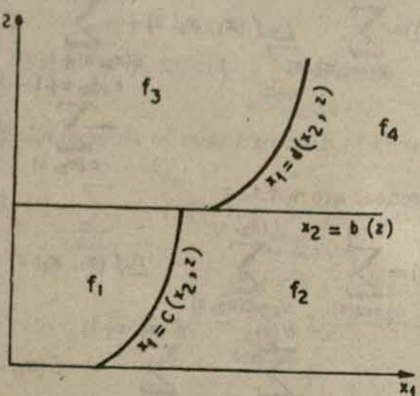

Fig. B-1

$$\triangle C(z)=\sum_{0}^{b}\sum_{0}^{c}\triangle f_1+\sum_{0}^{b}\sum_{c+1}^{\infty}\triangle f_2+\sum_{b+1}^{\infty}\sum_{0}^{d}\triangle f_3$$

$$+\sum_{b+1}^{\infty}\sum_{d+1}^{\infty}\triangle f_4$$

$$+\sum_{0}^{b}\sum_{c+1}^{c(x_2,\,z+1)}[f_1(x_1,x_2;z+1)-f_2(x_1,x_2;z+1)]$$

$$+\sum_{b+1}^{\infty}\sum_{d+1}^{d(x_2,\,z+1)}[f_3(x_1,x_2;z+1)-f_4(x_1,x_2;z+1)]$$

$$+\sum_{b+1}^{b(z+1)}\Big[\sum_{0}^{c(x_2,\,z+1)}f_1(x_1,x_2;z+1)+\sum_{c(x_2,\,z+1)+1}^{\infty}f_2(x_1,x_2;z+1)$$

$$-\sum_{0}^{d(x_2,\,z+1)}f_3(x_1\,x_2;z+1)-\sum_{d(x_2,\,z+1)-1}^{\infty}f_4(x_1,x_2;z+1)\Big]$$

It is possible that functions f_1, f_2, f_3, f_4 are such that all terms except the first four cancel out.

APPENDIX C

Differentiation of Integrals

Single Integration

Case 1. When x depends upon z and $f(x, z)$ has the same functional form in the region of integration. Let the function be defined as

$$F(z)=\int_{a(z)}^{b(z)} f(x, z).dx,$$

where $a(z)$ and $b(z)$ are the limits of integration. If $f(x, z)$ has a continuous derivative with respect to z throughout the region of integration and if the derivatives $\frac{d}{dz}[a(z)]$ and $\frac{d}{dz}[b(z)]$ exit, then

$$\frac{d}{dz}[F(z)]=\int_{a(z)}^{b(z)} \frac{\partial}{\partial z}[f(x, z)].dx+\left[f(x, z).\frac{dx}{dz}\right]_{a(z)}^{b(z)}$$

$$=\int_{a(z)}^{b(z)} \frac{\partial}{\partial z}[f(x, z)].dx+\left[f\{b(z), z\}.\frac{d}{dz}\{b(z)\}, \quad -f\{a(z), z\}.\frac{d}{dz}\{a(z)\}\right].$$

If $a(z)$ and $b(z)$ are constants, then $\frac{d}{dz}\{a(z)\}=0$ and $\frac{d}{dz}\{b(z)\}=0$.

$$\therefore \quad \frac{d}{dz}[F(z)]=\int_{a(z)}^{b(z)} \frac{\partial}{\partial z}[f(x, z)].dx$$

or $\quad \frac{dF}{dz}=\int_a^b \frac{\partial f(x, z)}{\partial z}.dx$, it being understood that F, a and b are functions of z.

This result holds good for $a=-\infty$ and $b=\infty$, provided the integral on the R.H.S. converges.

Case 2. When x depends upon z and $f(x, z)$ is defined in different forms in different parts of the region of integration. Let the function $f(x, z)$ be defined as

$$\begin{aligned} f(x, z) &= f_1(x, z) \quad &\text{for} \quad 0\leqslant x\leqslant b(z),\\ &= f_2(x, z) \quad &\text{for} \quad b(z)\leqslant x\leqslant \infty, \end{aligned}$$

and let it be required to find the differentiation of the function

$$F(z)=\int_0^{\infty} f(x, z).dx.$$

The above function can be differentiated only if $f_1(x, z)$ has a continuous derivative w.r.t. z in the region $0 \leqslant x \leqslant b(z)$ and $f_2(x, z)$ has a continuous derivative w.r.t. z in the region $x \geqslant b(z)$.

Now $$F(z)=\int_0^{b(z)} f_1(x, z).dx+\int_{b(z)}^{\infty} f_2(x, z).dx.$$

Using the result of case 1, we have

$$\frac{d}{dz}[F(z)]=\int_0^{b(z)} \frac{\partial}{\partial z} f_1(x, z).dx+\left[f_1(x, z).\frac{dx}{dz}\right]_0^{b(z)}$$

$$+\int_{b(z)}^{\infty} \frac{\partial}{\partial z} f_2(x, z).dx+\left[f_2(x, z).\frac{dx}{dz}\right]_{b(z)}^{\infty}$$

$$=\int_0^{b(z)} \frac{\partial f_1(x, z)}{\partial z}.dx+\int_{b(z)}^{\infty} \frac{\partial f_2(x, z)}{\partial z}.dx$$

$$+\left[f_1\{b(z), z\}\frac{d}{dz}\{b(z)\}+0+0-f_2\{b(z), z\}.\frac{d}{dz}\{b(z)\}\right]$$

$$=\int_0^{b(z)} \frac{\partial f_1(x, z)}{\partial z}.dx+\int_{b(z)}^{\infty} \frac{\partial f_2(x, z)}{\partial z}.dx$$

$$+\frac{d}{dz}\{b(z)\}[f_1\{b(z), z\}-f_2\{b(z), z\}]$$

or $$\frac{dF}{dz}=\int_0^{b} \frac{\partial f_1(x, z)}{\partial z}.dx+\int_b^{\infty} \frac{\partial f_2(x, z)}{\partial z}.dx$$

$$+\frac{db}{dz}\{f_1(b, z)-f_2(b, z)\},$$

where F, a and b are functions of z. Further, if the expression within brackets is identically zero, then

$$\frac{dF}{dz}=\int_0^{b} \frac{\partial f_1(x, z)}{\partial z}.dx+\int_b^{\infty} \frac{\partial f_2(x, z)}{\partial z}.dx.$$

Double Integration

Case 1. When $f(x, y\,;z)$ has the same functional form in the region of integration. Let

$$F(z)=\int_{a(z)}^{b(z)}\int_{c(y,z)}^{d(y,z)} f(x, y\,;z).dx.dy.$$

The above function can be differentiated if derivatives $\frac{\partial f}{\partial z}$, $\frac{\partial c}{\partial z}$ and $\frac{\partial d}{\partial z}$ exist and are continuous w.r.t. z throughout the region of integration. Let

$$G(y, z)=\int_{c(y,z)}^{d(y,z)} f(x, y; z).dx.$$

$\therefore$ The given function be written as

$$F(z)=\int_{a(z)}^{b(z)} G(y, z).dy \quad \text{or} \quad F=\int_a^b G(y, z).dy.$$

$$\text{Then } \frac{dF}{dz}=\int_a^b \frac{\partial G(y, z)}{\partial z}.dy+\left[G(y, z).\frac{dy}{dz}\right]_a^b$$

$$=\int_a^b \frac{\partial G(y, z)}{\partial z}.dy+G(b, z).\frac{db}{dz}-G(a, z)\frac{da}{dz}$$

$$=\int_a^b \frac{\partial G(y, z)}{\partial z}.dy+\frac{db}{dz}.G(b, z)-\frac{da}{dz}.G(a, z).$$

Substituting the value of $G(y, z)$ we get

$$\frac{\partial F}{\partial z}=\int_a^b\int_c^d \frac{\partial G(y, z)}{\partial z}.dx.dy+\int_a^b \frac{\partial d}{\partial z}.f(d, y; z).dx-\int_a^b \frac{\partial c}{\partial z}.f(c, y; z).dx$$

$$+\frac{db}{dz}\int_{c(b,z)}^{d(b,z)} f(x, b; z)dx-\frac{da}{dz}\int_{c(a,z)}^{d(a,z)} f(x, a; z)dx.$$

Case 2. When $f(x, y; z)$ is defined in different parts of the region of integration. Let a region R of the $x_1\,x_2$-plane in figure B·1 be divided into disjoint sub-regions R_i, where $\Sigma R_i=R$ and let there

be a function $f_i(x, y; z)$ for each subregion R_i. Let $\mathbf{F}_2$ be defined by

$$F(z)=\sum_i \iint_{R_i} f_i(x, y; z)\,dx\,dy.$$

Then $$\frac{dF(z)}{dz}=\sum_i \iint_{R_i} \frac{\partial f_i(x, y; z)}{\partial z}\,dx\,dy,$$

if the following conditions are satisfied:

(*i*) the boundary of the entire region R does not depend upon z,

(*ii*) if R_i and R_j are adjacent subregions, then $f_i(x, y; z) = f_j(x, y; z)$ at all the points (x, y) along the common boundary R_i and R_j.

TABLE C-1

Random Numbers Table

2181922396	2068577984	8262130892	8374856049	4837657422
1128105582	7295088579	9586111652	7055508767	5172382962
7112077556	3440672486	1882412963	0684012006	0633147925
6557477468	5435810788	9670852913	1291265730	2290031331
4199520858	3090908872	2039593181	5973470495	8076135523
3545174840	2275698645	8416549348	4676463101	5629367907
1749420382	4832630032	5670984959	5432114610	0666095693
9103161011	7413686599	1198757695	0414294470	9240121544
0764238934	7666127259	5263097712	5133648980	5111966912
3493969525	0272759769	0385998136	9999089966	1344056826
1292054466	0700014629	5169439659	8408705169	6574373193
4397426117	6488888550	4031652526	8123543276	6027534501
3807950579	9564268448	3457416988	1531027886	5116633717
4984768758	2389278610	3859431781	3643768456	5041314549
1340145867	9120831830	7228567652	1267173884	1320651658
0590453442	4800088084	1165628554	5407921254	9468932498
9566554338	5585265145	5089052204	9780623691	5795448061
7615116284	9172824179	5544814334	0016943666	2628538741
8508771938	4035554324	0840126299	4942059208	7875623913
6970024586	9324732696	1186263397	4425143189	3316653259
5799997185	0135968939	7678931194	1351031403	6002561840
6364375912	8383232768	1892850701	2323673751	3188881718
4165492027	6349104233	3382569662	4579426926	1513082455
0354683246	4765104877	8149224168	5468631609	6474393896
9130555058	5255147182	3519287786	2481675649	8907598697
5826984369	4725370390	9641916289	5049082870	7463807244
6285048453	3646121751	8436077768	2928794356	9956043516
7527791048	5765558107	8762562043	6185670830	6363845920
8976470693	0441608934	8749472723	2202271078	5897002653
2327991661	7936797054	9527542791	4711871173	8300978148
1182095589	5535798279	4764439855	6279247618	4446895088
3659397698	1056981450	8416606706	8234013222	6426813469
5924779358	1333750468	9434074212	5273692238	5902177065
3941092295	5726289716	3420847871	1820481234	0318831723
1955104281	0903099163	6827824899	6383872737	5901682626
2117595534	1634107293	8521057472	1471300754	3044151557
7471564123	7344613447	1128117244	3208461091	1699403490
8674262892	2809456764	5806554509	8224980942	5738031833
9061122871	0746980892	9285305274	6331989649	8764467686
6438538678	3049068967	6955157269	5482964330	2161984904
1834182305	6203476893	5937802079	3445280195	3694915658
1884227732	2923727501	8044389132	4611203081	6072112445
6791857341	6696243386	2219599137	3193884246	8224729918

TABLE C-2

Proportion of total area under the normal curve from ∞ to t,

where $t = \dfrac{x-\mu}{\sigma}$

t	$\psi(t)$	t	$\psi(t)$	t	$\psi(t)$	t	$\psi(t)$
0.00	0.5000	0.65	0.7422	1.30	0.9032	1.95	0.9744
0.01	0·5040	0.66	0.7454	1.31	0.9049	1.96	0.9750
0.02	0.5080	0.67	0.7486	1.32	0.9066	1.97	0.9756
0.03	0.5120	0.68	0.7517	1.33	0.9082	1.98	0.9761
0.04	0.5160	0.69	0.7549	1.34	0.9099	1.99	0.9767
0.05	0.5199	0.70	0.7580	1.35	0.9115	2.00	0.9772
0.06	0.5239	0.71	0·7611	1.36	0.9131	2.02	0.9783
0.07	0.5279	0.72	0.7642	1.37	0.9147	2.04	0.9793
0.08	0.5319	0.73	0.7673	1.38	0.9162	2.06	0.9803
0.09	0.5359	0.74	0.7703	1.39	0.9177	2.08	0.9812
0.10	0.5398	0.75	0.7734	1.40	0.9192	2.10	0.9821
0.11	0.5438	0.76	0.7764	1.41	0 9207	2.12	0.9830
0.12	0.5478	0.77	0.7794	1.42	0.9222	2.14	0.9838
0.13	0.5517	0.78	0.7823	1.43	0.9236	2.16	0.9846
0.14	0.5557	0.79	0.7852	1.44	0.9251	2.18	0.9854
0.15	0.5596	0.80	0.7881	1.45	0.9265	2.20	0.9861
0.16	0.5636	0.81	0.7910	1.46	0.9279	2.22	0.9868
0.17	0.5675	0.82	0.7939	1.47	0.9292	2.24	0.9875
0.18	0.5714	0.83	0.7967	1.48	0.9306	2.26	0.9881
0.19	0.5753	0.84	0.7995	1.49	0.9319	2.28	0.9887
0.20	0.5793	0.85	0.8023	1.50	0.9332	2.30	0.9893
0.21	0.5832	0.86	0.8051	1.51	0.9345	2.32	0.9898
0.22	0.5871	0.87	0.8078	1.52	0.9357	2.34	0.9904
0.23	0.5910	0.88	0.8106	1.53	0.9370	2.36	0.9909
0.24	0.5948	0.89	0.8133	1.54	0.9382	2.38	0.9913
0.25	0.5987	0.90	0.8159	1.55	0.9394	2.40	0.9918
0.26	0.6026	0.91	0.8186	1.56	0.9406	2.42	0.9922
0.27	0.6064	0.92	0.8212	1.57	0.9418	2.44	0.9927
0.28	0.6103	0.93	0.8238	1.58	0.9429	2.46	0.9931
0.29	0.6141	0.94	0.8264	1.59	0.9441	2.48	0.9934
0.30	0.6179	0.95	0.8289	1.60	0.9452	2.50	0.9938
0.31	0.6217	0.96	0.8315	1.61	0.9463	2.52	0.9941
0.32	0.6255	0.97	0.8340	1.62	0.9474	2.54	0.9945
0.33	0.6293	0.98	0.8365	1.63	0.9484	2.56	0.9948
0.34	0.6331	0.99	0.8389	1.64	0.9495	2.58	0.9951
0.35	0.6368	1.00	0.8413	1.65	0.9505	2.60	0.9953
0.36	0.6406	1.01	0.8438	1.66	0.9515	2.62	0.9956
0.37	0.6443	1.02	0.8461	1.57	0.9525	2.64	0.9959
0.38	0.6480	1.03	0.8485	1.68	0.9535	2.66	0.9961
0.39	0.6517	1.04	0.8508	1.69	0.9545	2.68	0.9963
0.40	0.6554	1.05	0.8531	1.70	0.9554	2.70	0.9965
0.41	0.6591	1.06	0.8554	1.71	0.9564	2.72	0.9967
0.42	0.6628	1.07	0.8577	1.72	0.9573	2.74	0.9969
0.43	0.6664	1.08	0.8599	1.73	0.9582	2.76	0.9971
0.44	0.6700	1.09	0.8621	1.74	0.9591	2.78	0.9973
0.45	0.6736	1.10	0.8643	1.75	0.9599	2.80	0.9974
0.46	0.6772	1.11	0.8665	1.76	0.9608	2.82	0.9976
0.47	0.6808	1.12	0.8686	1.77	0.9616	2.84	0.9977
0.48	0.6844	1.13	0.8708	1.78	0.9625	2.86	0.9979
0.49	0.6879	1.14	0.8729	1.79	0.9633	2.88	0.9980
0.50	0.6915	1.15	0.8749	1.80	0.9641	2.90	0.9981
0.51	0.6950	1.16	0.8770	1.81	0.9649	2.92	0.9982
0.52	0.6985	1.17	0.8790	1.82	0.9656	2.94	0.9984
0.53	0.7019	1.18	0.8810	1.83	0.9664	2.96	0.9985
0.54	0.7054	1.19	0.8830	1.84	0.9671	2.98	0.9986
0.55	0.7088	1.20	0.8849	1.85	0.9678	3.00	0·99865
0.56	0.7123	1.21	0.8869	1.86	0.9686	3.20	0.99931
0·57	0.7157	1.22	0.8888	1.87	0.9693	3.40	0.99966
0.58	0.7190	1.23	0.8907	1.88	0.9699	3.60	0.999841
0.59	0.7224	1.24	0.8925	1.89	0.9706	3.80	0.999928
0.60	0.7257	1.25	0.8944	1.90	0.9713	4.00	0.999968
0.61	0.7291	1.26	0.8962	1.91	0.9719	4.50	0.999997
0.62	0.7324	1.27	0.8980	1.92	0.9726	5.00	0.999997
0.63	0.7357	1.28	0.8997	1.93	0.9732		
0.64	0.7389	1.29	0.9015	1.94	0.9738		

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9566554338	5585265145	5089052204	9780623691	5795448061
7615116284	9172824179	5544814334	0016943666	2628538741
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6970024586	9324732696	1186263397	4425143189	3316653259
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4165492027	6349104233	3382569662	4579426926	1513082455
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5826984369	4725370390	9641916289	5049082870	7463807244
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8976470693	0441608934	8749472723	2202271078	5897002653
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3941092295	5726289716	3420847871	1820481234	0318831723
1955104281	0903099163	6827824899	6383872737	5901682626
2117595534	1634107293	8521057472	1471300754	3044151557
7471564123	7344613447	1128117244	3208461091	1699403490
8674262892	2809456764	5806554509	8224980942	5738031833
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1884227732	2923727501	8044389132	4611203081	6072112445
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